The Boltzmann collision equation in quantum field theory

Gregor Leiler
(supervisor Dr. Stefan Hollands)

Cardiff School of Mathematics
Cardiff University

24$^{\text{th}}$ LQP Workshop
26 – 27 June 2009, Leipzig

1. The Boltzmann equation
   - What is it and how is it used?
   - Current theoretical understanding
2. A derivation
   - The model
   - Bits and pieces of the derivation
   - Some results
3. Overview and conclusions
   - Baryogenesis revisited
   - Conclusions

The Boltzmann collision equation – Applications

A statndard tool in the non-equilibrium statistical mechanics toolbox

The "general relativistic quantum" Boltzmann equation in RW spacetime

$$\dot{f}_{\mathbf{p}_\psi}(t) + 3\frac{\dot{L}(t)}{L(t)} f_{\mathbf{p}_\psi}(t) = -\int \mathrm{dLIPS}\ \delta\begin{pmatrix}4 - \mathit{momentum} \\ \mathit{conservation}\end{pmatrix} \cdot \Big[|\mathcal{M}_{\psi+a+b+\cdots\to i+j+\ldots}|^2 f_{\mathbf{p}_\psi} f_{\mathbf{p}_a} f_{\mathbf{p}_b} \ldots (1 \pm f_{\mathbf{p}_i})(1 \pm f_{\mathbf{p}_j}) \ldots - |\mathcal{M}_{i+j+\ldots\to\psi+a+b+\ldots}|^2 f_{\mathbf{p}_i} f_{\mathbf{p}_j} \ldots (1 \pm f_{\mathbf{p}_\psi})(1 \pm f_{\mathbf{p}_a})(1 \pm f_{\mathbf{p}_b}) \ldots \Big]$$

dLIPS = Lorentz Invariant Phase Space measure,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + L^2(t)\mathrm{d}\mathbf{x}^2, \qquad f_{\mathbf{p}}(t) = \frac{1}{V(t)}\langle N_{\mathbf{p}}(t)\rangle$$

Applications (with successful quantitative predictions)

- diffusion of classical gasses
- computation of viscosity coefficients (ordinary liquids to quark–gluon plasma)
- nucleosynthesis and baryogenesis
- ...

The Boltzmann collision equation – Applications

A statndard tool in the non-equilibrium statistical mechanics toolbox

The "general relativistic quantum" Boltzmann equation in RW spacetime

$$\dot{f}_{\mathbf{p}_\psi}(t) + 3\frac{\dot{L}(t)}{L(t)} f_{\mathbf{p}_\psi}(t) = -\int \mathrm{dLIPS}\ \delta\begin{pmatrix} 4 - \textit{momentum} \\ \textit{conservation} \end{pmatrix} \cdot \Big[|\mathcal{M}_{\psi+a+b+\cdots\to i+j+\ldots}|^2 f_{\mathbf{p}_\psi} f_{\mathbf{p}_a} f_{\mathbf{p}_b} \ldots (1 \pm f_{\mathbf{p}_i})(1 \pm f_{\mathbf{p}_j}) \ldots - |\mathcal{M}_{i+j+\ldots\to\psi+a+b+\ldots}|^2 f_{\mathbf{p}_i} f_{\mathbf{p}_j} \ldots (1 \pm f_{\mathbf{p}_\psi})(1 \pm f_{\mathbf{p}_a})(1 \pm f_{\mathbf{p}_b}) \ldots \Big]$$

dLIPS = Lorentz Invariant Phase Space measure,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + L^2(t)\mathrm{d}\mathbf{x}^2, \qquad f_{\mathbf{p}}(t) = \frac{1}{V(t)}\langle N_{\mathbf{p}}(t)\rangle$$

Applications (with successful quantitative predictions)

- diffusion of classical gasses
- computation of viscosity coefficients (ordinary liquids to quark–gluon plasma)
- nucleosynthesis and baryogenesis
- ...

A peek into the literature

Standard references:

- E. W. Kolb & M. S. Turner – 1990, *The early universe*
- K. Huang – 1987, *Statistical mechanics*

Selected works on the quantum Boltzmann equation:

- L. Boltzmann – 1872, *Weitere studien über das Wärmegleichgewicht unter gasmolekülen*
- E. A. Uehling & G. E. Uhlenbeck – 1933, *Introduce the QBE*
- N. M. Hugenholtz – 1983, *Derivation of QBE for lattice Fermi gas*
- L. Erdős & H.-T. Yau – 1998, *Linear QBE for Lorentz gas*
- L. Erdős, M. Salmhofer & H-T. Yau – 2004, *Heuristic derivation of QBE*
- D. Benedetto, F. Castella, R. Esposito & M. Pulvirenti – 2004, *Quantum N body system – Wigner function approach*
- D. Buchholz – 2003, *Collisionless BE in QFT for free massless scalar field in LTE states*

The projection operator technique:

- L. Van Hove – 1955, 1959; L. Prigogine & R. Balescu – 1959; R. Zwanzig – 1960
- B. Robertson – 1967, 1970; K. Kawasaki & J. D. Gunton – 1973

Issues with the Boltzmann collision equation

Problems:

- What is a "particle" in CST?
- "Flat" or "curved" scattering amplitude?
- General covariance?
- Not an "exact" equation (unlike Schrödinger or Heiseneberg equations) – when does it provide a valid approximate description?
  - Separation of the time scales
  - Weak coupling and/or low density
- "Textbook" derivation requires *Stosszahlansatz* ("molecular chaos") – not really justified
  - Particle velocity distribution is assumed to be Gaussian at *all* times not only at the initial time
- "Weak coupling limit" derivations justify $|\mathcal{M}|^2$ in the Born approximation – Need to do better (e.g. baryogenesis)

This talk and future work

Aims of the talk:

- Heuristic/formal derivation of the BE from first principles – QFT on flat space
- Understand better the approximations that go into the derivation
- Systematic understanding of corrections

Work in progress:

- Generalization to curved spaces – curvature corrections
- Applications

The model

- $(1+1)$-D spacetime $\mathbb{R} \times \mathbb{S}^1$ with metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + L^2 \mathrm{d}\mathbf{x}^2$
  - momentum discretization
- Hamiltonian (self adjoint operator – Glimm & Jaffe):

$$H = H_0(t) + V(t) = L\frac{1}{2}\int_0^{2\pi} \mathrm{d}\mathbf{x}\Big[:\pi^2(t,\mathbf{x}): + L^2 : \big(\partial_{\mathbf{x}}\varphi(t,\mathbf{x})\big)^2 : + m^2 : \varphi^2(t,\mathbf{x}):\Big] +$$

$$+ L\frac{\lambda}{4!}\int_0^{2\pi} \mathrm{d}\mathbf{x} : \varphi^4(t,\mathbf{x}):$$

- Dynamics of an observable A described by Heisenberg equation of motion

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = i\delta\big[A(t)\big] = i\big[H, A(t)\big], \qquad A(t) = \alpha_t(A), \ \ \forall A \in \mathscr{A}$$

- Perturbation theory: interacting field $=$ formal power series in λ

Set up of the notation

Would like to derive an eqn of the form:

$$\partial_t f_{\mathbf{p}}(t) = C\left[\{f_{\mathbf{k}}(t)\}\right],$$

where:

$$f_{\mathbf{p}}(t) = \left\langle N_{\mathbf{p}}(t)\right\rangle_{\psi}/L = n_{\mathbf{p}}(t)/L,$$

$$N_{\mathbf{p}}(t) = a^{\dagger}_{\mathbf{p}}(t)a_{\mathbf{p}}(t), \qquad a_{\mathbf{p}}(t) := i\int_0^{2\pi} \mathrm{d}\mathbf{x} e^{-ipx} \overleftrightarrow{\partial_t}\varphi(t,\mathbf{x})$$

Heisenberg eqn contains info we don't need

- Idea: projection operator technique $\rightarrow$ introduce linear maps

$$\mathcal{P}_t : \mathscr{A} \mapsto \mathscr{A}, \qquad \mathcal{P}_t \circ \mathcal{P}_s = \mathcal{P}_s, \qquad \mathcal{Q}_t := id - \mathcal{P}_t$$

$$Y_{s,t} : \mathscr{A} \mapsto \mathscr{A}, \qquad \partial_t Y_{s,t} = Y_{s,t} \circ i\delta \circ \mathcal{Q}_t, \quad Y_{s,s} = id$$

(This work: perturbative solution for $Y_{s,t}$)

Outline of the derivation

1. Introduce the projectors $\mathcal{P}_t$ and $\mathcal{Q}_t$
2. A pre-Boltzmann equation
3. The scaling limit
4. The Boltzmann collision factor

The projector $\mathcal{P}_t$

Our $\mathcal{P}_t$ is adapted to the observables $\{N_{\mathbf{p}}(t)\}$ that we want to study

1. Introduce, $\forall t \in \mathbb{R}$, unique quasifree state $\langle \,\cdot\, \rangle_{\omega(t)}$ s.t., given $n_{\mathbf{p}}(t) \geq 0$

$$\langle N_{\mathbf{p}}(t)\rangle_{\omega(t)} = n_{\mathbf{p}}(t)$$

$$\Updownarrow \quad \text{(formally)}$$

$$\langle A\rangle_{\omega(t)} := \frac{1}{Z(t)}\mathrm{Tr}\left[\mathrm{e}^{-\sum_{\mathbf{q}}\lambda_{\mathbf{q}}(t)N_{\mathbf{q}}(t)}\, A\right], \qquad \lambda_{\mathbf{q}}(t) = \log\left(\frac{1+n_{\mathbf{q}}(t)}{n_{\mathbf{q}}(t)}\right)$$

2. Set $\Delta N_{\mathbf{p}}(t) := N_{\mathbf{p}}(t) - n_{\mathbf{p}}(t)\mathbf{1}$ and $C^{\omega(t)}_{\mathbf{pq}} := \langle \Delta N_{\mathbf{p}}(t)\Delta N_{\mathbf{q}}(t)\rangle_{\omega(t)}$

$$\mathcal{P}_t(A) := \langle A\rangle_{\omega(t)}\mathbf{1} + \sum_{\mathbf{p},\mathbf{q}\in\mathbb{Z}} (C^{\omega(t)}_{\mathbf{pq}})^{-1}\langle \Delta N_{\mathbf{q}}(t)A\rangle_{\omega(t)}\Delta N_{\mathbf{p}}(t), \quad \forall A \in \mathscr{A}$$

The projector $\mathcal{P}_t$

Our $\mathcal{P}_t$ is adapted to the observables $\{N_{\mathbf{p}}(t)\}$ that we want to study

1. Introduce, $\forall t \in \mathbb{R}$, unique quasifree state $\langle \ \cdot \ \rangle_{\omega(t)}$ s.t., given $n_{\mathbf{p}}(t) \geq 0$

$$\langle N_{\mathbf{p}}(t)\rangle_{\omega(t)} = n_{\mathbf{p}}(t)$$

$$\Updownarrow \quad \text{(formally)}$$

$$\langle A\rangle_{\omega(t)} := \frac{1}{Z(t)}\mathrm{Tr}\left[\mathrm{e}^{-\sum_{\mathbf{q}}\lambda_{\mathbf{q}}(t)N_{\mathbf{q}}(t)}\, A\right], \qquad \lambda_{\mathbf{q}}(t) = \log\left(\frac{1+n_{\mathbf{q}}(t)}{n_{\mathbf{q}}(t)}\right)$$

2. Set $\Delta N_{\mathbf{p}}(t) := N_{\mathbf{p}}(t) - n_{\mathbf{p}}(t)\mathbf{1}$ and $C^{\omega(t)}_{\mathbf{pq}} := \langle \Delta N_{\mathbf{p}}(t)\Delta N_{\mathbf{q}}(t)\rangle_{\omega(t)}$

$$\mathcal{P}_t(A) := \langle A\rangle_{\omega(t)}\mathbf{1} + \sum_{\mathbf{p},\mathbf{q}\in\mathbb{Z}} (C^{\omega(t)}_{\mathbf{pq}})^{-1}\langle \Delta N_{\mathbf{q}}(t) A\rangle_{\omega(t)}\Delta N_{\mathbf{p}}(t), \quad \forall A \in \mathscr{A}$$

The Robertson equation

With the above projector we can derive (the “Robertson eqn”):

$$\dot{n}_{\mathbf{p}}(t) = \int_{t_0}^{t} \mathrm{d}s \left\langle \alpha_s \circ i\delta \circ Y_{s,t} \circ i\delta \, N_{\mathbf{p}}(t_0) \right\rangle_{\omega(s)} .$$

- Further progress by solving the equation for $Y_{s,t}$ and expressing it in terms of $V(t) = \frac{\lambda}{4!} \int_0^{2\pi} \mathrm{d}\mathbf{x} \; :\varphi^4(t, \mathbf{x}):$

We define

$$B(E, \mathbf{p}, s) := E \int_{\mathbb{R}} \mathrm{d}\tau \, \mathrm{e}^{-iE\tau} \left\langle \mathcal{R}\left[N_{\mathbf{p}}(\tau); V(0) \otimes \exp_{\otimes}\left(-i \int_0^{\infty} V(T) \, \mathrm{d}T \right) \right] \right\rangle_{\omega(s)}$$

- $\mathcal{R}$ – retarded product
- $V(t)$ – interaction potential
- $D = 2$ – don't need renormalization

Some results – a pre-Boltzmann equation

A lengthy calculation yields

A pre-Boltzmann equation

$$\dot{n}_{\mathbf{p}}(t) = \int_{t_0}^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}E\ \mathrm{e}^{iE(t-s)} B(E,\mathbf{p},s) + \sum_{n=1}^{\infty} (-1)^n \int_{t_0}^{t} \mathrm{d}s \int_{s\le\tau_1\le\ldots\le\tau_n\le t} \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n\in\mathbb{Z}}$$

$$\cdot \left\{ \int_{\mathbb{R}} \mathrm{d}E\ \mathrm{e}^{iE(\tau_1-s)} B(E,\mathbf{k}_1,s) \left[\prod_{i=1}^{n-1} \frac{\partial}{\partial n_{\mathbf{k}_i}(\tau_i)} \int_{\tau_i}^{\tau_{i+1}} \mathrm{d}\tau_i' \int_{\mathbb{R}} \mathrm{d}E_i\ \mathrm{e}^{iE_i(\tau_i'-\tau_i)} B(E_i,\mathbf{k}_{i+1},\tau_i) \right] \cdot \right.$$

$$\left. \cdot \frac{\partial}{\partial n_{\mathbf{k}_n}(\tau_n)} \int_{\tau_n}^{t} \mathrm{d}\tau_n' \int_{\mathbb{R}} \mathrm{d}E_n\ \mathrm{e}^{iE_n(\tau_n'-\tau_n)} B(E_n,\mathbf{p},\tau_n) \right\}$$

Remarks:

- Exact (non–Markovian) equation
- “Rescattering” correction terms ($n \geq 1$)

Some results – Towards the Boltzmann collision factor

Next step: Relate $B(E, \mathbf{p}, s)$ to $\mathcal{S}$-matrix elements. We have

$$\dot{n}_{\mathbf{p}}(t) = \int_{t_0}^{t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}E \; \mathrm{e}^{iE(t-s)} B(E, \mathbf{p}, s) + \dots$$

Further progress: consider various limits

- Infinite volume limit: $L \to \infty$
- Scaling limit [Van Hove, Hugenholtz, ESY, . . .]
  - Weak coupling $(\lambda^2 t)$: $t \mapsto t/\epsilon$, $\lambda \mapsto \lambda\sqrt{\epsilon}$
  - Low density limit: $t \mapsto t/\epsilon$, $f_{\mathbf{p}}(t) \mapsto \epsilon^{\alpha} f_{\mathbf{p}}(t/\epsilon)$
  - Curved space: $L(t) \to L(\epsilon t)$

The “long time limit”: $B(E, \mathbf{p}, s) \mapsto B(0, \mathbf{p}, s)\delta(t - s)$
(Up to interchange of limits and integrals, ...)

Some results – the Boltzmann collision factor

After a long computation:

The Boltzmann collision factor

$$B(0,\mathbf{p},s) = 2\pi \sum_{\substack{r\to l \\ processess}} \int_{\mathbb{R}} \frac{\mathrm{d}\mathbf{p}_1}{2\omega(\mathbf{p}_1)} \cdots \frac{\mathrm{d}\mathbf{p}_r}{2\omega(\mathbf{p}_r)} \frac{\mathrm{d}\mathbf{q}_1}{2\omega(\mathbf{q}_1)} \cdots \frac{\mathrm{d}\mathbf{q}_l}{2\omega(\mathbf{q}_l)} \left|\widetilde{\mathcal{M}}(r\to l)\right|^2 \cdot$$

$$\cdot\, \delta^2\left(\sum_{i=1}^{r} p_i - \sum_{j=1}^{l} q_j\right)\left[\sum_{i=1}^{r}\delta(\mathbf{p}-\mathbf{p}_i) - \sum_{j=1}^{l}\delta(\mathbf{p}-\mathbf{q}_j)\right]\prod_{i=1}^{r} f_{\mathbf{p}_i}(s)\prod_{j=1}^{l}\left(1+f_{\mathbf{q}_j}(s)\right)$$

with $\quad f_{\mathbf{p}}(s) = \lim_{L\to\infty}\frac{n_{\mathbf{p}}(s)}{L}$

Remarks:

- Sum over *all* $r \to l$ scattering processess
- “Dressed” amplitude $\widetilde{\mathcal{M}}$ (to compute with Feynman rules)
  - $\widetilde{\Delta}_{F,t}(x-y) = \Delta_F(x-y) +$ correction depending on $f_{\mathbf{p}}(t)$
- All orders in λ
- Single scattering amplitude (CP invariant model)

An application – Baryogenesis revisited

Experimental "fact": maximal matter–anti-matter asymmetry.

Question: Why?

Answer: Baryogenesis (Sakharov – 1967)

- Baryon number violating interactions $\surd$ (GUT)

$$B = \begin{cases} +1 & \text{for baryons} \\ -1 & \text{for anti-baryons} \\ 0 & \text{for mesons} \end{cases}$$

- C and CP violation $\surd$ (Electroweak sector of SM)
- Thermal non–equilibrium $\surd$ (Expansion of the Universe)

★ Beqn used to trace evolution of $n_b(t, \mathbf{p}) - n_{\bar{b}}(t, \mathbf{p})$ (net baryon nr)

Crucial: *Loop effect* (invisible at tree level)

Overview

Heuristic Beqn		Our Beqn
Single process	vs	*All* scattering processess
Vacuum amplitude	vs	"Dressed" amplitude
Single scattering	vs	Rescattering correction terms
CP violating terms	vs	(Does not apply to φ^4 thy)

Conclusions

- (Formal) Derivation of BE in QFT model!
- Low density and/or weak coupling are crucial for the Boltzmann equation
- Non–Markovian equation if the nature of the quantum does not allow such limits
- Higher order corrections to the amplitude, i.e. beyond the Born approximation
- Reconsider the application to baryogenesis (loop effect)
- Framework adapted to deal with RW-spacetime
- Open issues:
  - Make formal steps *rigorous*! (lim's, convergence, domains, etc...)
  - Non perturbative derivation?