Non-perturbative aspects of gauge theories

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Content of lecture series

In the lecture course modern Renormalisation Group techniques are applied to strongly-correlated physics in QCD and Quantum Gravity The lecture course provides an introduction to the stronly-correlated physics of QCD and Quantum Gravity. The related physics problems are treated within the Functional Renormalisation Group (FRG), and a survey of alternative approaches is provided.

Outline

- The Functional RG
  - Derivation
  - Truncation schemes, optimisation & numerics
  - Fixed points in the Functional RG
- QCD
  - Introduction
  - Confinement & chiral symmetry breaking
  - Confinement-deconfinement phase transition at finite T
  - A glimpse at the QCD phase diagram
- Quantum Gravity
  - Introduction
  - RG approach to quantum gravity
  - Fixed point structure of quantum gravity
  - Cosmological applications

Literature

- ***Introductory reviews on FRG***

Aoki	Introduction to the Non-perturbative RG	Int.J.Mod.Phys.B14:1249-1326,2000
Berges, Tetradis, Wetterich	Non-Perturbative Renormalization Flow in Quantum Field Theory and Statistical Physics	Phys.Rept.363:223-386,2002
Polonyi	Lectures on the functional renormalization group method	Central Eur.J.Phys.1:1-71,2003

- ***Reviews on FRG in gauge theories & gravity***

Litim, Pawlowski	On gauge invariant Wilsonian flows	Proceedings 'The ERG', Word Scientific '99
Pawlowski	On Wilsonian flows in gauge theories	Habilitation thesis, Erlangen '02
Pawlowski	Aspects of the FRG	Annals Phys.322:2831-2915,2007
Gies	Introduction to the FRG and applications to gauge theories	Lecture notes
Reuter, Saueressig	FRG Equations, Asymptotic Safety, and Quantum Einstein Gravity	Lecture notes
Niedermaier, Reuter	The Asymptotic Safety Scenario in Quantum Gravity	Living review

FRG-reviews on various topics are listed as refs. [15]-[27] in 'Aspects of the FRG'.

- ***Reviews on DSEs in QCD***

Alkofer, von Smekal	The Infrared Behavior of QCD Green's Functions	Phys.Rept.353:281,2001
Fischer	Infrared Properties of QCD from Dyson-Schwinger equations	J.Phys.G32:R253-R291,2006

Literature, basics

- ***Textbooks on the renormalisation group and critical phenomena***

Amit	Field Theory, the Renormalization Group, and Critical Phenomena	World Scientific
Binney, Dowrick, Fisher, Newman	The Theory of Critical Phenomena, an Introduction to the Renormalization Group	Clarendon Press, Oxford
Cardy	Scaling and Renormalization in Statistical Physics	Cambridge University Press
Collins	Renormalization	Springer
Parisi	Statistical Field Theory	Addison-Wesley
Zinn-Justin	Quantum Field Theory and Critical Phenomena	Claredon Press, Oxford

- ***Quantum field theory, basics***

Haag	Local Quantum Physics	Springer, 1996
Itzykson, Zuber	Quantum Field Theory	McGraw-Hill
Peskin, Schroeder	An Introduction to Quantum Field Theory	Addison Wesley
Siegel	Fields	hep-th/9912205
Weinberg	The Quantum Theory of Fields, Vol. 1-2	Cambridge University Press

- *Quantum field theory, applications*

Kugo	Eichtheorie	Springer, 1997
Miransky	Dynamical Symmetry Breaking in Quantum Field Theories	World Scientific, 1993
Muta	Foundations of Quantum Chromodynamics	World Scientific, 1987
Pokorski	Gauge Field Theories	Cambridge, 1987
Wu-Ki Tung	Group Theory in Physics	World Scientific, 1985
Zinn-Justin	Quantum Field Theory and Critical Phenomena	Oxford, 1993

- *General relativity*

Carroll	Spacetime and Geometry	Addison Wesley
Göckeler & Schücker	Differential Geometry, gauge theories, and gravity	Cambridge University Press
Misner, Torne, Wheeler	Graviation	Freeman

I The Functional Renormalisation Group

Quantum field theories are given / determined by a complete set of correlation functions.

Example: scalar field theory with a real field $\varphi(x)$ in d dim.

$\boxed{\text{finite}}$ correlation functions:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle \quad , n \in \mathbb{N}_0$$

$n = 0$: $\langle 1 \rangle \overset{!}{=} 1$ normalised corr. fct.

$n = 1$: $\phi(x) := \langle \varphi(x) \rangle$ mean field

$n = 2$: $G(x,y) := \langle \varphi(x)\, \varphi(y) \rangle - \phi(x)\, \phi(y)$

propagator (connected 2 point fct.

$\vdots$

Generating functional: Euclidean space

finite $Z[J]$ with $\frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} = \langle \varphi(x_1) \cdots \varphi(x_n) \rangle$

$\Uparrow$

$Z[J]$ is the <u>renormalised</u> <u>finite</u> generating functional of normalised Green functions (correlation fcts.) of the theory.

Reminder: classical action

$$S[\varphi] = \frac{1}{2} \int d^d x \left(\partial_\mu \varphi(x)\, \partial_\mu \varphi(x) + m^2 \varphi(x)^2 \right)$$

$$+ \frac{1}{4} \int d^d x \; \lambda \; \varphi(x)^4$$

and

$$Z[J] = \frac{1}{N} \int [d\varphi]_{\text{ren}}\, e^{-S[\varphi] + \int d^d x\, J(x)\, \varphi(x)}$$

with e.g.

$$N = \int [d\varphi]_{\text{ren}}\, e^{-S[\varphi]} \quad , \; N = 1$$

- In the path integral representation the task is to define $\int d\varphi\, e^{-S}$.
- $Z[J]$ generates also disconnected Green functions.

  $\Rightarrow$ Schwinger functional $W[J]$:

$$W[J] = \ln Z[J] \qquad \text{finite}$$

generates connected Green functions
proof when deriving the flow (FRG)

- $\Gamma[\phi]$ generates 1PI Green functions

$$\Gamma[\phi] = \sup_J \left\{ \int d^dx\, J(x)\phi(x) - W[J] \right\}$$

$$\Rightarrow \quad \phi(x) = \left.\frac{\delta W}{\delta J(x)}\right|_{J_{sup}} \qquad (\text{if differentiable})$$

$$\frac{\delta \Gamma}{\delta \phi(x)} = \int d^dx' \frac{\delta J_{sup}(x')}{\delta \phi(x)} \phi(x') + J_{sup}(x) - \int d^dx' \frac{\delta J_{sup}(x')}{\delta \phi(x)} \cdot \left.\frac{\delta W}{\delta J}\right|_{J_{sup}}$$

$$= J_{sup}(x)$$

TBD proof with flow

$$\int d^d x' \frac{\delta^2 W[J]}{\delta J(x)\, J(x')} \quad \frac{\delta^2 \Gamma}{\delta\phi(x')\,\delta\phi(y)} \left[= \delta^{(d)}(x-y) \right]$$

$$= \int d^d x' \left(\frac{\delta}{\delta J(x)} \phi(x') \right) \frac{\delta}{\delta\phi(x')} J(y)$$

$$= \int d^d x' \quad \frac{\delta}{\delta J(x)} J(y) \quad = \delta^{(d)}(x-y)$$

or with $\Gamma^{(n)}(x_1, \dots, x_n) = \dfrac{\delta^n \Gamma}{\delta\phi(x_1)\cdots\delta\phi(x_n)}$

$$W^{(n)}(x_1, \dots, x_n) = \frac{\delta^n W}{\delta J(x_1)\cdots\delta J(x_n)}$$

$$\boxed{\int d^d x' \cdot W^{(2)}(x, x')\, \Gamma^{(2)}(x', y) = \delta^{(d)}(x-y)}$$

and $G(x,y) = W^{(2)}(x,y) = 1/\Gamma^{(2)}(x,y)$

The above relations are valid in the presence of non-vanishing fields/currents, e.g.

$$\Gamma^{(2)} = \Gamma^{(2)}[\phi](x_1, x_2)$$

• functional relations (instead of path integral)

Quantum equations of motion [Dyson-Schwinger eq. DSE]

$$\int [d\varphi]_{\text{ren}} \frac{\delta}{\delta\varphi(x)} \left\{ e^{-S[\varphi] + \int d^d x\, J(x)\varphi(x)} \right\} = 0$$

$$\Rightarrow \quad \langle J(x) \rangle_J - \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_J = 0$$

$$\Rightarrow \quad \boxed{J(x) = \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_J}$$

Important relation:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \left(\frac{\delta}{\delta J(x_1)} + \phi(x_1) \right) \langle \varphi(x_2) \cdots \varphi(x_n) \rangle_J$$

remember: $\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \frac{1}{Z[J]} \int [d\varphi]_{\text{ren}} \varphi(x_1) \cdots \varphi(x_n) e^{-S + \int J\varphi}$

$$\Rightarrow \langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \prod_{i=1}^{n} \left(\frac{\delta}{\delta J(x_i)} + \phi(x_i) \right)$$

Use
$$\frac{\delta}{\delta J(x_i)} = \int d^d x \, \frac{\delta \phi(x')}{\delta J(x_i)} \frac{\delta}{\delta \phi(x')} = \int d^d x \, G(x_i, x') \frac{\delta}{\delta \phi(x')}$$

$$= G \cdot \frac{\delta}{\delta \phi}(x_i)$$

$$\Rightarrow \boxed{ \frac{\delta \Gamma}{\delta \phi(x)} = \frac{\delta S}{\delta \varphi(x)} \left[\varphi(x) = G \frac{\delta}{\delta \phi}(x) + \phi(x) \right] }$$

S action of real scalar field:

$$\frac{\delta S}{\delta \varphi(x)} = -\partial_\mu^2 \varphi(x) + m^2 \varphi(x) + \lambda \varphi(x)^3$$

$$\Big|_{\varphi = G\frac{\delta}{\delta\phi} + \phi} = -\partial_\mu^2 \phi(x) + m^2 \phi(x) + \lambda \phi(x)^3$$

$$+ \lambda \left[\left(G \frac{\delta}{\delta \phi} + \phi \right)^3 - \phi^3 \right]$$

General DSE (including symmetry ID's)

$$\int d\varphi \, \frac{\delta}{\delta\varphi(x)} \left\{ \Psi[\varphi] \, e^{-S[\varphi] + \int J \varphi \, d^d x} \right\} = 0$$

see 'Aspects of the FRG', chapter II

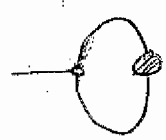
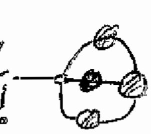
$$\Rightarrow \left.\frac{\delta S}{\delta \varphi(x)}\right|_{\varphi = G\frac{\delta}{\delta\phi}+\phi} = \frac{\delta S[\phi]}{\delta\phi(x)} + \lambda\left(G\frac{\delta}{\delta\phi}(x)\,\phi^2_{(x)} + \phi(x)\,G\frac{\delta}{\delta\phi}(x)\phi(x)\right)$$

$$+ \lambda\left(G\frac{\delta}{\delta\phi}\right)^2\phi$$

$$= \frac{\delta S[\phi]}{\delta\phi(x)} + 3\lambda\, G(x,x)\;\phi(x)$$

$$-\lambda \prod_i \int d^d x_i\, G(x, x_i')\;\Gamma^{(3)}(x_1,x_2,x_3)$$

Diagrammatically:

$= \frac{\delta S}{\delta\phi(x)} + \frac{1}{2}$  $- \frac{1}{3!}$ 

with

$= 1/\Gamma^{(2)}[\phi](x,y)$

$= \Gamma^{(n)}[\phi](x_1, \dots, x_n)$

$= S^{(n)}[\phi](x_1, \dots, x_n)$

I-1 Derivation

Heuristic idea: Kadanoff block-spinning in continuum

Define

$$Z_k[J] = \int [d\varphi]_{p^2 \gtrsim k^2} \, e^{-S[\varphi] + \int d^dx \, \varphi(x) J(x)}$$

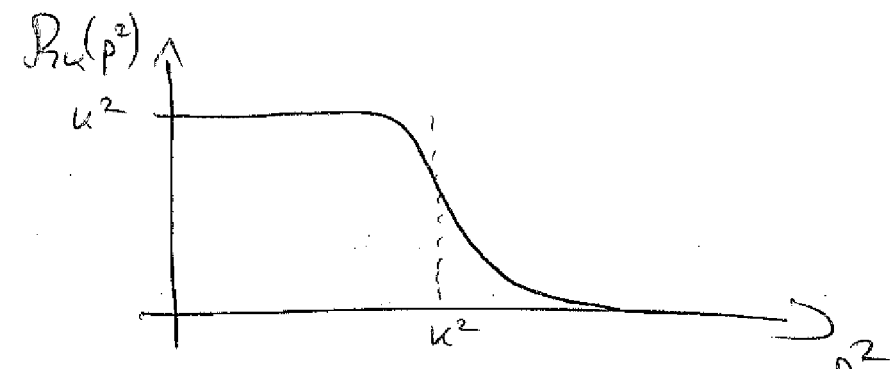
suppression of infrared (IR) modes

Practically

$$[d\varphi]_{ren,\, p^2 \gtrsim k^2} = [d\varphi]_{ren} \, e^{-\Delta S_k[\varphi]}$$

with

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \, \varphi(p) \, R_k(p^2) \, \varphi(-p)$$



Fourier transformation

$$\varphi(x) = \int \frac{d^d p}{(2\pi)^d} \tilde{\varphi}(p)\, e^{ip\cdot x}$$

$$\Rightarrow \quad \varphi(p) = \int d^d x\, \varphi(x)\, e^{-ipx}$$

In particular:

$$\int d^d x\, d^d y\, \varphi(x)\, f(x,y)\, \varphi(y)$$

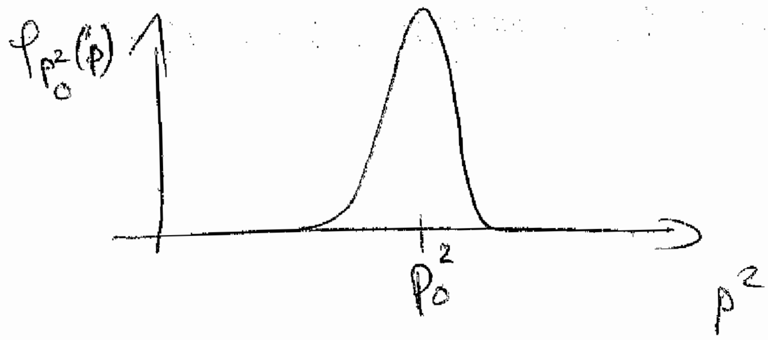
$$= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \varphi(p)\, \varphi(q) \int d^d x\, d^d y\, f(x,y)\, e^{ipx} e^{iqy}$$

$$= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \varphi(p)\, \varphi(q) \cdot f(-p,-q)$$

Regulator: $\quad R_k(x,y) = R_k(-\partial_x^2)\, \delta^{(d)}(x-y)$

$$\Rightarrow \quad R_k(-p,-q) = R_k(p^2)\, \delta^{(d)}(p+q)$$

$\Rightarrow$ page I-7



$$\Delta S_k[\varphi_{p_0^2 \ll k^2}] \simeq \frac{1}{2}\left(\int d^d p \; \varphi_{p_0^2}(p)\,\varphi_{p_0^2}(-p)\right) k^2 .$$

mass-suppression

$$\Delta S_k[\varphi_{p_0^2 \gg k^2}] \simeq 0$$

$$\Rightarrow \boxed{\text{IR suppressed}}$$

2nd lecture

Limits:

- UV: $k \to \infty$, all momentum modes suppressed

  ΔS_k dominates S

  $\Rightarrow$ Gaussian path integral

- IR: $k \to 0$, no modes suppressed

  $\Delta S_k \to 0$

  $Z_k \to Z$

Question: $\int [d\varphi]_{ren}\, e^{-\Delta S_k[\varphi]}$

renormalised measure?

Not necessarily naively

Formally correct:

(1) $Z[J]$ finite renormalised gen. funct.

of " " Green fcts.

(2) $\frac{\delta^n Z}{\delta J^n}$ exists for all n

((1) assumes existence of theory and

(2) 'good' choice of field variable $\varphi(x)$)

(3)

$$\boxed{Z_k[J] = e^{-\Delta S_k[\frac{\delta}{\delta J}]}\, Z[J]}$$

$$\Big[\simeq e^{-\Delta S_k[\frac{\delta}{\delta J}]} \int d\varphi\, e^{-S[\varphi] + \int J\varphi}$$

$$= \int d\varphi\, e^{-S[\varphi] - \Delta S_k[\varphi] + \int J\varphi}$$

$$e^{f[\frac{\delta}{\delta J}]}\, e^{\int J\varphi} = e^{f[\varphi]} \quad \Big]$$

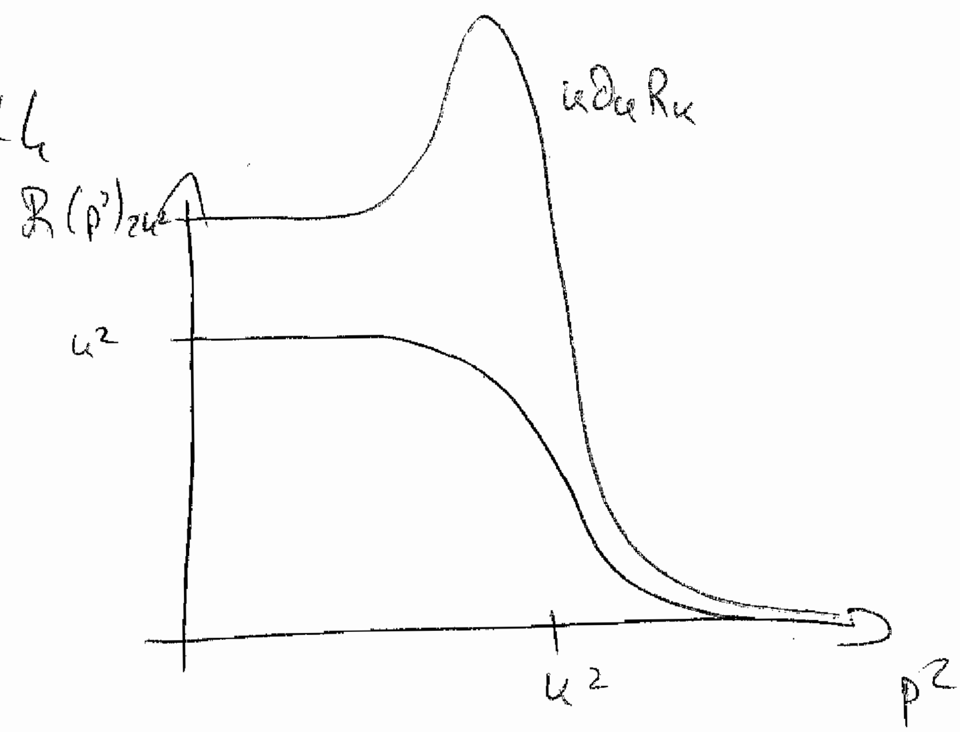
Flow equation:

$$k\,\partial_k Z_k[J] = -\left(k\partial_k \Delta S_k\left[\frac{\delta}{\delta J}\right]\right)\underbrace{e^{-\Delta S_k\left[\frac{\delta}{\delta J}\right]} \mathcal{Z}_k[J]}_{Z_k[J]}$$

$$= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\,\frac{\delta}{\delta J(p)}\,k\partial_k R(p^2)\,\frac{\delta}{\delta J(-p)}\,Z_k[J]$$

$$\Rightarrow \boxed{k\,\partial_k Z_k[J] = -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\,\frac{\delta^2 Z_k[J]}{\delta J(p)\,\delta J(-p)}\,k\partial_k R_k(p^2)}$$

with



Flow of Schwinger functional $\boxed{t = \ln k}$

- $$\frac{1}{Z_k}\, k\partial_k Z_k = k\partial_k \ln Z_k = \partial_t W_k$$

- $$\frac{1}{Z_k[J]} \frac{\delta^2 Z_k[J]}{\delta J(p)\,\delta J(-p)} = \frac{\delta^2 W_k}{\delta J(p)\,\delta J(-p)} + \phi(p)\,\phi(-p)$$

with $$\phi(p) = \frac{\delta W}{\delta J(p)}$$

and $$\frac{\delta^2 \ln Z_k}{\delta J(p)\,\delta J(-p)} = \frac{\delta}{\delta J(p)} \frac{1}{Z_k} \frac{\delta Z_k}{\delta J(-p)} = \frac{1}{Z} \frac{\delta^2 Z_k}{\delta J(p)\,\delta J(-p)}$$

$$- \frac{1}{Z_k}\frac{\delta Z_k}{\delta J(p)} \frac{1}{Z_k}\frac{\delta Z_k}{\delta J(-p)}$$

$\Rightarrow$

$$\boxed{\partial_t W_k[J] = -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left[W_k^{(2)}(p,-p) + \phi(p)\phi(-p) \right] \cdot \partial_t R_k}$$

Flow of effective action

$$\Gamma_k[\phi] = \sup_J \left\{ \int d^dx\, J(x)\phi(x) - W_k[J] \right\}$$

$$- \Delta S_k[\phi]$$

Flow: $(J = J_{sup}[\phi])$

$$\partial_t\big|_\phi \Gamma_k[\phi] = \int d^dx\, \partial_t\big|_\phi J(x) \underbrace{\Big[\phi(x) - \overbrace{\frac{\delta W_k[J]}{\delta J(x)}}^{\phi(x)} \Big]}_{0}$$

$$- \partial_t\big|_J W_k[J] - \partial_t\big|_\phi \Delta S_k[\phi]$$

$$= \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \left[W_k^{(2)}(p,-p) + \phi(p)\phi(-p) - \phi(p)\phi(-p) \right]$$

$$\cdot\, \partial_t R_k$$

$$\Rightarrow \boxed{\partial_t \Gamma_k[\phi] = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \left[W_k^{(2)}(p,-p) \right] \partial_t R_k(p)}$$

Relation between ϕ - der. of Γ_k

and J - der. of W:

(i) $\Gamma_k[\phi] + \Delta S_k[\phi]$ Legendre trafo of W_k

$\Rightarrow$ I-3 - I-4:

$$\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\phi(x)} = J_{\sup_k}(x)$$

$$\frac{\delta W_k}{\delta J(x)} = \phi(x)$$

I-3a:

$$\int d^d x' \, \frac{\delta^2 W_k[J]}{\delta J(x)\,\delta J(x')} \, \frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi(x')\,\delta\phi(y)} = \delta^{(d)}(x-y)$$

$$\boxed{\Rightarrow \int d^d x' \, W_k^{(2)}(x, x')\left(\Gamma^{(2)} + R_k\right)(x', y) = \delta^{(d)}(x-y)}$$

with $\Delta S_k[\phi] = \frac{1}{2}\int d^d x \quad \phi(x)\, R_k(x,y)\, \phi(y)$

$$\Rightarrow \boxed{G_k(x,y) = W_k^{(2)}(x,y) = \frac{1}{\Gamma_k^{(2)} + R_k}(x,y)}$$

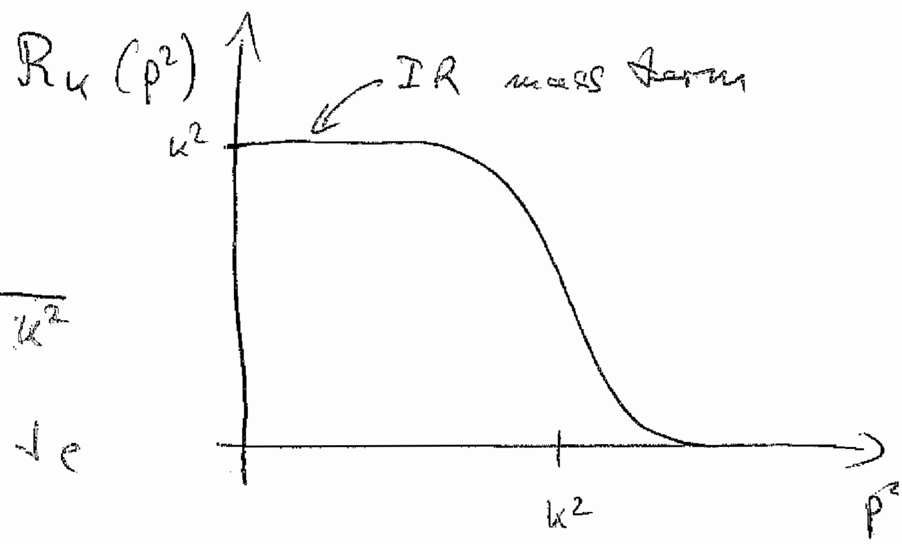
e.g. $R_k(x,y) = R_k(-\partial_x^2)\, \delta^{(d)}(x-y)$

final flow:

$$\boxed{\partial_t \Gamma_k[\phi] = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\,\frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(p,-p)\,\partial_t R_k(p^2)}$$

Finiteness

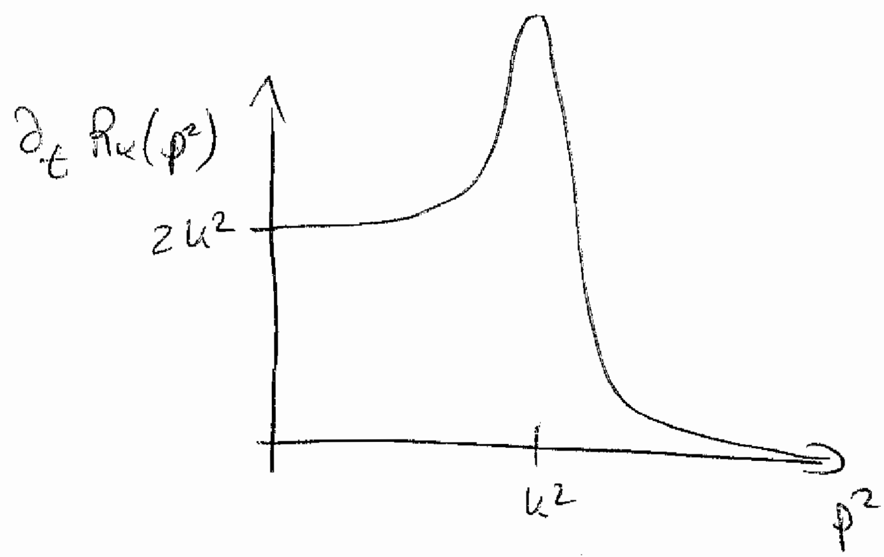
(i) IR :



$$\frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \xrightarrow{p^2/k^2 \to 0} \frac{1}{\Gamma_k^{(2)} + k^2}$$

$\Rightarrow \partial_t \Gamma_k$ IR-finite

(ii) UV :



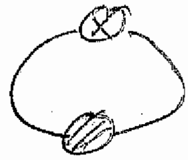
$$p^d \, \frac{1}{\Gamma_k^{(2)}[\phi]}\, R_k(p^2) \xrightarrow{p^2/k^2 \to \infty} 0$$

$\boxed{\Gamma_k, \int d^dp}$

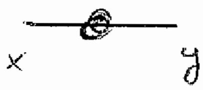
$\Rightarrow \partial_t \Gamma_k$ UV-finite

3rd lecture

Diagrammatically: (DSE I-6)

$$\partial_t \Gamma_k = \frac{1}{2}$$ 

with

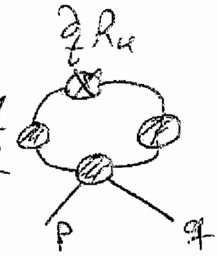


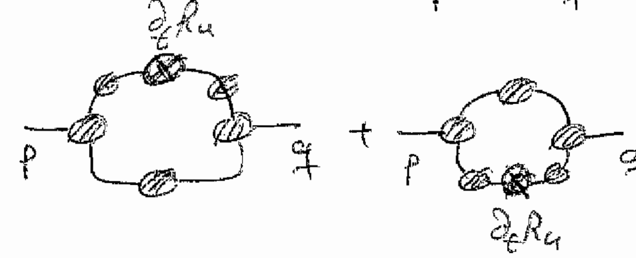
$$= \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(x,y)$$

$$\otimes = \partial_t R_k$$

$$= \Gamma^{(n)}[\phi](x_1, \ldots, x_n)$$

Examples:

$$\frac{\delta}{\delta\phi(p)} \frac{\delta}{\delta\phi(q)} \partial_t \Gamma_k[\phi] = \partial_t \Gamma_k^{(2)}[\phi](p,q) = -\frac{1}{2}$$ 

$$+ \frac{1}{2}\Bigg[$$ 

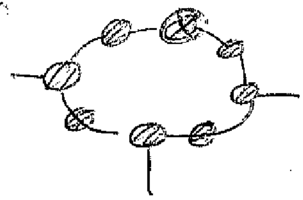
$$\Bigg]$$

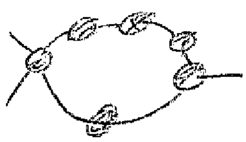
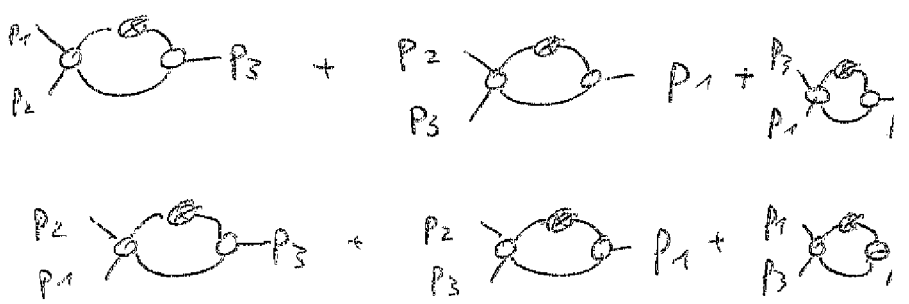
$\sim \Gamma^{(5)}$ $\qquad\qquad$ $\sim \Gamma^{(4)}$

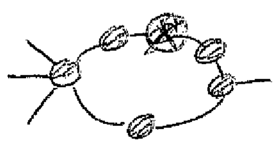
$\partial_t \Gamma^{(3)} = -\frac{1}{2}$  $+ \frac{1}{2}\Big($  $\Big)$

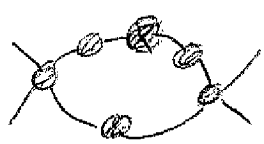
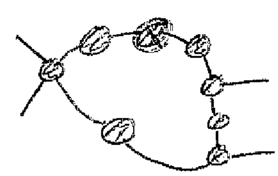
$p_1 \quad p_2 \quad p_3$

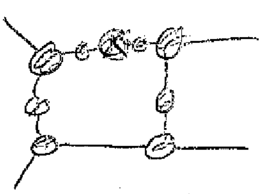
$\sim \Gamma^{(3)}$ $\qquad$ permut.

$-\frac{1}{2}\Big($  $\Big)$

e.g.: $\Big($  $\Big) =$ 

$\partial_t \Gamma^{(4)} = -\frac{1}{2}$  $+ \frac{1}{2}\Big($  $\Big)$

$+ \frac{1}{2}\Big($  $\Big)$ $\quad \frac{1}{2}\Big($  $\Big)$

$+ \frac{1}{2}\Big($  $\Big)$

Remarks:

(1) Finiteness, see I-14.

(2) Flow equation for $\partial_t \Gamma_k[\phi]$

+ initial condition $\Gamma_\Lambda[\phi]$ at

some (UV/IR) scale Λ provide

definition of the quantum field theory

related to Γ_Λ.

(a) perturbative renormalisability

$$\lim_{\Lambda \to \infty} \Gamma_\Lambda \sim \underset{\substack{\uparrow \\ \text{bare action}}}{S_{cl}} \quad \text{(all perturb. relevant terms)}$$

(b) non-perturbative renormalisability

$$\lim_{\Lambda \to \infty} \Gamma_\Lambda = \Gamma_{\text{fixed-point}} \quad \text{(includes (a))}$$

(3) No restriction to momentum cut-off:

$$\underset{\text{time scale}}{R_k(t,t')}\,, \quad \underset{\text{vertex/coupl. reg, nPI-reg}}{\Delta S_k \sim \int R_k(x_1,\dots,x_n)\,\phi(x_1)\dots\phi(x_n)}$$

... Wetterich: integro-diff. eq. for Γ_k

$$e^{-\Gamma_k[\phi]} = \int d\varphi\, e^{-(S[\varphi] + \Delta S_k[\varphi]) + \int J\cdot(\varphi-\phi)} \cdot e^{+\Delta S_k[\phi]}$$

$$\Rightarrow\ e^{-\Gamma_k[\phi]} = \int d\varphi\, e^{-S[\varphi] - \Delta S_k[\varphi] + \int\left(\frac{\delta \Gamma_k}{\delta\phi} + \frac{\delta \Delta S_k}{\delta\phi}\right)(\varphi-\phi)} \cdot e^{\Delta S_k[\phi]}$$

$$= \int d\varphi\, e^{-S[\varphi] + \Delta S_k[\varphi-\phi] + \int \frac{\delta\Gamma_k}{\delta\phi}(\varphi-\phi)}$$

$$\Rightarrow\ \boxed{e^{-\Gamma_k[\phi]} = \int d\varphi\, e^{-S[\varphi+\phi] + \Delta S_k[\varphi] + \int \frac{\delta \Gamma_k}{\delta\phi}\cdot\varphi}}$$

$k \to \infty$: $\Delta S_k[\varphi] \longrightarrow \infty$: Saddle point exp. becomes exact

$$\Rightarrow\ e^{-\Gamma_k[\phi]} \simeq e^{-S[\phi] + \text{ren.} + \mathcal{O}(1/k)}$$

(4) regulator term ($\sim \phi^2$)

$$\Delta S_k[\phi]$$

may break symmetries; in particular non-linear symmetries like

(a) non-Abelian gauge symmetries

QCD

(b) diffeomorphisms

gravity

Appendix

Zero-dim Example:

assumes existence of Taylor expansion

$$\Gamma[x;R] = \frac{1}{2}\alpha x^2 + \lambda/_{4!}\, x^4 + \sum_{n=5}^{\infty} \lambda_n/_{n!}\, x^n$$

$$R = 10^3 : \quad \alpha = 1 \Rightarrow \Gamma[x;10^3] = 1/_2\, x^2 + \lambda/_{4!}\, x^4$$

$$\partial_R \left(\frac{1}{2}\alpha x^2 + \lambda/_{4!}\, x^4 \right) = \frac{1}{2} \frac{1}{\alpha + \lambda/_2\, x^2 + R}$$

$+ \lambda_n$-terms

Taylorexpansion:

$$\partial_x^2 \partial_R \Gamma \Big|_{x=0} = \partial_R \alpha = -\frac{1}{2}\lambda \frac{1}{(\alpha+R)^2} \qquad \left[-\frac{1}{2} \right]$$

$\partial_R R = 1$

$$\partial_x^4 \partial_R \Gamma \Big|_{x=0} = \partial_R \lambda = 3\lambda^2 \frac{1}{(\alpha+R)^3} \qquad \left[3 \right]$$

$\Rightarrow$ plots

'Functional' RG flows for integrals

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'Functional' RG flows for integrals

generating function

$$Z[j] = \int dx \, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

- $R \to \infty$: $\quad Z[j;R] \to \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 + j\,x\right)$
- $R \to 0$: $\quad Z[j;R=0] = Z[j]$
- $j = 0$: $\quad Z[0;R] = \frac{2}{\sqrt{1+R}}\, e^{\frac{3(1+R)^2}{4\lambda}} \sqrt{\frac{3(1+R)^2}{4\lambda}}\, K\left(\frac{1}{4}, \frac{3(1+R)^2}{4\lambda}\right)$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}Rx^2$

$$\partial_R \ln Z[j;R]\,, \qquad \partial_R \Gamma[x;R] = -\partial_R \ln Z[j;R] - \frac{1}{2}x^2\,,$$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \ln Z[j;R] = -\langle \frac{1}{2}x^2 \rangle_j$$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \ln Z[j;R] = -\frac{1}{2}\left(\partial_j^2 \ln Z[j] + (\partial_j \ln Z[j])^2\right)$$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2\Gamma[x;R] + R}$$

- $\partial_j^2 \ln Z[j] = \dfrac{1}{\partial_x^2\Gamma[x;R] + R}$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2\Gamma[x;R] + R}$$

- boundary condition: $\Gamma[x; R \to \infty] = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4$

'Functional' RG flows for integrals

$$\ln Z[j] = \ln \int dx\, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

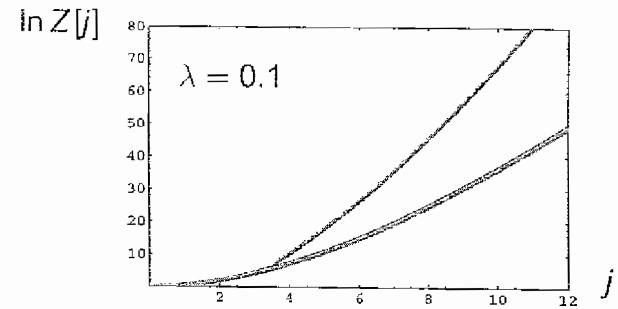
- asymptotic perturbative series with optimal order

$$n_{\rm opt}(j) \leq n_{\rm opt}(0) \sim \frac{3}{2\lambda}$$

- flow with boundary condition
  - $\Gamma[x, R = 10^3] = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4$
  - $\Gamma[x = \pm 10^2, R] = \Gamma[x = \pm 10^2, R = 10^3]$
- numerical integration of $\ln Z[j]$

'Functional' RG flows for integrals

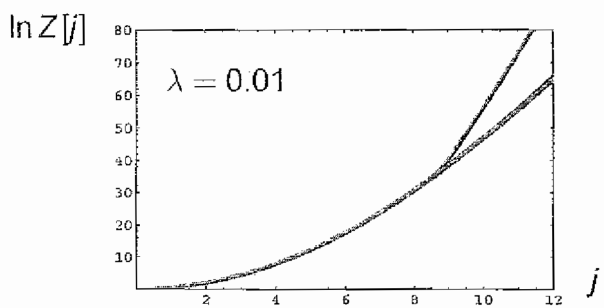
$$\ln Z[j] = \ln \int dx\, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$



- perturbative expansion: $n = 6$
- flow
- numerical integration

'Functional' RG flows for integrals

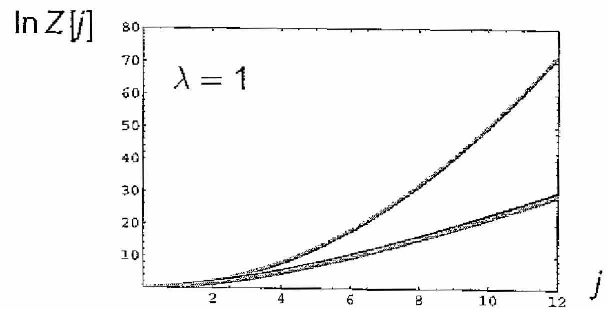
$$\ln Z[j] = \ln \int dx\, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$



- perturbative expansion: $n = 26$
- flow
- numerical integration

'Functional' RG flows for integrals

$$\ln Z[j] = \ln \int dx\, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$



- perturbative expansion: $n = 0$
- flow
- numerical integration

'Functional' RG flows for integrals: truncations

$$\partial_R \Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2 \Gamma[x;R] + R}$$

- parameterisation

requires convergence of Taylor expansion

$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \frac{1}{4!}\lambda_6[R]x^6 + \sum_{n=3}^{\infty}\frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$

- initial conditions at $R_{\rm in}$

$$\alpha[R_{\rm in}] = 1\,, \qquad \lambda[R_{\rm in}] = \lambda\,, \qquad \lambda_{2n}[R_{\rm in}] = 0 \quad \forall n > 2\,.$$

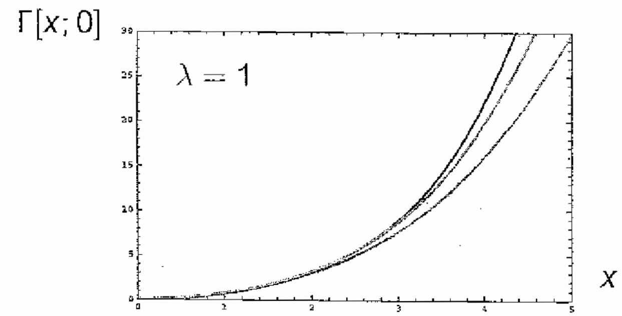
- truncation (i): $\lambda_{2n>4}[R] \equiv 0$ (ii): $\lambda_{2n>6}[R] \equiv 0$

- flows for coefficients

$$\partial_x^n \Gamma[0,R] = \partial_x^n \left[\frac{1}{2}\frac{1}{\partial_x^2 \Gamma[x;R] + R}\right]_{x=0}$$

'Functional' RG flows for integrals: truncations

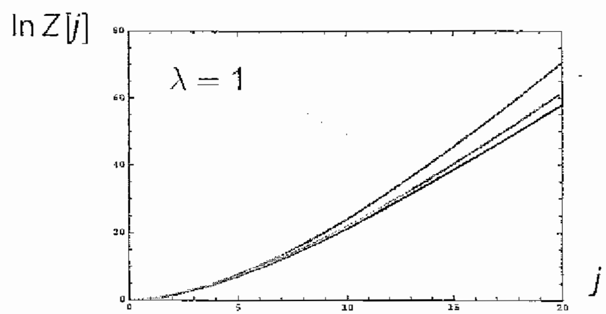
$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}}\frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$



- $N_{\max} = 2$: $\lambda_{2n>4} \equiv 0$
- $N_{\max} = 3$: $\lambda_{2n>6} \equiv 0$
- numerical integration

'Functional' RG flows for integrals: truncations

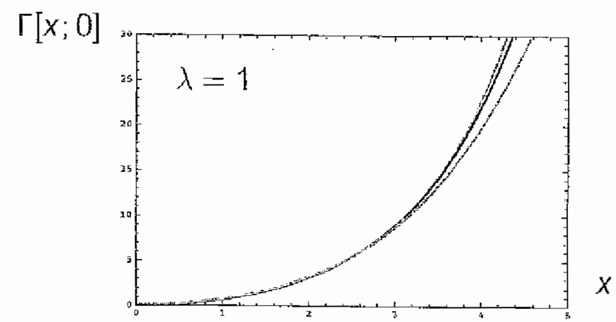
$$\ln Z[j] = \ln \int dx\, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$



- truncation (i): $\lambda_6 \equiv 0$, $\lambda_{2n>6} \equiv 0$
- truncation (ii): $\lambda_6 \neq 0$, $\lambda_{2n>6} \equiv 0$
- numerical integration

'Functional' RG flows for integrals: truncations

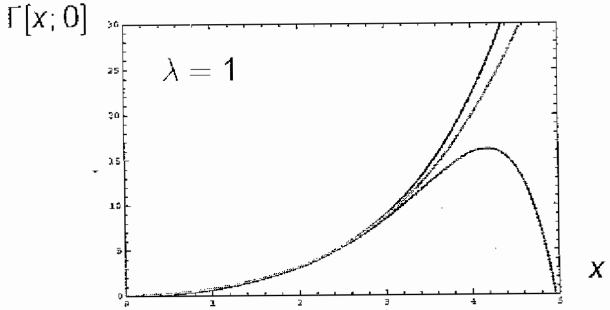
$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}}\frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$



- $N_{\max} = 4$
- $N_{\max} = 3$
- numerical integration

'Functional' RG flows for integrals: truncations

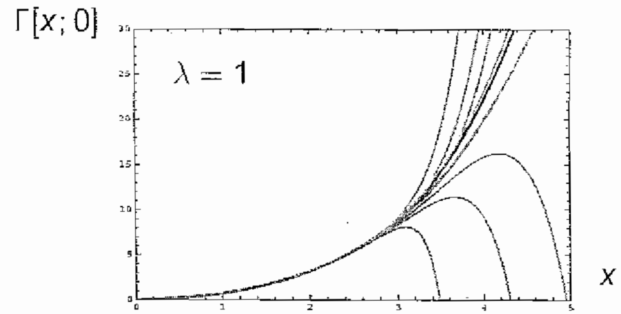
$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}} \frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$



— $N_{\max} = 5$
— $N_{\max} = 3$
— numerical integration

'Functional' RG flows for integrals: truncations

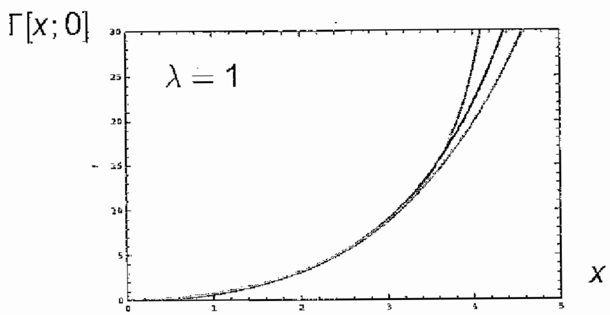
$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}} \frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$



— $N_{\max} = 4 - 10$
— $N_{\max} = 3$
— numerical integration

'Functional' RG flows for integrals: truncations

$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}} \frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$

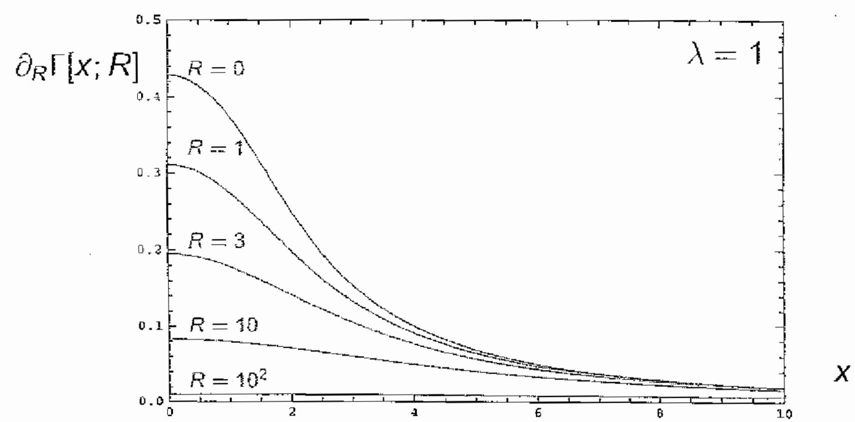


— $N_{\max} = 7$
— $N_{\max} = 3$
— numerical integration

'Functional' RG flows for integrals: truncations

full flow

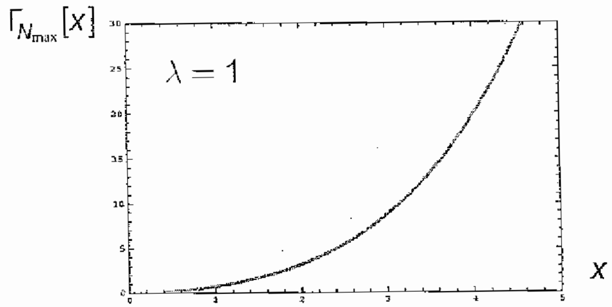
$$\partial_R\Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2\Gamma[x;R]+R}$$



'Functional' RG flows for integrals: truncations

- rapid convergence for large x:

$$\Gamma_{N_{\max}}[x;R]=\frac{1}{2}x^2+\frac{1}{4!}\lambda x^4+\frac{1}{2}\log(1+\tfrac{1}{2}\lambda x^2+R)+\sum_{n=0}^{N_{\max}}\frac{1}{(2n)!}\frac{\Delta\lambda_{2n}[R]\,x^{2n}}{(1+\frac{1}{2}\lambda x^2+R)^{n+2}}$$

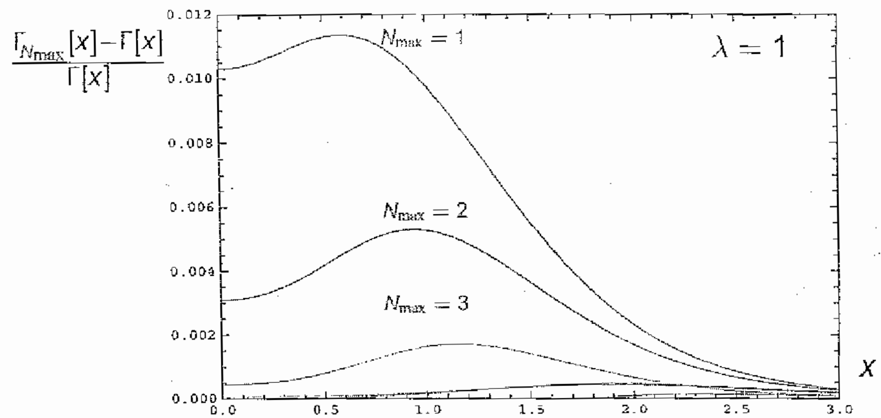


- 1-loop perturbation theory: $\Delta\lambda_{2n}\equiv 0$. $\forall n$
- $N_{\max}=0$
- numerical integration

'Functional' RG flows for integrals: truncations

- rapid convergence for large x:

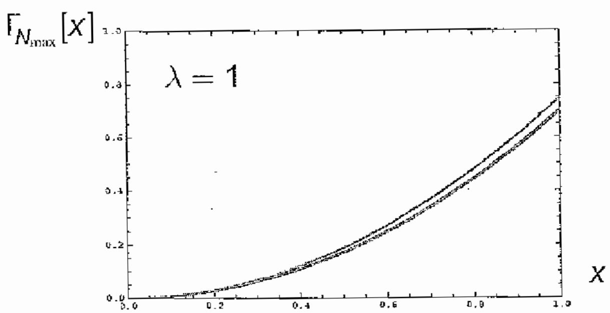
$$\Gamma_{N_{\max}}[x;R]=\frac{1}{2}x^2+\frac{1}{4!}\lambda x^4+\frac{1}{2}\log(1+\tfrac{1}{2}\lambda x^2+R)+\sum_{n=0}^{N_{\max}}\frac{1}{(2n)!}\frac{\Delta\lambda_{2n}[R]\,x^{2n}}{(1+\frac{1}{2}\lambda x^2+R)^{n+2}}$$



'Functional' RG flows for integrals: truncations

- rapid convergence for large x:

$$\Gamma_{N_{\max}}[x;R]=\frac{1}{2}x^2+\frac{1}{4!}\lambda x^4+\frac{1}{2}\log(1+\tfrac{1}{2}\lambda x^2+R)+\sum_{n=0}^{N_{\max}}\frac{1}{(2n)!}\frac{\Delta\lambda_{2n}[R]\,x^{2n}}{(1+\frac{1}{2}\lambda x^2+R)^{n+2}}$$

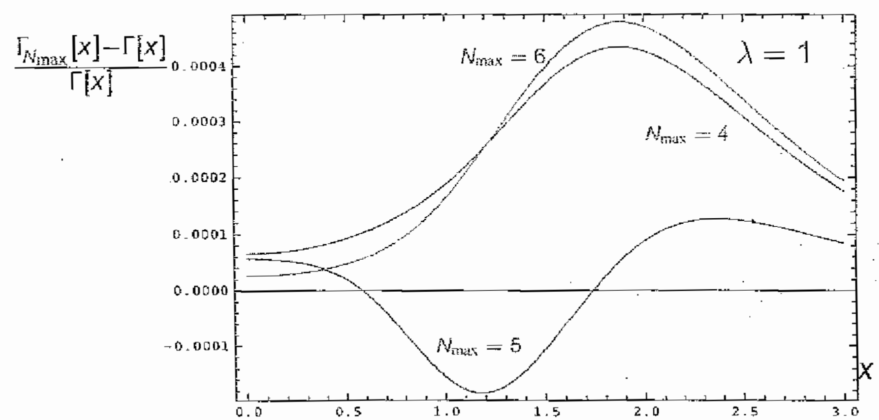


- 1-loop perturbation theory: $\Delta\lambda_{2n}\equiv 0$, $\forall n$
- $N_{\max}=0$
- numerical integration

'Functional' RG flows for integrals: truncations

- rapid convergence for large x:

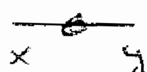
$$\Gamma_{N_{\max}}[x;R]=\frac{1}{2}x^2+\frac{1}{4!}\lambda x^4+\frac{1}{2}\log(1+\tfrac{1}{2}\lambda x^2+R)+\sum_{n=0}^{N_{\max}}\frac{1}{(2n)!}\frac{\Delta\lambda_{2n}[R]\,x^{2n}}{(1+\frac{1}{2}\lambda x^2+R)^{n+2}}$$



I-2 Truncation Schemes, optimisation & numerics

(i) Perturbation theory

(a) 1-loop: $\partial_t \Gamma_k^{1\text{-loop}} = \frac{1}{2}$  Loop

$\Rightarrow$ 

$= \dfrac{1}{S^{(2)}[\phi] + R_k}(x,y)$

and hence

$$\partial_t \Gamma_k^{1\text{-loop}}[\phi] = \frac{1}{2} \mathrm{Tr} \frac{1}{S^{(2)}[\phi] + R_k} \partial_t R_k$$

k-dep

$$= \frac{1}{2} \mathrm{Tr}\, \partial_t \ln\left(S^{(2)}[\phi] + R_k\right)$$

($\mathrm{Tr}\, \partial_t \neq \partial_t \mathrm{Tr}$, strictly speaking)

Integration:

$$\Gamma_k^{1\text{-loop}}[\phi] = \Gamma_\Lambda^{1\text{-loop}}[\phi] + \int_\Lambda^k \frac{dk'}{k'} \partial_{t'} \Gamma_{k'}^{1\text{-loop}}[\phi]$$

renormalisation

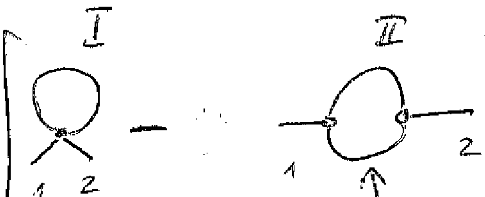
$$\Rightarrow \Gamma_k^{1\text{-loop}}[\phi] = \Gamma_\Lambda^{1\text{-loop}}[\phi] + \frac{1}{2}\,\mathrm{Tr}\Big[\ln\big(S^{(2)}[\phi]+R_k\big) - \ln\big(S^{(2)}[\phi] + R_\Lambda\big)\Big]$$

finite

e.g.:

$$\Gamma_k^{1\text{-loop}\,(2)}[\phi] = \Gamma_\Lambda^{1\text{-loop}\,(2)}[\phi]$$

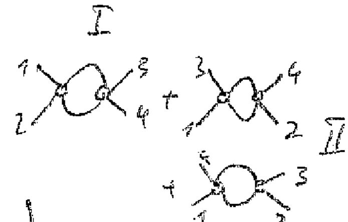
$$+ \frac{1}{2}\Big[\ \ \Big]_\Lambda^k$$



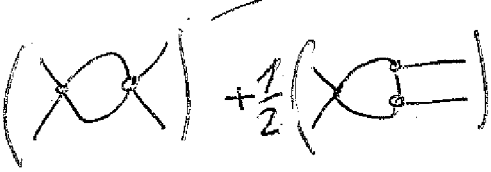
$$\frac{1}{S^{(2)}[\phi]+R_k}$$

$$\left(\Gamma_k^{1\text{-loop}}[0](p,q) = \frac{1}{2}\Big[\ \ \Big]_\Lambda^k\right) + \Gamma_\Lambda^{1\text{-loop}(2)}[0](p,q)$$

$$\Gamma_k^{1\text{-loop}\,(4)}[\phi] = \frac{\delta^2}{\delta\phi^2}\,\Gamma_k^{1\text{-loop}\,(2)}[\phi]$$

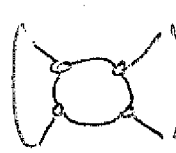


$$= \Big[-\frac{1}{2}\big(\ \ \big) + \frac{1}{2}\big(\ \ \big)$$

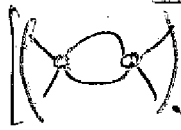


see pages I-74(a,b)

$$-\frac{1}{2}\big(\ \ \big)\Big]_\Lambda^k + \Gamma_\Lambda^{1\text{-loop}\,(4)}[\phi]$$



$$\Rightarrow \Gamma_k^{1\text{-loop}\,(4)}[0](p_1,\ldots,p_4) = -\frac{1}{2}\Big[\ \ \Big]_\Lambda^k + \Gamma_\Lambda^{1\text{-loop}(4)}[0](p_1,\ldots,p_4)$$



Renormalisation:

(1) Γ_k indep. of Λ!

$$\Rightarrow \Lambda\partial_\Lambda \Gamma_k = 0$$

$$= \Lambda\partial_\Lambda \Gamma_\Lambda + \frac{1}{2}\,\mathrm{Tr}\Big[\ln\big(\Gamma^{(2)}[\phi]+R_k\big)\Big]_\Lambda^k$$

(2) Λ-dep. of Γ_Λ is fixed by Flow

$\Rightarrow$ Renormalisation is

(α) adjusting Λ-indep. of Γ_k, $k \neq \Lambda$
~ regularisation

(β) fixing Λ-indep parts of Γ_Λ
~ renormalisation conditions

(3) extends trivially to full flow

Example: β-function in ϕ^4-theory in 4-dim I – 19 a

$$S[\phi] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \, \phi(p) \left(p^2 + m^2 \right) \phi(-p)$$

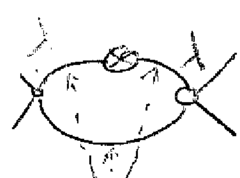
$$+ \underline{\underline{\frac{\lambda}{4!}}} \int_{p_1, \dots, p_4} \phi(p_1) \cdots \phi(p_4) \cdot (2\pi)^4 \, \delta^{(4)}(p_1 + \cdots + p_4)$$

with $\int_p = \int \frac{d^d p}{(2\pi)^d}$, here $d = 4$.

Inserting into $\partial_t \Gamma^{(4)} \big|_{\phi = 0} = \partial_t \lambda = \underline{\underline{\dot{\lambda}}}$

1-Loop: rhs: $\Gamma^{(2)} = S^{(2)} = \left(p^2 + m^2 \right) \delta^{(4)}(p+q)$

$$\Gamma^{(4)} = S^{(4)} = \lambda \, (2\pi)^4 \, \delta^{(4)}(p_1 + \cdots + p_4)$$

. I-14b $\Rightarrow$ $\dot{\Gamma}^{(4)} = 3 \cdot$ 

$\frac{1}{p_i^2 + m^2}$

$$= 3 \lambda^2 \int \frac{d^4 p}{(2\pi)^4} \left(\frac{1}{p^2 + m^2 + R_k} \dot{R}_k(p^2) \frac{1}{p^2 + m^2} \right. \left. \cdot \frac{1}{p^2 + m^2 + R_k} \right)$$

$$d p\, p = \tfrac{1}{2} d p^2$$

$$\mathring{\Gamma}^{(4)} = \mathring{\lambda} = 3\lambda^2 \cdot \int \frac{d\Omega_4}{(2\pi)^4} \cdot \frac{1}{2} \int_0^\infty dp^2\, p^2 \left(\frac{1}{p^2 + m^2 + R_k(p^2)} \right)^3 \mathring{R}_k(\ldots$$

Introduce $\quad x = p^2/k^2$

$$R_k(p^2) = p^2\, r(x)$$

$$\Rightarrow \mathring{R}_k(p^2) = -2p^2\, x\, r'(x) = -2k^2\, x r'(x)$$

$$\int d\Omega_4 = \Omega_4$$

$$\Rightarrow \quad \mathring{\lambda} = -\frac{2}{2}\, 3\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx\; x^3 \left(\frac{1}{x(1+r) + \hat{m}^2} \right)^3 r'$$

$$= -3\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx \left(\frac{1}{1+r + \hat{m}^2/x} \right)^3 r'$$

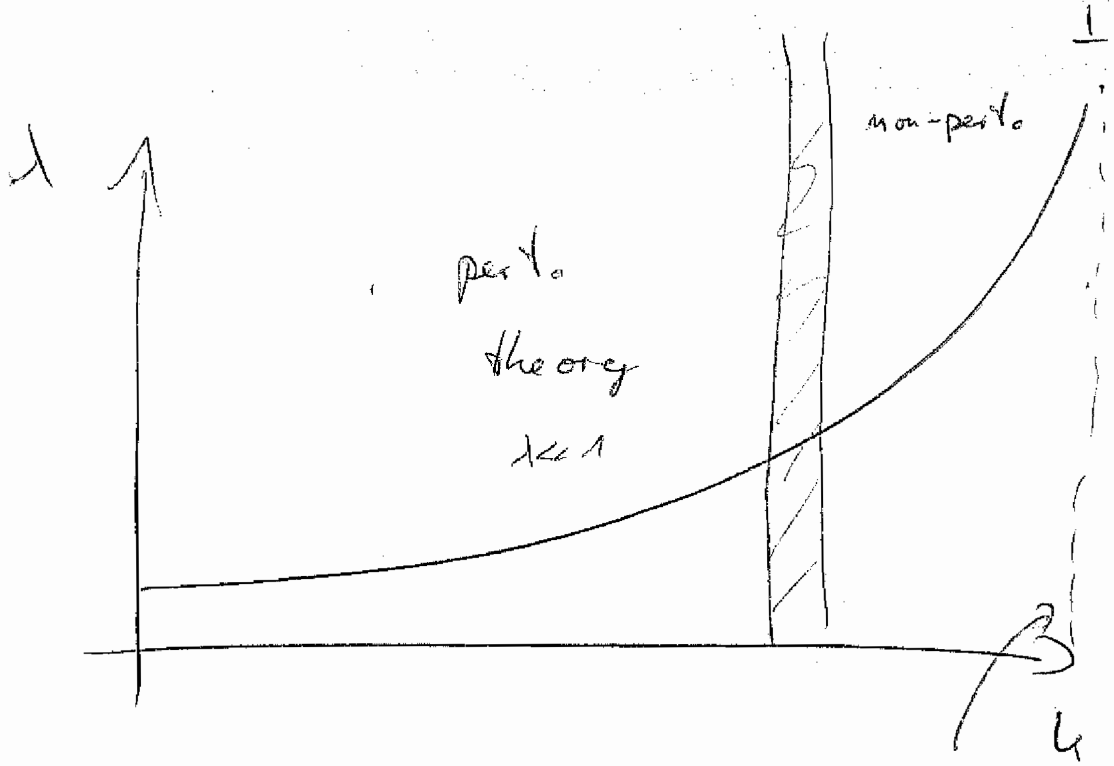
$$= \frac{3}{2}\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx \left[\frac{\partial}{\partial x} \frac{1}{(1+r+\hat{m}^2/x)^2} - \frac{\hat{m}^2}{x^2} \frac{1}{(1+r+\hat{m}^2/x)^3} \right]$$

$$= \frac{3}{2}\lambda^2 \frac{\Omega_4}{(2\pi)^4} + \hat{m}^2\text{-corrections}$$

$\Omega_4 = 2\pi^2$ $\qquad\qquad$ $\hat{m}^2 = m^2/k^2 + \lambda^2\text{-terms}$

$$\Rightarrow \quad \boxed{\partial_t \lambda \simeq \frac{3}{16\pi^2}\lambda^2} \qquad \rightarrow \lambda(t) = \frac{\lambda_0}{1 + \frac{3}{16\pi^2}(t_0 - t)\lambda_0}$$

Landau ...



Landau - pole

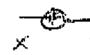
triviality: demand finite λ for $t \to \infty$

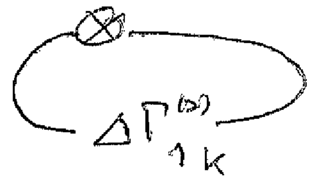
- ϕ^4-theory valid at all scales
- no Landau pole

$$\Rightarrow \boxed{\lambda_{\text{phys}} = \lambda_{k=0} \hat{=} 0}$$

[as $t_0 \to \infty$]

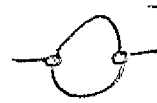
(b) 2-loop:

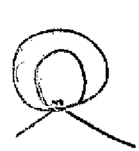
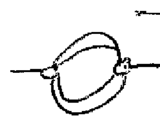
$$\partial_t \Gamma_k^{2\text{-loop}} = \frac{1}{2}$$  with  $= \frac{1}{\Gamma^{1\text{-loop}(2)}[\phi]+R_k}(x,y)$

$$= \frac{1}{2}$$  $- \frac{1}{2}$ 

1-loop

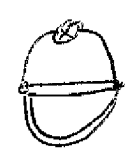
with $\Delta\Gamma^{(2)}_{1k} = \Gamma_k^{1\text{-loop}(2)} - S^{(2)}$

$$= \frac{1}{2}\Big[$$  $-$  $\Big]_\Lambda^k + \left(\Gamma_\Lambda^{1\text{-loop}(2)} - S^{(2)}\right)$

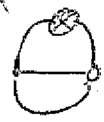
$$= \frac{1}{2}\Big[$$  $-$  $\Big]$

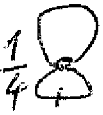
$= \frac{1}{2}\Big($  $-$  $\Big) - \frac{1}{2}\Big($  $-$  $\Big)$

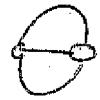
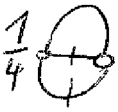
It follows

$$\partial_t \Gamma_k^{2\text{-loop}} = \frac{1}{2}$$  $- \frac{1}{4}$  $+ \frac{1}{4}$ 

$$\Rightarrow \partial_t \Gamma_k^{2\text{-loop}} = \partial_t(1\text{-loop}) - \frac{1}{4}\Big[$$  $$-$$  $$\Big]$$

$$+\frac{1}{4}\Big[$$  $$-$$  $$\Big]$$

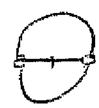
$$= \partial_t(1\text{-loop}) + \partial_t \Big\{ \frac{1}{8}$$  $$-\frac{1}{4}$$ 

$$-\frac{1}{12}$$  $$+\frac{1}{4}$$  $$\Big\}$$

$$\Rightarrow \Gamma_k^{2\text{-loop}} = \Gamma_\Lambda^{2\text{-loop}} + \int_\Lambda^k \frac{dk'}{k'}\, \partial_{t'}\Gamma_{k'}$$

$$= S_{cl} + (1\text{-loop})_{ren}$$

$$+ \frac{1}{8}$$  $$- \frac{1}{4}$$ 

$$- \frac{1}{12}$$  $$+ \frac{1}{4}$$  $$+\left(\Gamma_\Lambda^{2\text{-loop}} - S - \Gamma_\Lambda^{1\text{-loop}}\right)$$

$$\Rightarrow \Delta\Gamma_k^{2\text{-loop}\,(2)} \Rightarrow 3\text{-loop}$$

(ii) Effective Potential approximation

(zeroth order derivat. expansion)

Effective Potential: ϕ_c constant examples: I-22a

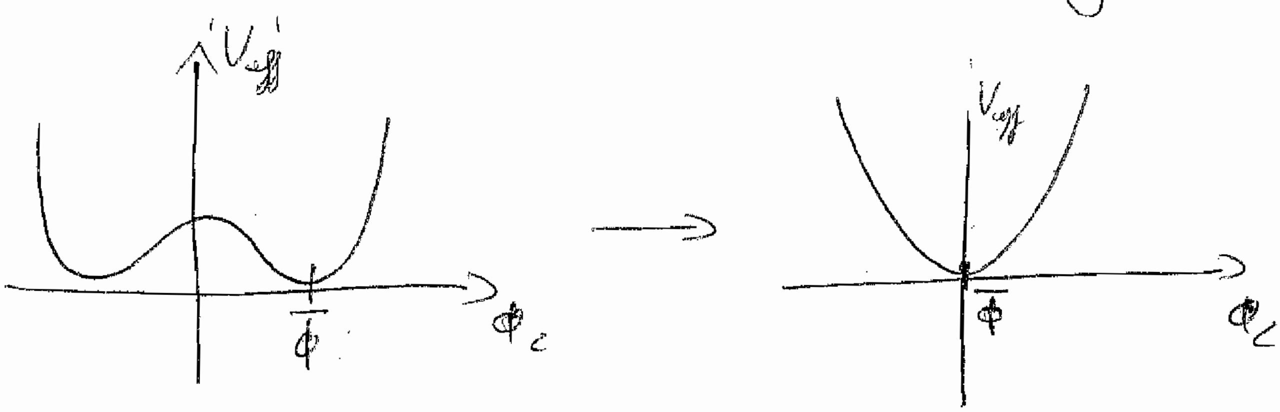
$$\mathrm{Vol}_d \cdot V_k[\phi_c] := \Gamma_k[\phi_c] \qquad (V_{\mathrm{eff}\,k})$$

$\dim V_k = d$

quantum equ. of classical pot.

$$\left.\frac{\partial V_k}{\partial \phi_c}\right|_{\bar\phi} = 0 \quad \text{approximates ground state}$$

e.g. order parameter of symmetry breaking



broken phase symmetric phase

$$V_{\mathrm{eff}} = V_{k=0}$$

Examples:

(a) classical action

$$S_{cl}[\phi] = \frac{1}{2}\int d^dx\, \partial_\nu\phi\,\partial_\nu\phi + \int d^dx \left\{ \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\}$$

$$\Rightarrow S_{cl}[\phi_c] = \left\{ \frac{m}{2}\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 \right\} \underbrace{\int d^dx}_{\mathrm{Vol}_d}$$

(b) local potential approximation (LPA)
[0th derivative expansion]

$$\Gamma_k[\phi] = \frac{1}{2}\int d^dx\, \partial_\nu\phi\,\partial_\nu\phi + \int d^dx\, V_k[\phi(x)]$$

$$\Rightarrow \Gamma_k[\phi_c] = \mathrm{Vol}_d\, V_k[\phi_c]$$

full flow for $V_k[\phi_c]$:

hs requires $\Gamma_k^{(2)}[\phi_c](p,q)$:

$$\Gamma_k^{(2)}[\phi_c](p,q) = \left(Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c]\right)(2\pi)^d \delta^{(d)}(p+q)$$

see p. I-23a

$$\Rightarrow \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{1}{\Gamma_k^{(2)}[\phi_c]+R_k}(p,-p)\,\partial_t R_k(p^2)$$

see p. I-23a

$$= \underset{\substack{\uparrow \\ (2\pi)^d \delta^{(d)}(p+q)}}{\mathrm{Vol}_d}\, \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{1}{Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c] + R_k(p^2)}\,\partial_t R_k(p^2)$$

lhs: $\partial_t \Gamma_k[\phi_c] = \mathrm{Vol}_d \cdot \partial_t V_k[\phi_c]$

$$\Rightarrow \boxed{\partial_t V_k[\phi_c] = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{1}{Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c] + R_k(p^2)}\,\partial_t R_k(p^2)}$$

full flow, not closed because of Z_k

$$\Gamma_k^{(2)}[\phi_c](p,q) = \left(Z_k(p^2,\phi_c)\, p^2 + \partial_{\phi_c}^2 V_k[\phi_c]\right)(2\pi)^d\, \delta^{(d)}(x-y)$$

e.g. from $\Gamma_k[\phi] = \frac{1}{2}\int d^dx\; Z_k(-\partial^2, \phi(x))\,\partial_\mu\phi(x)\,\partial_\mu\phi(x)$

$$+ \int d^dx\; V_k[\phi(x)]$$

with

$$\left.\frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_c} = -Z_k(-\partial_x^2, \phi_c)\;\partial_x^2\;\delta(x-y)$$

$$+ \partial_{\phi_c}^2 V_k[\phi_c]\;\delta^{(d)}(x-y)$$

$$\frac{1}{\Gamma_k^{(2)}[\phi_c] + R_k}(p,q) = \frac{1}{p^2 Z_k(p^2,\phi_c) + V_k[\phi_c] + R_k(p^2)}\,\delta^{(d)}(p+q)$$

$$R_k(p,q) = R_k(p^2)\,\delta^{(d)}(p+q)$$

$$\mathring{R}_k(p,q) = \mathring{R}_k(p^2)\,\delta^{(d)}(p+q)$$

$$(2\pi)^d\,\delta^{(d)}(p=0) = \left.\int d^dx\; e^{ipx}\right|_{p=0} = Vol_d$$

0th order deriv. expansion:

$$Z_k(p^2, \phi_c) = 1 \quad \Leftarrow \text{ flow closed}$$

- good low energy (momentum) approximation
$\Leftrightarrow$ requires mass-scales!

regulator choise:

$$R_k = (k^2 - p^2)\,\Theta(k^2 - p^2) \qquad \text{optimised}$$
$$\dot{R}_k = 2k^2\,\Theta(k^2 - p^2) \qquad \text{cut-off}$$

! for 0th order der. expans.!

$$\Rightarrow \partial_t V_k[\phi_c] = \frac{\overbrace{\Omega_d}^{\int d\Omega_d}}{(2\pi)^d} \cdot \int_0^k dp\, p^{d-1} \frac{k^2}{k^2 + \partial^2_{\phi_c} V_k}$$

$$= \frac{1}{d} \cdot \frac{\Omega_d}{(2\pi)^d} \frac{k^{d+2}}{k^2 + \partial^2_{\phi_c} V_k[\phi_c]}$$

with

$$\Omega_d = 2\pi^{d/2} / \Gamma[d/2]$$

Example: flow of λ in $d=4$:

$$V_k = \frac{1}{2} m_k^2 \phi_c^2 + \frac{\lambda_k}{4!} \phi_c^4 + \frac{\lambda_{6k}}{6!} \phi_c^6 + \dots$$

$$\Rightarrow \partial_{\phi_c}^2 V_k = m_k^2 + \lambda_k/2\, \phi_c^2 + \frac{\lambda_{6k}}{4!} \phi_c^4 + \dots$$

$$\partial_{\phi_c}^4 \dot{V}_k \Big|_{\phi_c=0} = \dot{\lambda}_k = \partial_{\phi_c}^4 \Big|_{\phi_c=0} \frac{1}{2} \frac{1}{16\pi^2} \frac{k^6}{\left(k^2 + m_k^2 + \frac{\lambda_k}{2}\phi_c^2 + \frac{\lambda_{6k}}{4!}\phi_c^4 + \dots\right)}$$

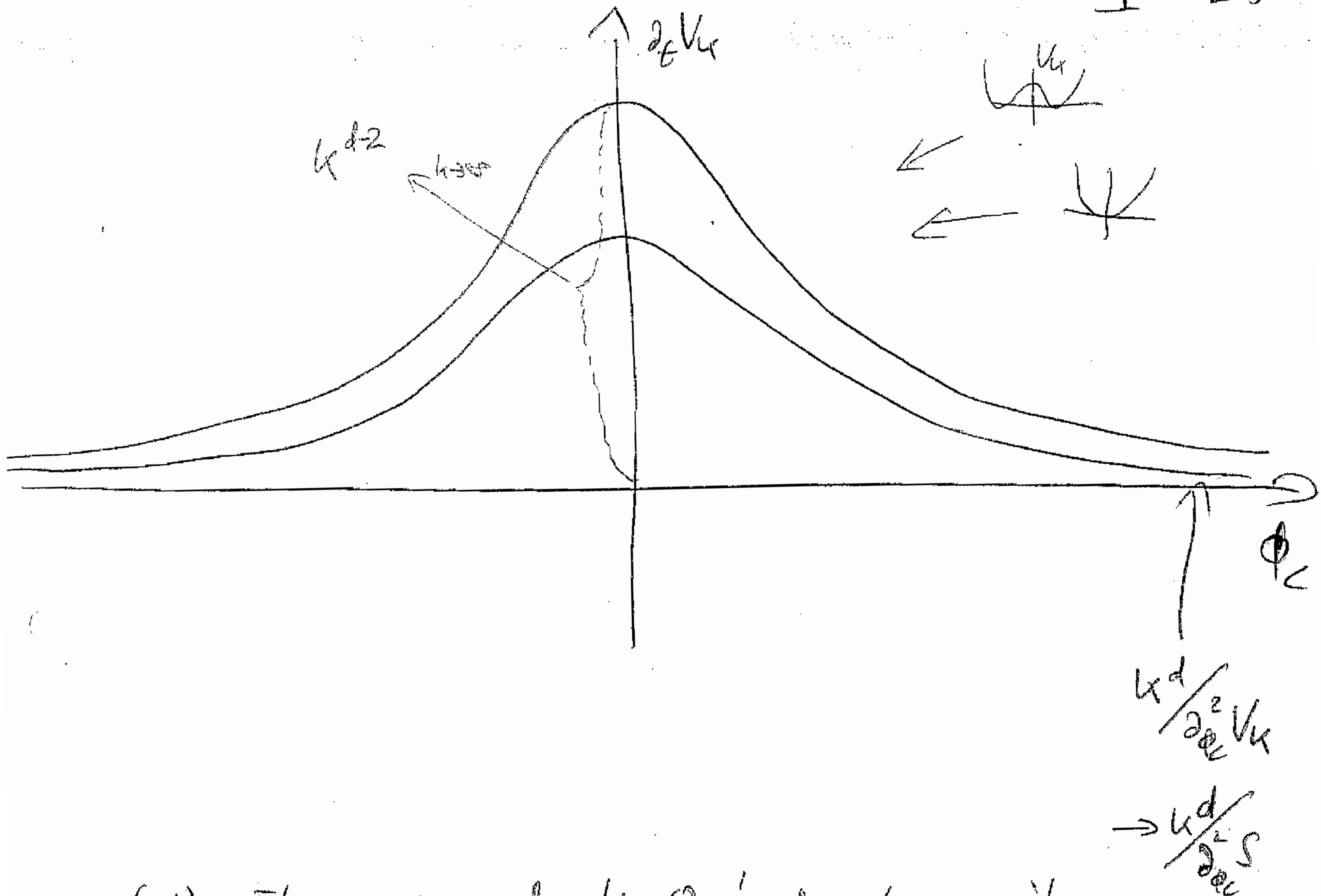
$$= \frac{6}{2} \frac{1}{16\pi^2} \lambda_k^2 \frac{1}{(1+\underbrace{\hat{m}_k^2}_{= m_k^2/k^2})^3}$$

$$- \frac{1}{2} \frac{1}{16\pi^2} \lambda_{6k} k^2 \frac{1}{(1+\hat{m}_k^2)^2}$$

$\hat{m}_k^2 \sim 0$, $k^2 \lambda_{6k} \sim 0$:

$$\boxed{\dot{\lambda}_k = 3 \frac{1}{16\pi^2} \lambda_k^2}$$

see page I-19b
pert. theory



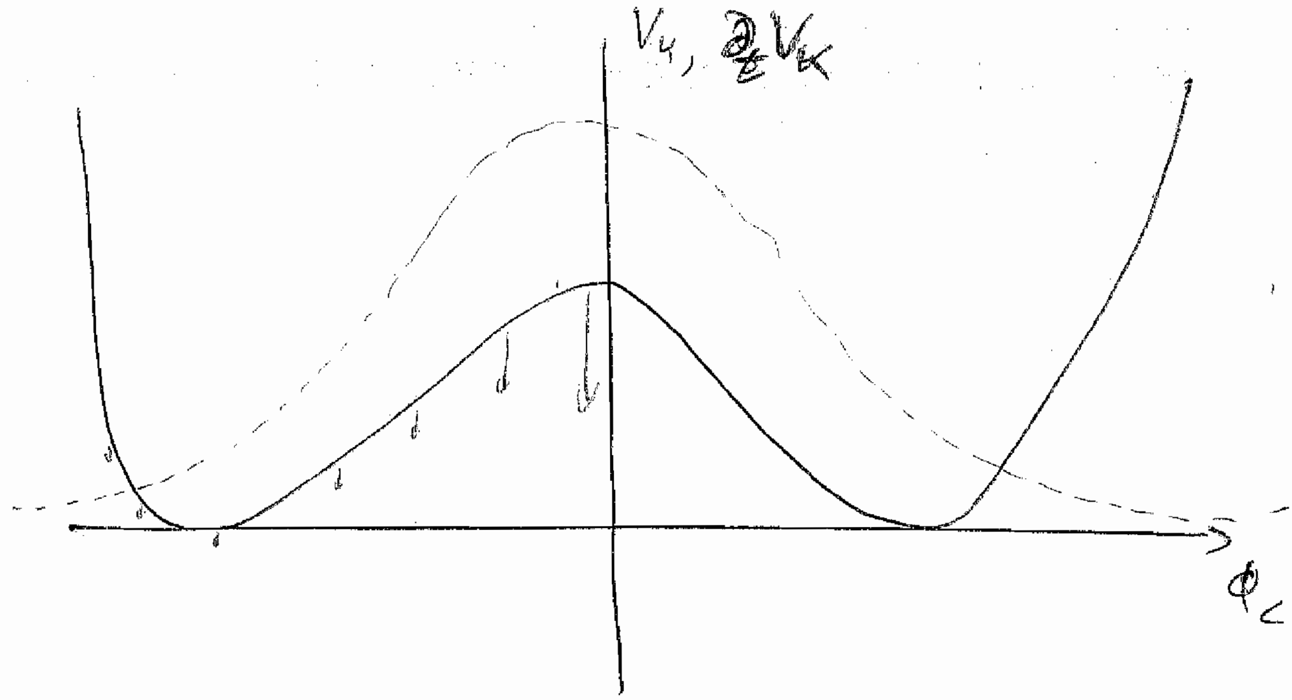
(α) Flow towards $k=0$ 'enforced' convexity:

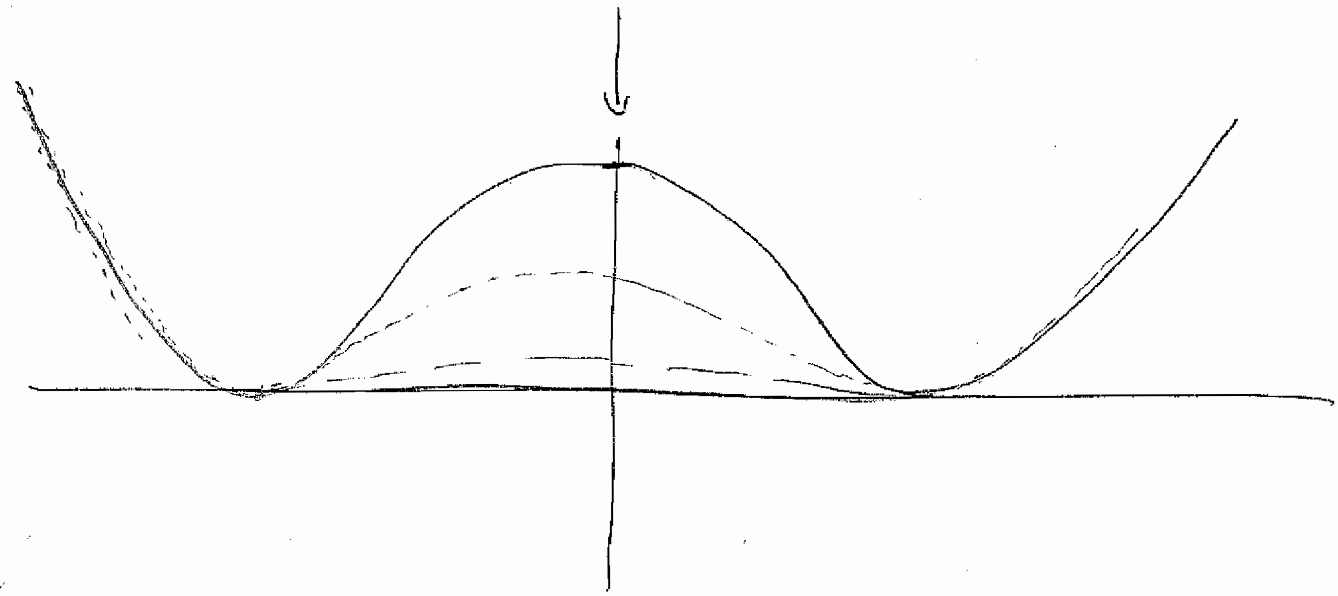
(i) $$\partial^2_{\phi_c} V[\phi_{c_1}] < \partial^2_{\phi_c} V[\phi_{c_2}]$$

$$\Rightarrow \quad \frac{1}{k^2 + \partial^2_{\phi_c} V[\phi_{c_1}]} > \frac{1}{k^2 + \partial^2_{\phi_c} V[\phi_{c_2}]}$$

(ii) $$\partial^2_{\phi_c} V[\phi_{c_{sing}}] + k^2 \to 0$$

$$\Rightarrow \quad \frac{1}{k^2 + \partial^2_{\phi_c} V[\phi_{c_{sing}}]} \to ↯$$





(B) Flow towards $k \to \infty$ leads to non-convexity:

$$\lim_{k\to\infty} m_k^2 < 0$$

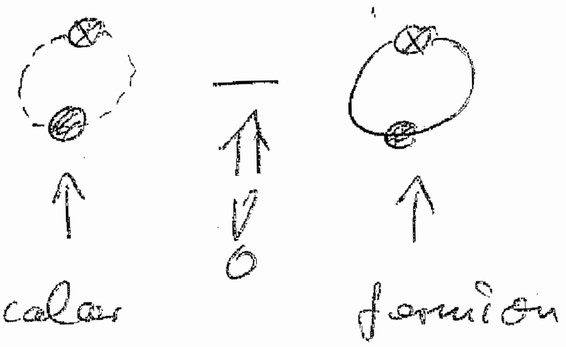
(i) $\Gamma_k + \Delta S_k$ <u>has</u> to be convex ("Legendre trafo")

(ii) poles $(\Gamma_k^{(2)}+R_k)$ $\boxed{(\Gamma_k^{(2)}+R_k)(p,q) \geq 0}$ gets singular:

$\boxed{\text{Theory has no UV-completion}}$

(iii) a glimpse at fermions: ψ

$\partial_t \Gamma_k[\phi,\psi] = \frac{1}{2}$

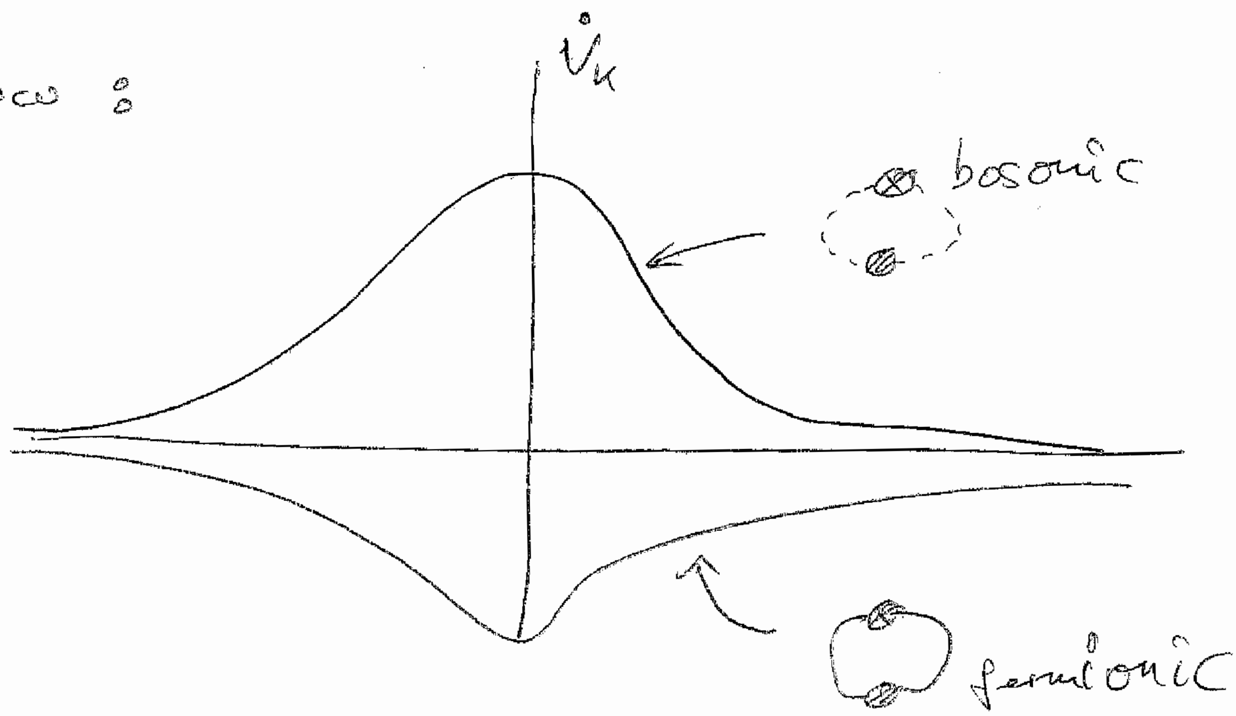


$\Rightarrow$ adds to flow of effective potential

$$\Gamma_k \sim \bar{\psi}\, F[\phi]\, \psi \implies \frac{\delta}{\delta\psi}\frac{\delta}{\delta\bar{\psi}}\Gamma_k\Big|_{\psi=\bar{\psi}=0} = F[\phi]$$

see p. I-27a

Flow:



- scalar (bosonic) flow is symmetry-restoring
- fermionic flow is symmetry-breaking

$$\Gamma_\psi[\phi, \psi] = \int \frac{d^d p}{(2\pi)^d} \; \bar{\psi} (\not{p} + m) \psi$$

$$+ \int d^d x \; \bar{\psi} F(\phi) \psi$$

+ bosonic - terms

I-3 Fixed points in the functional RG

Remark: (a) flows vanish identically for $k \to 0$:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \dot{R}_k \Big|_{R_k \hat{=} 0} \hat{=} 0$$

(b) Fixed points have to be searched for in dimensionless quantities:

$\boxed{\text{total rescaling of the theory}}$

as $\quad g(\text{scales} \to \lambda \cdot \text{scales}) = \lambda^{d_g} g(\text{scales})$

e.g. scalar theory in 4 d: $\hat{g}$ are dim-less

$$\lambda \to \hat{\lambda} = \lambda$$

$$m^2 \to \hat{m}^2 = m^2/k^2$$

$$\phi \to \hat{\phi} = \phi/k$$

$$V_k[\phi] \to \hat{V}_k[\hat{\phi}] = \frac{1}{k^d} \cdot V_k[\hat{\phi} \cdot k]$$

Fixed point: $\partial_t \hat{g}_i = \beta_i(\vec{\hat{g}})$

$$\beta_i(\vec{\hat{g}}_*) \doteq 0$$

e.g. $\vec{\hat{g}} = (\hat{m}, \hat{\lambda})$

Stability: expansion about $\vec{\hat{g}}_*$:

$$\hat{g}_i = \hat{g}_{*i} + \delta \hat{g}_i$$

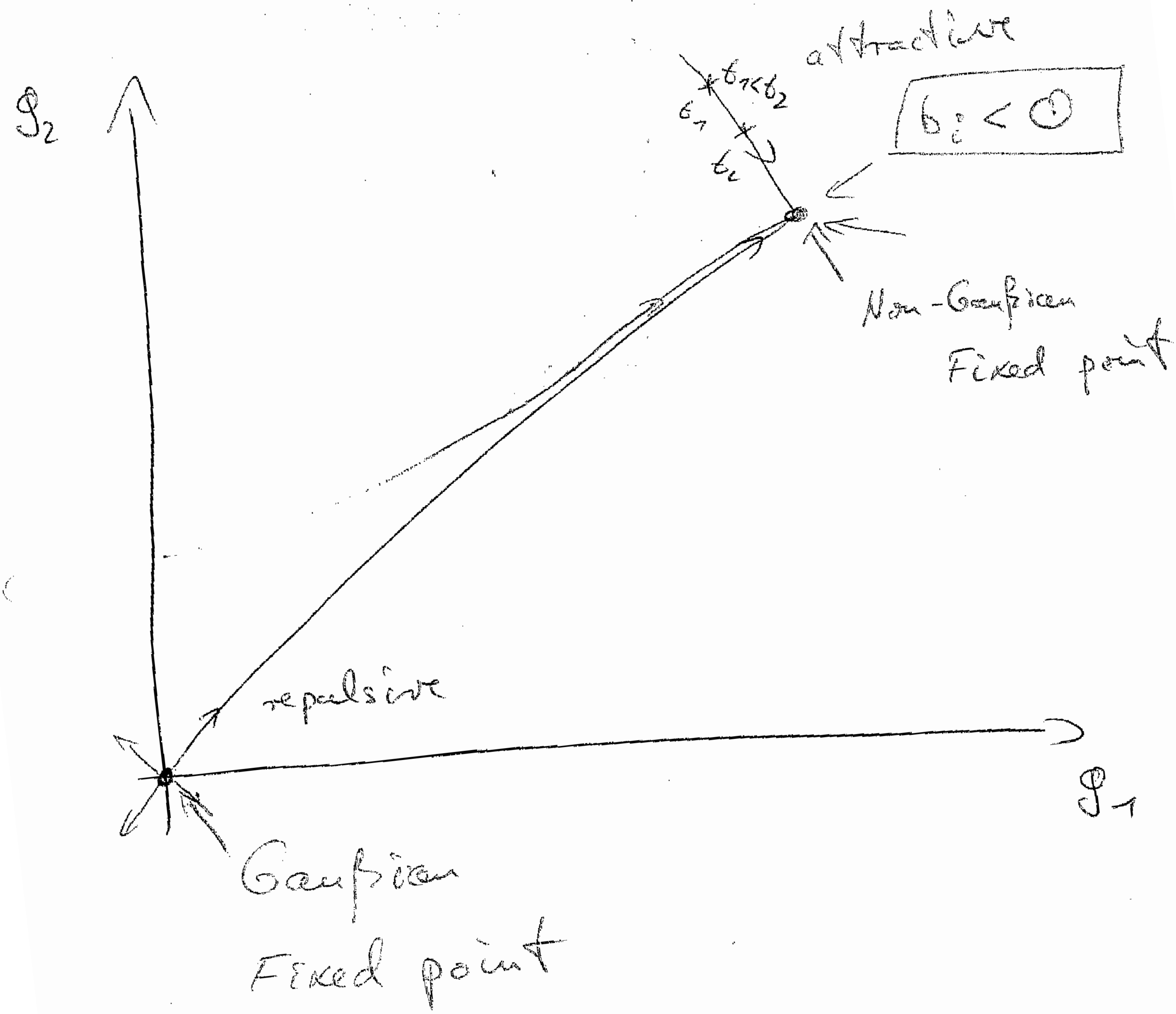
$$\Rightarrow \partial_t \hat{g}_i = \underbrace{\beta_i(\vec{\hat{g}}_*)}_{0} + B_{ij}(\vec{\hat{g}}_*)\,\delta\hat{g}_j + \mathcal{O}(\delta\hat{g}^2)$$

with $B_{ij} = \dfrac{\partial \beta_i}{\partial \hat{g}_j}$

Diagonalise: $\partial_t \bar{g}_i = b_i \bar{g}_i$ (no sum)

with $B \cdot \vec{e}_i = b_i$, $\delta\vec{\hat{g}} = \sum_i \bar{g}_i \vec{e}_i$

$$\Rightarrow \boxed{\bar{g}_i = e^{b_i t} \cdot \bar{g}_{0i}}$$


g_2
attractive
$b_1 < b_2$
b_1
b_2
$b_i < 0$
Non-Gaußian
Fixed point
repulsive
g_1
Gaußian
Fixed point

b_i complex