Statistics Inversion in Curved Spacetime

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Motivation

- Minkowski spacetime $(\mathbb{R}^d, \eta)$
  - Observable alg $\mathscr{CCR}(\mathbb{R}^d)$ for $m = 0$.
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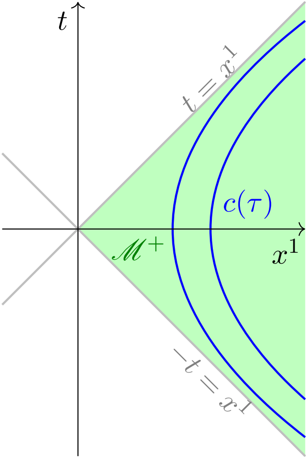

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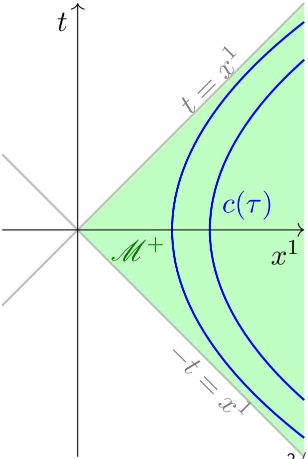
- Bisognano and Wichmann (1975, 1976); Davies (1975); Fulling (1973); Unruh (1976) effect (Buchholz & Verch, 2015, 2016)

$$\beta^{-1} = {}^{\kappa}\!/_{2\pi} \sqrt{-\upeta\big(c'(\tau), c'(\tau)\big)}$$



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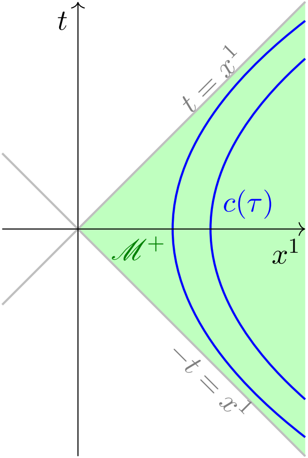

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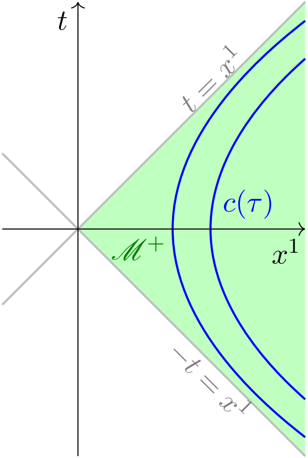

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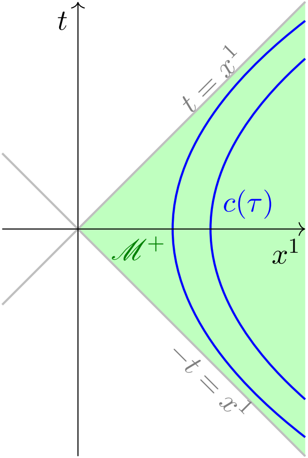

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- $\mathcal{R}_3 = 1/2(1 + \mathrm{e}^{-\beta E})$ ... "Fermi-Dirac statistics" (Takagi, 1985, 1986).

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- $\mathbb{G}$ independent of β.

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Statistics inversion (Ooguri, 1986)

$$\mathcal{F}_{\tau\mapsto E}\mathbb{W}^{+}_{\beta}(\tau)=\mathbb{P}(a)\mathbb{P}(E)/(1+\mathrm{e}^{-\beta H})$$

Huygens' Operators

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- G Lagrangian distribution $I^{-3/2}(\mathscr{M} \times \mathscr{M}, C')$ of order (at most) $-3/2$ associated with geodesic relation (Duistermaat & Hörmander, 1972)

$$C' := \{(x, \xi; y, \eta) \in \mathrm{T}^*_0\mathscr{M} \times \mathrm{T}^*_0\mathscr{M} | \exists s \in \mathbb{R} : (x, \xi) = \Phi_s(y, \eta)\}$$

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- $\mathsf{G} \in I^{-3/2}(\mathscr{M} \times \mathscr{M}, C')$ (Duistermaat & Hörmander, 1972).
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- $\Box$ op. of Huygens' type if (Friedlander, 1975; Günther, 1988)

$$\operatorname{supp} \mathsf{G} \subseteq C_{\mathscr{M} \times \mathscr{M}} := \{(x, y) \in \mathscr{M} \times \mathscr{M}\}$$

Huygens' Operators

- $(\mathscr{M}, \mathrm{g})$ globally hyperbolic $d \geq 3$-dimensional spacetime.
- Klein-Gordon op. $\Box := -\operatorname{div} \circ \operatorname{grad} + m^2 + \lambda \mathsf{scal} : C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M})$
- Pauli-Jordan op. $G := G^{\mathrm{ret}} - G^{\mathrm{adv}} : C^\infty_{\mathrm{c}}(\mathscr{M}) \to C^\infty_{\mathrm{sc}}(\mathscr{M})$.
- $\mathsf{G} \in I^{-3/2}(\mathscr{M} \times \mathscr{M}, C')$ (Duistermaat & Hörmander, 1972).
- $C' := \{(x, \xi; y, \eta) \in \mathrm{T}^*_0\mathscr{M} \times \mathrm{T}^*_0\mathscr{M} | \exists s \in \mathbb{R} : (x, \xi) = \Phi_s(y, \eta)\}$.
- $\Box$ op. of Huygens' type if (Friedlander, 1975; Günther, 1988)

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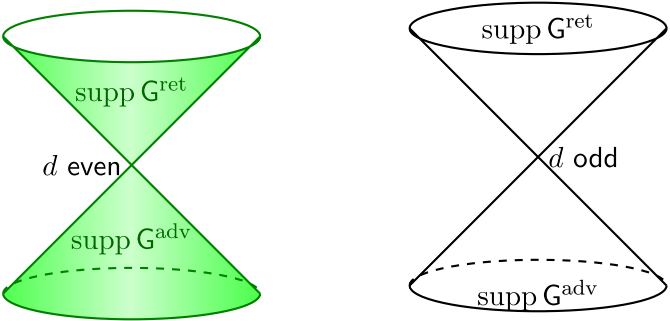

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- $\Box$ Huygens' op $\Leftrightarrow U_{d/2-1}(x,y) = 0$ (Günther, 1988).

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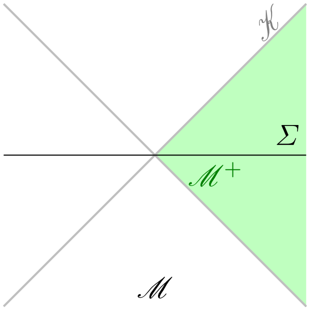
Statistics inversion in curved spacetime

- $(\mathscr{M}, \mathrm{g}, \Xi, \mathscr{K})$ globally hyperbolic stationary spacetime w. stationary bifurcate Killing horizon



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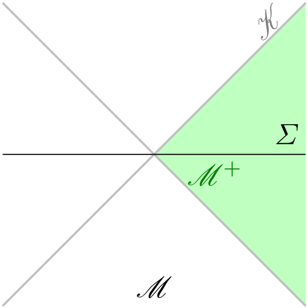

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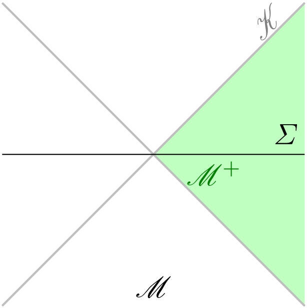

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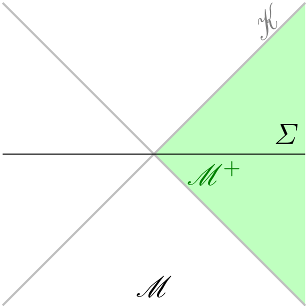

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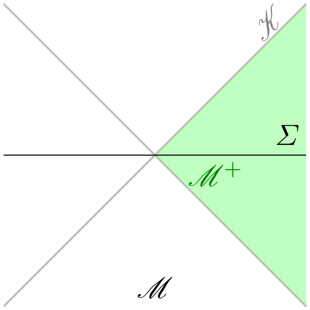

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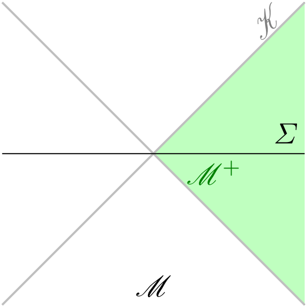

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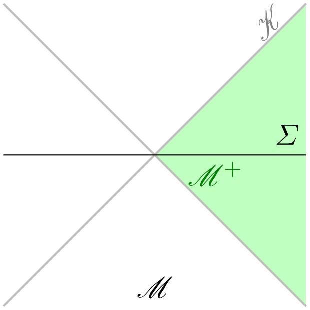

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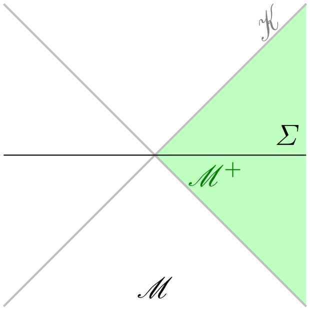

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Summary

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Alles Gute zum Geburtstag!

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- No statistics inversion for fermions (Louko & Toussaint, 2016).

Alles Gute zum Geburtstag!

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Thank you for your kind attention! Questions & Comments?

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