GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Exercises: Introduction to RG

from: P. Kopietz, L. Bartosch, F. Schütz
Introduction to the Functional Reonormalization Group
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Sheet 2

Exercise 2 (Migdal-Kadanoff RG for the Ising chain)

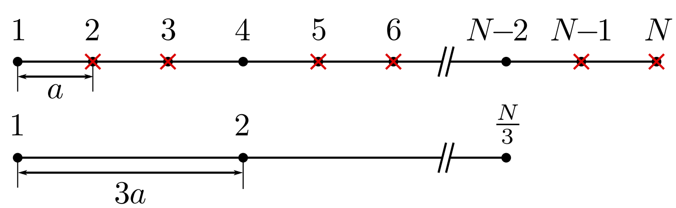
In order to calculate the partition function of the Ising model in one dimension (the so-called Ising chain) one can use the transfer matrix method. The partition function is then given by

$$\mathcal{Z} = \mathrm{Tr}[\mathbf{T}^N]$$

where the transfer matrix is defined as

$$\mathbf{T} = \begin{pmatrix} e^{g} & e^{-g} \\ e^{-g} & e^{g} \end{pmatrix} ,$$

where we introduced the dimensionless coupling constant $g = \beta J$.
One can exactly solve the Ising chain using the real-space RG, keeping only every b-th spin and tracing over intermediate spins. The figure below illustrates this for $b = 3$.



a) To calculate the partition function for $h = 0$ it is advantageous to diagonalize the matrix $\mathbf{T}$. Using $\mathrm{Tr}[ABC] = \mathrm{Tr}[BCA]$ one can calculate all quantities also in the diagonalized basis. Calculate the eigenvalues of the matrix $\mathbf{T}$.

b) For $h = 0$, $g = \beta J$, show that $\mathbf{T}^b = \mathrm{const} \times \mathbf{T}'$ with

$$\mathbf{T}' = \begin{pmatrix} e^{g'} & e^{-g'} \\ e^{-g'} & e^{g'} \end{pmatrix} .$$

and the recursion relation $g'(g) = \mathrm{Artanh}(\tanh^b g)$.
Hint: You might find it advantageous to operate with the diagonalized matrix $\mathbf{T}$.

c) * Rewrite the recursion relation in terms of $y \equiv e^{-2g}$ and $y' \equiv e^{-2g'}$ as $y'(y)$. In order to determine the RG β-function one performs infinitestimal transformations, setting $b = e^{\delta l} \approx 1 + \delta l$ and takes the limit $\delta l \to 0$. Using this procedure one gets the differential equation describing the flow. Determine the fixed points of this equation and sketch the flow of y under repeated transformations.

d) * Linearize $y'(y)$ around the unstable fixed point $y = 0$ by expanding in powers of b to show that $y' \approx by$. Argue that the correlation length fulfills $\xi(y) = b\xi(y') \approx b\xi(by)$. By an appropriate choice of b show that $\xi \propto y^{-1} = e^{2g} = e^{2\beta J}$.