

Exercises: Introduction to RG
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Sheet 1

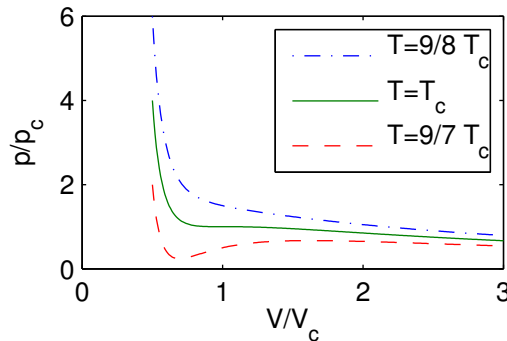
Exercise 1 (Critical exponents of the van der Waals gas)

To take into account interaction corrections to an ideal gas of atoms, van der Waals proposed the following equation of state (with $a, b > 0$),

$$\left(p + a \left(\frac{N}{V} \right)^2 \right) (V - Nb) = Nk_B T .$$

Roughly speaking, the coefficient b represents effects due to a hard core repulsive interaction and decreases the effective volume of the system. At larger distances, the interaction between the atoms is attractive and leads to a reduction of the pressure. Because the interaction is always between pairs of molecules, this correction is expected to be proportional to the square of the density $n = N/V$.

- a) Using the van der Waals equation of state it is possible to describe the discontinuous liquid-gas transition and the critical end point. A plot of $p(V)$ for representative values of T is shown below.



Rewrite the equation of state as a cubic polynomial in V and argue that at the critical point (p_c, V_c, T_c) the van der Waals equation of state reduces to $(V - V_c)^3 = 0$. Compare coefficients and show that $a = 3p_c(V_c/N)^2$ and $b = 1/3V_c/N$.

- b) Calculate the free energy $F(T, V)$ by integrating $-p(T, V)$ along an isotherm with respect to V , i.e.

$$F(T, V) = - \int_{V_0}^V p(V') dV' + \text{const.}(T) .$$

You may adjust the temperature-dependent additive constant by comparing your result in the limit $V \rightarrow \infty$ with that of an ideal gas

$$F(T, V)_{\text{ideal}} = Nk_B T \ln \frac{h^3}{(2\pi m k_B T)^{3/2} V} + Nk_B T .$$

Using the relation $C_V = -T \left(\frac{\partial^2 F}{\partial T^2} \right)_V$ calculate the specific heat at constant volume. What do you obtain for the critical exponent α defined by $C_V \propto |t|^{-\alpha}$?

- c) The exponent γ is defined by $\kappa_T \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T = -\frac{1}{V} \left(\frac{\partial p}{\partial V} \right)_{T, V=V_c}^{-1} \propto |t|^{-\gamma}$ (with $V = V_c$). The exponent δ is defined at $T = T_c$ by $p - p_c \propto (n - n_c)^\delta \propto -(V - V_c)^\delta$. Calculate the critical exponents γ and δ .
- d) * As you might have noticed, the van der Waals equation of state predicts regions in the phase diagram where the compressibility $\kappa_T \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T$ does not satisfy $\kappa_T \geq 0$. This instability is of course unphysical and can be traced back to the fact that the van der Waals equation does not allow for phase separation. A simple remedy is the Maxwell construction: Draw a line parallel to the V -axis which cuts the graph $p(V)$ in such a way that the two areas enclosed by this line and $p(V)$ are equal. In the inner part of the graph (ranging from V_{liquid} to V_{gas}) the pressure $p(V)$ is now replaced by the horizontal line. Justify the Maxwell construction by considering an isotherm of $F(T, V) = - \int p dV + \text{const}(T)$. By allowing for phase separation you can now reduce the free energy $F(T, V)$ for $V_{\text{liquid}} < V < V_{\text{gas}}$, turning the free energy convex. Apply the Maxwell construction to your above sketch and mark the coexistence curve in the p - V -diagram (where F is non-analytic).
- e) * To obtain the critical exponent β write $V = V_c(1 + v)$ and $T = T_c(1 + t)$ and expand $p(T, V)$ for small t and v . You need to keep terms up to order $\mathcal{O}(t, tv, v^3)$. You should justify this later on. Apply the Maxwell construction to obtain the coexistence curve. Go ahead and calculate the coefficient β defined by $V_{\text{gas}} - V_{\text{liquid}} \propto |t|^\beta$ with $t \leq 0$.

Verify the identity

$$e^{-\frac{x^4}{4!}} = \int_{-\infty}^{\infty} dy e^{-\frac{x^2}{2} - \frac{ix^2 y}{4}}$$

which is the simplest analogon to the mathematical operations performed in the Hubbard Stratonovich transformation.