

## Multifractality of Self-Avoiding Walks on Percolation Clusters

Viktoria Blavatska<sup>1,2,\*</sup> and Wolfhard Janke<sup>1,+</sup>

<sup>1</sup>*Institut für Theoretische Physik and Centre for Theoretical Sciences (NTZ), Universität Leipzig,  
Postfach 100 920, D-04009 Leipzig, Germany*

<sup>2</sup>*Institute for Condensed Matter Physics of the National Academy of Sciences of Ukraine, 79011 Lviv, Ukraine*  
(Received 8 May 2008; revised manuscript received 28 July 2008; published 18 September 2008)

We consider self-avoiding walks on the backbone of percolation clusters in space dimensions  $d = 2, 3, 4$ . Applying numerical simulations, we show that the whole multifractal spectrum of singularities emerges in exploring the peculiarities of the model. We obtain estimates for the set of critical exponents that govern scaling laws of higher moments of the distribution of percolation cluster sites visited by self-avoiding walks, in a good correspondence with an appropriately summed field-theoretical  $\varepsilon = 6 - d$  expansion [H.-K. Janssen and O. Stenull, Phys. Rev. E **75**, 020801(R) (2007)].

DOI: [10.1103/PhysRevLett.101.125701](https://doi.org/10.1103/PhysRevLett.101.125701)

PACS numbers: 64.60.al, 07.05.Tp, 64.60.ah

When studying physical processes on complicated fractal objects, one often encounters the interesting situation of the coexistence of a family of singularities, each associated with a set of different fractal dimensions [1]. In these problems, the conventional scaling approach cannot describe the system. Instead, an infinite set of critical exponents is needed to characterize the different moments of the distribution of observables, which scale independently. These peculiarities are usually referred to as multifractality [2]. The multifractal spectrum can be used to provide information on the subtle geometrical properties of a fractal object, which cannot be fully described by its fractal dimensionality. Indeed, clusters generated by diffusion-limited aggregation [3] and percolation clusters have the same fractal dimensions, but a completely different geometrical structure, which can be clarified, e.g., by studying the multifractality of the voltage distribution in percolation clusters [4] and the growth probability distribution in diffusion-limited aggregation [5,6]. Multifractal properties arise also in many different contexts, for example, in studies of turbulence in chaotic dynamical systems and strange attractors [2,7], human heartbeat dynamics [8], Anderson localization transition [9], etc.

To understand the common roots underlying this phenomenon, it is worthwhile to consider the generic case of self-avoiding walks (SAWs) on fractal clusters. It is well established that configurational properties of SAWs on a regular lattice are governed by scaling laws; e.g., for the averaged end-to-end distance  $\langle r \rangle$  of a SAW with  $N$  steps one finds in the asymptotic limit  $N \rightarrow \infty$ :

$$\langle r \rangle \sim N^{\nu_{\text{SAW}}}, \quad (1)$$

where  $\nu_{\text{SAW}}$  is a universal exponent depending only on the space dimension  $d$ .

The scaling of SAWs changes crucially when the underlying lattice has a fractal structure. Indeed, new critical exponents were found for SAWs residing, e.g., on a Sierpinski gasket and Sierpinski carpet [10]. A related problem arises when studying SAWs on disordered lattices

with concentration  $p$  of structural defects very close to the percolation threshold  $p_c$ . In this case, an incipient cluster of pure sites can be found in the system. The diameter of a typical cluster below  $p_c$  is characterized by the correlation length  $\xi$ , which diverges as  $\xi \sim (p - p_c)^{-\nu_p}$  with a universal exponent  $\nu_p$ . Note that percolation clusters are fractal objects (see Table I) and apparently change the universality class of residing SAWs; the scaling (1) holds in this case with an exponent  $\nu_{p_c} \neq \nu_{\text{SAW}}$ . Aiming to study the scaling of SAWs on a percolative lattice, we are interested rather in the backbone of percolation cluster: the structure left when all “dangling ends” are eliminated from the cluster. Infinitely long chains can only exist on the backbone of the cluster.

Although the behavior of SAWs on percolative lattices has served as a subject of numerous numerical [16–21] and analytical [22–25] studies since the early 1980s, not enough attention has been paid to clarifying the multifractality of the problem. Following an early idea of Meir and Harris [17], it was only recently proven in field-theoretical studies [24,25] that the exponent  $\nu_{p_c}$  alone is not sufficient to completely describe the peculiarities of SAWs on percolation clusters. Instead, a whole spectrum  $\nu^{(q)}$  of multifractal exponents emerges [25]:

$$\nu^{(q)} = \frac{1}{2} + \left( \frac{5}{2} - \frac{3}{2^q} \right) \frac{\varepsilon}{42} + \left( \frac{589}{21} - \frac{397}{14 \times 2^q} + \frac{9}{4^q} \right) \left( \frac{\varepsilon}{42} \right)^2, \quad (2)$$

with  $\varepsilon = 6 - d$ . Note that putting  $q = 0$  in (2), we restore an estimate for the dimension  $d_{p_c}^B$  of the underlying back-

TABLE I. Fractal dimensions of percolation cluster  $d_{p_c}^F$  and backbone of the percolation cluster  $d_{p_c}^B$  and correlation length critical exponent  $\nu_p$  for different space dimensions  $d$ .

$d$	$d_{p_c}^F$	$d_{p_c}^B$	$\nu_p$
2	91/49 [11]	$1.650 \pm 0.005$ [12]	4/3 [13]
3	$2.51 \pm 0.02$ [14]	$1.86 \pm 0.01$ [12]	$0.875 \pm 0.008$ [15]
4	$3.05 \pm 0.05$ [14]	$1.95 \pm 0.05$ [12]	$0.69 \pm 0.05$ [12]

bone of percolation clusters found previously [26] via  $\nu^{(0)} = 1/d_{p_c}^B$ , whereas  $q \rightarrow \infty$  gives the percolation correlation length exponent  $\nu^{(\infty)} = \nu_p$ , which restores, e.g., the result of Amit [27].  $\nu^{(1)}$  gives us the exponent  $\nu_{p_c}$ , governing the scaling law for the averaged end-to-end distance of SAWs on the backbone of percolation clusters. This is a major step forward, but the reliability of a two-loop  $\varepsilon$  expansion in  $\varepsilon = 6 - d$  remains *a priori* questionable in physical dimensions  $d = 2, 3$  where  $\varepsilon$  is large. To settle this question, we report in this Letter a careful computer simulation study of SAWs on percolation clusters. We primarily aim to obtain precise numerical estimates for the multifractal exponents  $\nu^{(q)}$  in a wide range of  $q$  and find the related spectrum of singularities on the underlying fractal cluster, thus going into deeper understanding of peculiarities of the model.

We consider site percolation on regular lattices of edge lengths up to  $L_{\max} = 400, 200, 50$  in dimensions  $d = 2, 3, 4$ , respectively. Each site of the lattice is occupied randomly with probability  $p_c$  and empty otherwise. To extract the backbone of a percolation cluster, we apply the algorithm proposed in Ref. [28]. First, the starting point—“seed”—is chosen at the center of the cluster. Then, for all the sites on the borders of the lattice, the shortest paths to the “seed” are found, forming the so-called elastic backbone [29]. Finally, by adding all “loops” (paths of sites connected to the elastic backbone in two places at least) we obtain the geometric backbone of the cluster. We construct 1000 clusters in each space dimension.

Starting on the “seed” of a single cluster, we construct a SAW on it, applying the pruned-enriched chain-growth algorithm [30]. We let a trajectory of SAW grow step by step, until it reaches some prescribed distance (say  $R$ ) from the starting point. Then, the algorithm is stopped, and a new SAW grows from the same starting point. In such a way, we are interested in constructing different possible trajectories with fixed end-to-end distance, as is shown schematically in Fig. 1. For each lattice size  $L$ , we change  $R$  up to  $\approx L/3$  to avoid finite-size effects, since close to

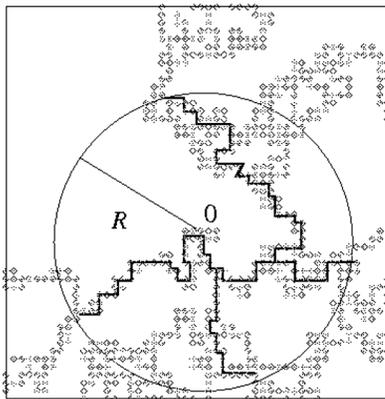


FIG. 1. Different SAW trajectories with fixed end-to-end distance  $R$  on the backbone of a percolation cluster in  $d = 2$ .

lattice borders the structure of the backbone of percolation clusters is distorted and thus can falsify the SAW statistics.

Let us denote by  $K(R)$  the total number of constructed SAW trajectories between 0 and  $R$  (we perform  $\sim 10^6$  SAWs for each value of  $R$ ). Then, for each site  $i$  of the backbone we sum up the portion of trajectories, passing through this site. In such a way, we prescribe a weight  $w(i) = K(i)/K(R)$  to each site  $i \in R$ , and thus receive some “population,” occupying the underlying fractal cluster. The distribution  $n(w)$  of weights, prescribed to the sites of the backbone of a percolation cluster, visited by SAWs with fixed end-to-end distance  $R = 160$  in two dimensions, is shown in Fig. 2.

The multifractal moments  $M^{(q)}$  are defined as follows:

$$M^{(q)} = \sum_{i \in R} w(i)^q. \quad (3)$$

Averaged over different configurations of the constructed backbones of percolation clusters, they scale as

$$\overline{M^{(q)}} \sim R^{1/\nu^{(q)}}, \quad (4)$$

with exponents  $\nu^{(q)}$  that do not depend on  $q$  in a linear or affine fashion, implying that SAWs on percolation clusters are multifractals. To estimate the numerical values of  $\nu^{(q)}$  on the basis of data obtained by us (see Fig. 3), linear least-squares fits are used.

It is clear that at  $q = 0$  we just count the number of sites of the cluster of linear size  $R$ , and thus  $1/\nu^{(0)}$  corresponds to the fractal dimension of the backbone  $d_{p_c}^B$ . Our results give  $d_{p_c}^B(d = 2) = 1.647 \pm 0.006$ ,  $d_{p_c}^B(d = 3) = 1.865 \pm 0.006$ ,  $d_{p_c}^B(d = 4) = 1.946 \pm 0.006$ , in very good agreement with Table I. At  $q = 1$ , we restore the value of the exponent  $\nu_{p_c}$ , governing the scaling law of the end-to-end distance for SAWs on the backbone of percolation clusters. We obtain  $\nu^{(1)}(d = 2) = 0.779 \pm 0.006$ ,  $\nu^{(1)}(d = 3) = 0.669 \pm 0.006$ ,  $\nu^{(1)}(d = 4) = 0.591 \pm 0.006$ , in perfect agreement with our recent estimates [21] based on the scaling of the end-to-end distance with the number of SAW steps. At  $q \rightarrow \infty$ , the so-called “red sites” of the backbone

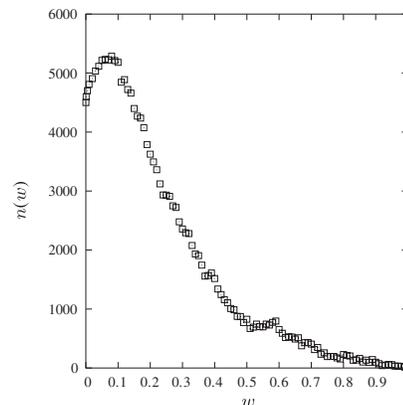


FIG. 2. Number of sites  $n(w)$  of the backbone of a percolation cluster with weights  $w$ , visited by SAWs with fixed end-to-end distance  $R = 160$ , in  $d = 2$  dimensions.

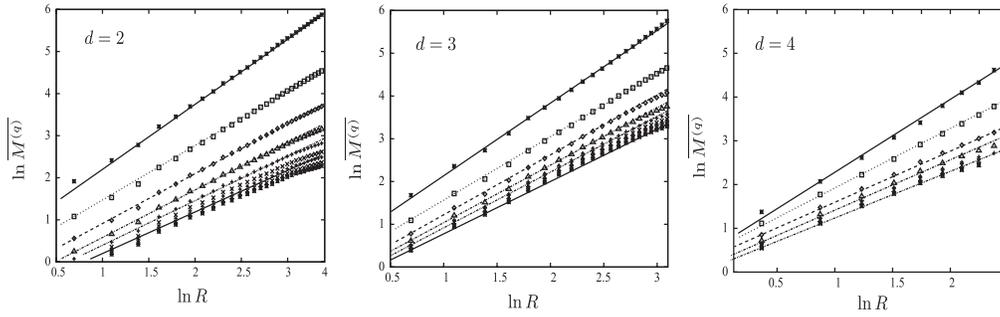


FIG. 3. Averaged moments  $\overline{M^{(q)}}$  as function of  $R$  in double logarithmic scale in  $d = 2, 3, 4$ . In each case,  $q = 0, 1, 2, 3, 4, \dots$  going from above. Lines are guides to the eyes.

are mainly taken into account—the singly connected sites, such that cutting off one of them will produce a disconnection of the cluster. These sites are most often visited by SAWs, and thus have the maximum weights. It was proven [31] that the number of “red sites” within a sphere of radius  $R$  (cf. Fig. 1) scales as  $N_{\text{red}}(R) \sim R^{1/\nu_p}$ . Indeed, as it follows from our data, at large  $q$  the value of the exponent  $\nu^{(q)}$  tends to  $\nu_p$ , the percolation correlation length exponent; cf. Table I. So, in two limiting cases ( $q \rightarrow 0$ ,  $q \rightarrow \infty$ ) we restore the corresponding multifractal exponents of the voltage distribution on the backbone of percolation clusters [4]. Note that in the latter problem, the exponent governing the scaling of the moment with  $q = 1$  gives the resistance exponent; e.g.,  $\nu_R^{(1)}(d = 2) \simeq 0.97$  [4].

Because of the long tail of the distribution  $n(w)$ , the precision of our estimates decreases with increasing  $q$ .

This problem turns out to be especially crucial when exploring the moments with negative powers  $q$ : the sites with small probabilities to be visited, which are determinant in negative moments, are very difficult to probe.

Our estimates of the exponents  $\nu^{(q)}$  for different  $q$  are presented in the upper panel of Fig. 4. These values appear to be in an astonishingly perfect correspondence with analytical estimates down to  $d = 2$  dimensions, derived by applying Padé approximation to the  $\varepsilon = 6 - d$  expansion (2), presenting the given series as ratio  $[m]/[n]$  of two polynomials of degree  $m$  and  $n$  in  $\varepsilon$ . We used the  $[1]/[2]$  approximant, because it appears to be most reliable in restoring the known estimates in limiting cases ( $q = 0$ ,  $q \rightarrow \infty$ ). A direct use of the expression (2) gives worse results, especially for low dimensions  $d$  where the expansion parameter  $\varepsilon = 6 - d$  is large. To contrast our results

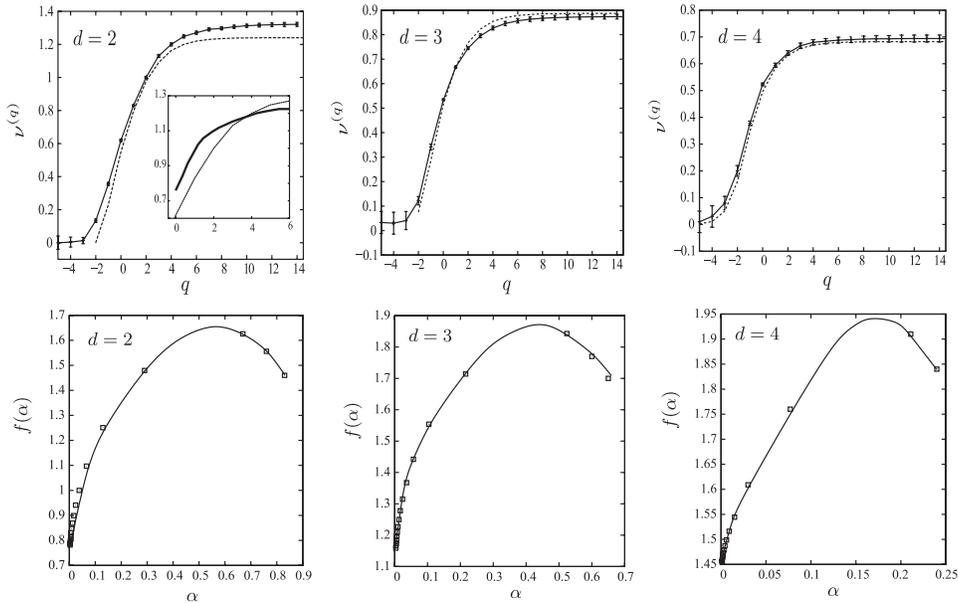


FIG. 4. Top: Spectrum of multifractal exponents  $\nu^{(q)}$  as function of  $q$  in  $d = 2, 3, 4$ . Dotted lines present  $[1]/[2]$  Padé approximants to the analytical results of Janssen and Stenull [Eq. (2)]. Inset for  $d = 2$ : Comparison with the exactly known multifractal spectrum for the related, but different, problem of diffusive particles at the surface of percolation clusters [32] (upper bold line). Bottom: Spectral function  $f(\alpha)$  in  $d = 2, 3, 4$ . The maximum value of  $f(\alpha)$  gives the fractal dimension  $d_{p_c}^B$  of the underlying backbone of percolation cluster.

for  $d = 2$  with the exactly known multifractal spectrum for the harmonic measure describing diffusing particles at the surface of percolation clusters [32], we have added the inset in Fig. 4. That this is not a theory for the present system, however, is apparent already from the 0th moment: in our case it gives the “bulk” fractal dimension  $d_{pc}^B$  of the percolation cluster, whereas in the latter case it corresponds to its accessible external perimeter.

It is well known that the set of exponents governing scaling of multifractal moments of the type (4) is related to the spectrum of singularities  $f(\alpha)$  of the fractal measure [5], called also the spectral function. The physical meaning of  $f(\alpha)$  in our problem is that the number  $N_R(\alpha)$  of sites  $i$ , where the weight  $w(i)$  scales as  $R^{-\alpha}$ , behaves as

$$N_R(\alpha) \sim R^{f(\alpha)}. \quad (5)$$

The singularity spectrum  $f(\alpha)$  is given by the Legendre transform:

$$f(\alpha) = q\alpha - \tau(q), \quad \alpha(q) = \frac{d\tau(q)}{dq}, \quad (6)$$

with  $\tau(q) = 1/\nu^{(q)}$ . Spectral functions  $f(\alpha)$ , obtained on the basis of our results, are given in the lower panel of Fig. 4. The general properties of  $f(\alpha)$  are as follows: it is positive on an interval  $[\alpha_{\min}, \alpha_{\max}]$ , where  $\alpha_{\min} = \lim_{q \rightarrow +\infty} \tau(q)/(q-1)$ ,  $\alpha_{\max} = \lim_{q \rightarrow -\infty} \tau(q)/(q-1)$ . The maximum value of the spectral function gives the fractal dimension of the underlying structure, which in our case corresponds to the fractal dimension  $d_{pc}^B$  of the backbone of percolation clusters.

To conclude, we have shown numerically that SAWs residing on the backbone of percolation clusters give rise to a whole spectrum of singularities, thus revealing multifractal properties. To completely describe peculiarities of the model, the multifractal scaling should be taken into account. We have found estimates for the exponents, governing different moments of the weight distribution, which scale independently, in surprisingly good coincidence with two-loop  $\varepsilon$  expansions. The behavior of the spectral function, describing the frequency of observation of a set of singularities on the underlying backbone of percolation clusters, is analyzed as well.

V.B. is grateful for support through the Alexander von Humboldt foundation. We thank B. Waclaw for useful discussions.

\*viktorija@icmp.lviv.ua; blavatska@itp.uni-leipzig.de

<sup>†</sup>Wolfhard.Janke@itp.uni-leipzig.de

- [1] For review see, e.g., H. E. Stanley and P. Meakin, *Nature* (London) **335**, 405 (1988).
- [2] B. B. Mandelbrot, *J. Fluid Mech.* **62**, 331 (1974).
- [3] T. A. Witten and L. M. Sander, *Phys. Rev. Lett.* **47**, 1400 (1981).
- [4] L. de Arcangelis, S. Redner, and A. Coniglio, *Phys. Rev. B* **31**, 4725 (1985); *Phys. Rev. B* **34**, 4656 (1986); R. Rammal, C. Tannous, and A. M. S. Tremblay, *Phys. Rev. A* **31**, 2662 (1985); R. Blumenfeld and A. Aharony, *J. Phys. A* **18**, L443 (1985).
- [5] T. C. Halsey, P. Meakin, and I. Procaccia, *Phys. Rev. Lett.* **56**, 854 (1986); T. C. Halsey *et al.*, *Phys. Rev. A* **33**, 1141 (1986).
- [6] M. H. Jensen *et al.*, *Phys. Rev. E* **65**, 046109 (2002); M. H. Jensen, J. Mathiesen, and I. Procaccia, *Phys. Rev. E* **67**, 042402 (2003).
- [7] H. G. Hentschel and I. Procaccia, *Physica* (Amsterdam) **8D**, 435 (1983); R. Benzi *et al.*, *J. Phys. A* **17**, 3521 (1984); A. Brandenburg *et al.*, *Phys. Rev. A* **46**, 4819 (1992); D. Queiros-Conde, *Phys. Rev. E* **64**, 015301 (2001); L. Biferale *et al.*, *Phys. Rev. Lett.* **93**, 064502 (2004).
- [8] P. Ch. Ivanov *et al.*, *Nature* (London) **399**, 461 (1999).
- [9] M. Schreiber and H. Grussbach, *Phys. Rev. Lett.* **67**, 607 (1991); H. Grussbach and M. Schreiber, *Phys. Rev. B* **51**, 663 (1995); A. Mildenerger, F. Evers, and A. D. Mirlin, *Phys. Rev. B* **66**, 033109 (2002); A. D. Mirlin *et al.*, *Phys. Rev. Lett.* **97**, 046803 (2006).
- [10] D. Dhar, *J. Math. Phys.* (N.Y.) **19**, 5 (1978); *J. Phys.* (Paris) **49**, 397 (1988); F. D. Reis and R. Riera, *J. Phys. A* **28**, 1257 (1995); A. Ordemann, M. Porto, and H. E. Roman, *Phys. Rev. E* **65**, 021107 (2002); *J. Phys. A* **35**, 8029 (2002); I. Živić, S. Milošević, and B. Djordjević, *J. Phys. A* **38**, 555 (2005).
- [11] S. Havlin and D. Ben Abraham, *Adv. Phys.* **36**, 695 (1987).
- [12] C. Moukarzel, *Int. J. Mod. Phys. C* **9**, 887 (1998).
- [13] M. P. M. den Nijs, *J. Phys. A* **12**, 1857 (1979); B. Nienhuis, *J. Phys. A* **15**, 199 (1982).
- [14] P. Grassberger, *J. Phys. A* **19**, 1681 (1986).
- [15] P. N. Strenski, R. M. Bradley, and J. M. Debierre, *Phys. Rev. Lett.* **66**, 1330 (1991).
- [16] K. Kremer, *Z. Phys. B* **45**, 149 (1981); S. B. Lee and H. Nakanishi, *Phys. Rev. Lett.* **61**, 2022 (1988); S. B. Lee, H. Nakanishi, and Y. Kim, *Phys. Rev. B* **39**, 9561 (1989); K. Y. Woo and S. B. Lee, *Phys. Rev. A* **44**, 999 (1991); H. Nakanishi and S. B. Lee, *J. Phys. A* **24**, 1355 (1991); S. B. Lee, *J. Korean Phys. Soc.* **29**, 1 (1996).
- [17] Y. Meir and A. B. Harris, *Phys. Rev. Lett.* **63**, 2819 (1989).
- [18] P. Grassberger, *J. Phys. A* **26**, 1023 (1993).
- [19] M. D. Rintoul, J. Moon, and H. Nakanishi, *Phys. Rev. E* **49**, 2790 (1994).
- [20] A. Ordemann *et al.*, *Phys. Rev. E* **61**, 6858 (2000).
- [21] V. Blavatska and W. Janke, *Europhys. Lett.* **82**, 66006 (2008).
- [22] Y. Kim, *J. Phys. C* **16**, 1345 (1983); *J. Phys. A* **20**, 1293 (1987).
- [23] K. Barat, S. N. Karmakar, and B. K. Chakrabarti, *J. Phys. A* **24**, 851 (1991).
- [24] C. von Ferber *et al.*, *Phys. Rev. E* **70**, 035104(R) (2004).
- [25] H.-K. Janssen and O. Stenull, *Phys. Rev. E* **75**, 020801(R) (2007).
- [26] H. K. Janssen and O. Stenull, *Phys. Rev. E* **61**, 4821 (2000).
- [27] D. J. Amit, *J. Phys. A* **9**, 1441 (1976).
- [28] M. Porto *et al.*, *Phys. Rev. E* **56**, 1667 (1997).
- [29] S. Havlin *et al.*, *J. Phys. A* **17**, L957 (1984).
- [30] M. N. Rosenbluth and A. W. Rosenbluth, *J. Chem. Phys.* **23**, 356 (1955); P. Grassberger, *Phys. Rev. E* **56**, 3682 (1997).
- [31] A. Coniglio, *Phys. Rev. Lett.* **46**, 250 (1981); *J. Phys. A* **15**, 3829 (1982).
- [32] B. Duplantier, *Phys. Rev. Lett.* **82**, 3940 (1999).