PATH-INTEGRAL DERIVATION OF LARGE-ORDER BEHAVIOUR OF PERTURBATION THEORY FOR ANISOTROPIC ANHARMONIC OSCILLATORS

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The large-order behaviour of perturbation coefficients for anisotropic anharmonic oscillators with potential $V(x, y)=\frac{1}{2}(x^2+y^2)+\lambda(x^4+2cx^2y^2+y^4)$ is rederived by means of the path-integral approach. This derivation turns out to be much simpler than the original one by Banks, Bender and Wu, applying multidimensional WKB techniques.

1. Introduction

Several years ago Bender and Wu [1] applied WKB techniques to the calculation of large-order terms in the Rayleigh–Schrödinger perturbation series for various anharmonic oscillators. For isotropic potentials, most of their results have been rederived from path-integral approaches [2] (a very short derivation is given in ref. [3]). It is generally appreciated that the latter calculations are conceptually more transparent, allowing, e.g., for simple scaling and symmetry arguments. In most cases they are also technically simpler and, furthermore, they can be generalized to field theory (for reviews see ref. [4]; see also the forthcoming introductory text [3]).

Applying rather involved *multidimensional* WKB techniques, Banks, Bender and Wu (BBW) [5] were able to extend their approach to anisotropic problems. As a typical example they considered the Hamiltonian ($\boldsymbol{r}=(x, y)$)

$$H=\tfrac{1}{2}\boldsymbol{p}^2+\tfrac{1}{2}\boldsymbol{r}^2+\lambda(x^4+2cx^2y^2+y^4) \tag{1}$$

and derived the large-order behaviour of perturbation expansions in λ for the ground-state energy of (1),

$$E_0=\sum_{k=0} E_{0,k}\lambda^k\,, \tag{2}$$

depending on the anisotropy parameter $c\geqslant -1$ (for $c<-1$, the potential is not bounded from below). Their calculation is based on a sequence of ingenious transformations which are difficult to motivate, combined with various approximations which require careful justifications. Finally they end up with a Riccati equation which can be further transformed into an associated Legendre equation whose solutions are known.

The purpose of this brief note is to show that all such complications are avoided in the path-integral approach. In the isotropic case this is well known, but, when applied to the anisotropic Hamiltonian (1), the advantages are even more impressive. Working with Langer's [6] formulation (which is closely related to Lipatov's [7]) and making use of known results for the one-dimensional anharmonic oscillator, we shall reproduce the results of BBW very easily in a few lines of calculation. In particular the leading behaviour of the large-order terms can be understood by simple symmetry arguments.

2. Path-integral approach

To start let us briefly recall the general framework of the path-integral approach in Langer's formulation. It is based on the path-integral representation of the quantum partition function

$$Z=\int \mathcal{D}^2 r \exp(-\mathcal{A}[\boldsymbol{r}]) \xrightarrow{\beta\to\infty} \exp(-\beta E_0)\ , \tag{3}$$

where

$$\mathcal{A}[\boldsymbol{r}]=\int_{-\beta/2}^{\beta/2} \mathrm{d}\tau\ [\tfrac{1}{2}\dot{\boldsymbol{r}}^2+\tfrac{1}{2}\boldsymbol{r}^2+\lambda(x^4+2cx^2y^2+y^4)] \tag{4}$$

is the Euclidean action corresponding to the Hamiltonian (1), $\beta\equiv 1/k_B T$ is the inverse temperature, and $\mathcal{D}^2 r\equiv\prod_{n=1}^{N}\mathrm{d}^2 r_n/2\pi\epsilon$, denotes the usual path-integral measure on a sliced "time"-axis with spacing $\epsilon=\beta/N$. The paths in (3) are assumed to be "periodic", i.e., $\boldsymbol{r}(-\beta/2)=\boldsymbol{r}(\beta/2)$.

For positive coupling λ, the system is stable and Z is real. For negative coupling λ, however, the system becomes unstable and Z develops an exponentially small imaginary part ($\propto \exp(-1/a|\lambda|)$) related to the decay-rate. For small $\lambda<0$, this can be computed in a saddle-point approximation by an expansion around non-trivial "critical bubble" or "instanton" solutions which extremize the action ($\propto 1/a|\lambda|$). By taking the large β limit in (3), one finds then immediately the decay-rate Γ of the ground-state resonance,

$$-\frac{1}{\beta}\frac{\operatorname{Im} Z}{\operatorname{Re} Z} \xrightarrow{\beta\to\infty} \tfrac{1}{2}\Gamma=\operatorname{Im} E_0 \propto \exp(-1/a|\lambda|)\ , \tag{5}$$

and by means of the dispersion relation [1,5,8]

$$E_{0,k}=\frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\lambda\, \frac{\operatorname{Im} E_0(\lambda+\mathrm{i}0)}{\lambda^{k+1}} \tag{6}$$

the corresponding large-order behaviour of the coefficients in the expansion (2). In eq. (5) the fact was used that $\operatorname{Im} Z\propto\exp(-\beta)\exp(-1/a|\lambda|)$ is much smaller than $\operatorname{Re} Z=\exp\{-\beta[1+\mathrm{O}(\lambda)]\}$ (which results from locally stable fluctuations around $\boldsymbol{r}=0$). Let us now apply this approach to the Hamiltonian (1) and determine along these lines the precise form of $\operatorname{Im} E_0$ as $\lambda\to 0^-$ and thus $E_{0,k}$ as $k\to\infty$.

2.1. *Case $-1\leqslant c<1$*

For $-1\leqslant c<1$, the potential term x^4+y^4 in (4) is dominant compared with $2cx^2y^2$, so that the "tunneling-paths" of extremal action [#1] are obviously straight lines along the coordinate axis. Along these rays, say

$$x_c(\tau)\equiv u_c(\tau)\geqslant 0,\quad y_c(\tau)\equiv 0\ , \tag{7}$$

the action

$$\mathcal{A}_c\equiv\mathcal{A}[u_c,0]=\int \mathrm{d}\tau\ (\tfrac{1}{2}\dot{u}_c^2+\tfrac{1}{2}u_c^2+\lambda u_c^4)=\frac{1}{3|\lambda|} \tag{8}$$

is extremized, in the large β limit, by the well-known "critical bubble" solution [2]

$$u_c(\tau)=\sqrt{\frac{1}{2|\lambda|}}\,\frac{1}{\operatorname{ch}(\tau-\tau_0)} \tag{9}$$

[#1] More precisely these paths can be characterized as functional *saddle-points with least action.*

of the one-dimensional anharmonic oscillator. Any choice of τ_0 breaks spontaneously translational invariance (for simplicity, we shall set $\tau_0=0$ in the sequel). This gives rise to a Nambu–Goldstone zero-mode which will be taken into account by a careful treatment of the fluctuations around the "critical buble". Their leading contribution is found by expanding the full action (4) in the deviations $\delta x=x-x_c$, $\delta y=y-y_c$ up to quadratic order,

$$\delta\mathcal{A}\equiv\mathcal{A}-\mathcal{A}_c=\int d\tau\,[\tfrac{1}{2}(\delta\dot{x}^2+\delta\dot{y}^2)+(\delta x^2+\delta y^2)+\lambda(\delta x\,\delta y)M(\delta x\,\delta y)^{\mathrm{T}}+...] \tag{10}$$

with M denoting the matrix

$$M\equiv\begin{pmatrix} 6x_c^2+2cy_c^2 & 4cx_cy_c \\ 4cx_cy_c & 6y_c^2+2cx_c^2 \end{pmatrix}. \tag{11}$$

Inserting the solution (7), (9), M is automatically diagonalized and (10) simplifies to

$$\delta\mathcal{A}=\frac{1}{2}\int d\tau\left[\delta x\left(-\frac{d^2}{d\tau^2}+1-\frac{6}{\mathrm{ch}^2\tau}\right)\delta x+\delta y\left(-\frac{d^2}{d\tau^2}+1-\frac{2c}{\mathrm{ch}^2\tau}\right)\delta y+...\right]. \tag{12}$$

The Gaussian path-integral over the *longitudinal* fluctuations δx coincides with that appearing in the one-dimensional oscillator problem and is therefore known [2]. It contains a negative eigenmode, this being responsible for the expected imaginary part of Z, and the already mentioned zero eigenmode $\propto\dot{u}_c(\tau)$, associated with translations of the "critical bubble" along the τ-axis. The well-known result is [2]

$$\begin{aligned} f_{\delta x}&\equiv\int\mathcal{D}\,\delta x\exp\left[-\frac{1}{2}\int d\tau\,\delta x\left(-\frac{d^2}{d\tau^2}+1-\frac{6}{\mathrm{ch}^2\tau}\right)\delta x\right] \\ &=\det\left(-\frac{d^2}{d\tau^2}+1-\frac{6}{\mathrm{ch}^2\tau}\right)^{-1/2}=-\frac{i}{2}\frac{1}{\sqrt{3}}6\beta\sqrt{\frac{\mathcal{A}_c}{2\pi}}\,e^{-\beta/2}, \end{aligned} \tag{13}$$

with the factor $-(i/2)(1/\sqrt{3})$ coming from the negative mode, $\beta\sqrt{\mathcal{A}_c/2\pi}$ from the zero mode, and 6 from all other modes with positive eigenvalues.

The *transversal* fluctuations δy do not contain any negative or zero modes. To see this we consider first the limit $c\to 1$ in which the operator governing the transversal fluctuations approaches that of an isotropic oscillator. This operator is known [2] to contain just one zero eigenmode $\propto u_c(\tau)$, associated with the rotational invariance in this limit. Since the potential-well $1-2c/\mathrm{ch}^2\tau$ becomes shallower for $c<1$, all eigenvalues must be positive in this case. To evaluate the fluctuation determinant we proceed as follows. First we write

$$f_{\delta y}=\det\left(-\frac{d^2}{d\tau^2}+1-\frac{2c}{\mathrm{ch}^2\tau}\right)^{-1/2}\equiv f(z=1)Z_{\mathrm{osc}}, \tag{14}$$

where

$$Z_{\mathrm{osc}}\equiv\det\left(-\frac{d^2}{d\tau^2}+1\right)^{-1/2}=\frac{1}{2\,\mathrm{sh}(\beta/2)}\xrightarrow{\beta\to\infty}e^{-\beta/2} \tag{15}$$

is the partition function of the harmonic oscillator and

$$f(z)\equiv\left(\frac{\det[-d^2/d\tau^2+z-s(s+1)/\mathrm{ch}^2\tau]}{\det(-d^2/d\tau^2+z)}\right)^{-1/2}, \tag{16}$$

with $s(s+1)\equiv 2c$. To this ratio of determinants we then apply a general formula in the theory of Fredholm determinants stating that [2]

$$f(z)=\left(\frac{\Gamma(\sqrt{z}-s)\Gamma(\sqrt{z}+1+s)}{\Gamma(\sqrt{z})\Gamma(\sqrt{z}+1)}\right)^{1/2}, \tag{17}$$

where $\Gamma(\ )$ is the Gamma function. In more physical terms, this result can be derived (see ref. [3]) by relating $f(z)$ to the transmission amplitude of one-dimensional scattering at the potential $z-s(s+1)/\mathrm{ch}^2\tau$. Inserting $z=1$, we obtain

$$f_{\delta y}=[-s(s+1)\Gamma(-s)\Gamma(1+s)]^{1/2}\mathrm{e}^{-\beta/2}=\left(\frac{s(s+1)\pi}{\sin\pi s}\right)^{1/2}\mathrm{e}^{-\beta/2}. \tag{18}$$

Finally, combining (13) and (18), and multiplying with a factor 4 for the four equivalent rays $(\pm u_c, 0)$, $(0, \pm u_c)$, we get for $-1\leqslant c<1$ (recall (5) and notice that all β dependent factors cancel)

$$\begin{aligned}\mathrm{Im}\,E_0 &\xrightarrow{\lambda\to 0^-} 4\frac{|f_{\delta x}|f_{\delta y}}{\beta\mathrm{e}^{-\beta}}\exp(-\mathscr{A}_c)\\ &=4\left(\frac{s(s+1)\pi}{\sin\pi s}\right)^{1/2}\sqrt{\frac{3}{2\pi}}\sqrt{\frac{1}{3|\lambda|}}\,\mathrm{e}^{-1/3|\lambda|}=\left(\frac{48c}{\sin\pi s}\right)^{1/2}\sqrt{\frac{1}{3|\lambda|}}\,\mathrm{e}^{-1/3|\lambda|},\end{aligned} \tag{19}$$

and by means of the dispersion relation (6)

$$E_{0,k}\xrightarrow{k\to\infty}-\frac{1}{\pi}\left(\frac{48c}{\sin\pi s}\right)^{1/2}(-3)^k\Gamma(k+\tfrac{1}{2}), \tag{20}$$

with $s(s+1)=2c$. Expressing s in terms of c, the argument of the square root in (19), (20) can be rewritten as

$$\frac{48c}{\sin\pi s}=\frac{48c}{-\cos[\pi\sqrt{2(c+\frac{1}{8})}]}>0$$

(when $c<-\frac{1}{8}$, cos is analytically continued to cosh). The dependence of the square-root factor on c is shown in fig. 1. These results are of course in agreement with BBW, who have checked them against exact perturbation coefficients in high order.

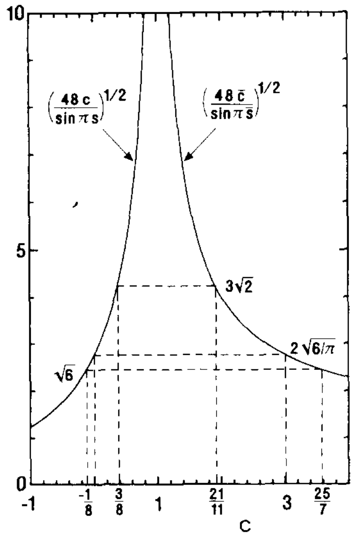

Fig. 1. The square-root factor in eqs. (20) and (23) versus the anisotropy parameter c. The parameters s, $\bar{s}$ and $\bar{c}$ are determined from $s(s+1)=2c$ and $\bar{s}(\bar{s}+1)=2\bar{c}=2(3-c)/(1+c)$, respectively. By means of the last equation, the curves for $-1\leqslant c<1$ and $c>1$ can be mapped onto each other. The rotationally symmetric case $c=1$ is an isolated discontinuity.

2.2. Case $c>1$

Actually, formula (20) also solves the problem for $c>1$. This special property of the action (4) was already noted and exploited by BBW. They observed that under the orthogonal transformation

$$x=\frac{1}{\sqrt{2}}(\bar{x}+\bar{y}),\quad y=\frac{1}{\sqrt{2}}(\bar{x}-\bar{y}) \tag{21}$$

the action (4) maps onto itself with "renormalized" parameters

$$\lambda\to\bar{\lambda}=\tfrac{1}{2}(1+c)\lambda,\quad c\to\bar{c}=\frac{3-c}{1+c}=1+2\frac{1-c}{1+c}, \tag{22}$$

satisfying $-1<\bar{c}<1$ for $\infty>c>1$. Hence, applying (20) to the transformed action, one has for $c>1$

$$E_{0,k}\lambda^k \xrightarrow{k\to\infty} -\frac{1}{\pi}\left(\frac{48\bar{c}}{\sin\pi\bar{s}}\right)^{1/2}(-3)^k\Gamma(k+\tfrac{1}{2})\bar{\lambda}^k=-\frac{1}{\pi}\left(\frac{48\bar{c}}{\sin\pi\bar{s}}\right)^{1/2}[-\tfrac{3}{2}(1+c)]^k\Gamma(k+\tfrac{1}{2})\lambda^k, \tag{23}$$

with $\bar{s}(\bar{s}+1)=2\bar{c}=2(3-c)/(1+c)$.

In view of applications to more general anisotropic anharmonic oscillators it is instructive to derive this result once more from a direct calculation. For $c>1$, the potential term $2cx^2y^2$ in (4) is dominant, and the paths of extremal action are therefore along the two diagonals in the xy-plane. Along these diagonal rays, say

$$\frac{1}{\sqrt{2}}u_c(\tau)\equiv x_c(\tau)=y_c(\tau)\geqslant 0, \tag{24}$$

the action simplifies again to that of a one-dimensional anharmonic oscillator,

$$\mathscr{A}_c\equiv\mathscr{A}[u_c/\sqrt{2},u_c/\sqrt{2}]=\int d\tau\,[\tfrac{1}{2}\dot{u}_c^2+\tfrac{1}{2}u_c^2+\tfrac{1}{2}\lambda(1+c)u_c^4]=[\tfrac{3}{2}(1+c)|\lambda|]^{-1}, \tag{25}$$

which is extremized by the "critical bubble" solution

$$u_c(\tau)=\sqrt{\frac{1}{(1+c)|\lambda|}}\,\frac{1}{\mathrm{ch}(\tau-\tau_0)}. \tag{26}$$

The matrix M governing the coupling between δx and δy fluctuations becomes now (see (11))

$$M=\frac{u_c^2}{2}\begin{pmatrix}6+2c & 4c\\ 4c & 6+2c\end{pmatrix}, \tag{27}$$

with eigenvalues $M_1=6(1+c)$ and $M_2=2(3-c)$. It can be diagonalized by a 45° rotation to new coordinates

$$\xi=\frac{1}{\sqrt{2}}(\delta x+\delta y),\quad \eta=\frac{1}{\sqrt{2}}(\delta x-\delta y), \tag{28}$$

which measure fluctuations parallel and orthogonal to the diagonal. Inserting $u_c(\tau)$ from (26), we now obtain, instead of (12),

$$\delta\mathscr{A}=\frac{1}{2}\int d\tau\left[\xi\left(-\frac{d^2}{d\tau^2}+1-\frac{6}{\mathrm{ch}^2\tau}\right)\xi+\eta\left(-\frac{d^2}{d\tau^2}+1-\frac{2(3-c)/(1+c)}{\mathrm{ch}^2\tau}\right)\eta+\dots\right]. \tag{29}$$

The *longitudinal* fluctuations ξ give precisely the same determinant as in (13). For the second operator governing the *transversal* fluctuations η we observe that

$$2\frac{3-c}{1+c}=2+4\frac{1-c}{1+c}<2\quad\text{for } c>1\,.$$

By the argument given above, this implies again a strictly positive spectrum. Hence, application of the Fredholm formula (17) with

$$\bar{s}(\bar{s}+1)\equiv 2\frac{3-c}{1+c}=2\bar{c}$$

leads to the same result as in (18), with s replaced by $\bar{s}$. Consequently, the final results reads for $c>1$,

$$\operatorname{Im} E_0 \xrightarrow{\lambda\to 0^-} \left(\frac{48\bar{c}}{\sin\pi\bar{s}}\right)^{1/2}\sqrt{\frac{1}{\frac{3}{2}(1+c)|\lambda|}}\exp\left(-\frac{1}{\frac{3}{2}(1+c)|\lambda|)}\right), \tag{30}$$

and by means of the dispersion relation (6) we recover precisely (23).

2.3. Case $c=1$

The rotationally symmetric case $c=1$ is an isolated discontinuity. Although formulae (20) and (23) approach each other in the limit $c\to 1$, they are both divergent (see fig. 1). The reason for this discontinuity is the higher symmetry for $c=1$. As mentioned above, this leads to a rotationally zero-mode $\propto u_c(\tau)$ being responsible for an additional action factor $\sqrt{\mathscr{A}_c/2\pi}\propto 1/\sqrt{|\lambda|}$ in eq. (19). This in turn modifies the leading large-order behaviour of $E_{0,k}$ in eq. (20) from $\Gamma(k+\frac{1}{2})$ to $\Gamma(k+1)$, so that for $c=1$

$$E_{0,k} \xrightarrow{k\to\infty} -\frac{1}{\pi}6(-3)^k\Gamma(k+1)\,, \tag{31}$$

with the constant prefactor taken from ref. [2].

3. Discussion

Let us finally mention that it is straightforward to extend the present considerations to the slightly more general anisotropic potentials of the type $\lambda(ax^4+2cx^2y^2+by^4)$ considered also by BBW. Without loss of generality we may assume $a>b\geqslant 0$. Then, for $-\sqrt{ab}\leqslant c<a$, we can use formula (19) with $\lambda\to a\lambda$ (or, equivalently, eq. (20) with $(-3)^k\to(-3a)^k$) and $c\to c/a$. In addition we have to multiply the r.h.s. of eqs. (19) and (20) by a factor $\frac{1}{2}$, because for $a>b$ the paths along the y-direction no longer contribute to the leading behaviour (the discrete Z_4 symmetry is reduced to Z_2). The case $c>a$ can be discussed in a similar way.

In summary, we have seen once again that the path-integral approach is the ideal tool for deriving large-order formulae, in particular for anisotropic problems. The calculations remain as straightforward as in the one-dimensional case. They are conceptually more transparent and technically much simpler than the original derivation by BBW based on multidimensional WKB techniques.

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