

## Fat and thin Fisher zeroes\*

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We show that it is possible to determine the locus of Fisher zeroes in the thermodynamic limit for the Ising model on planar (“fat”)  $\phi^4$  random graphs and their dual quadrangulations by matching up the real part of the high- and low-temperature branches of the expression for the free energy. Similar methods work for the mean-field model on generic, “thin” graphs. Series expansions are very easy to obtain for such random graph Ising models.

### 1. INTRODUCTION

One of the more remarkable results to emerge from the study of various statistical mechanical models coupled to two-dimensional quantum gravity is a solution of the Ising model in field [1]. In discrete form the coupling to gravity takes the form of the spin models living on an annealed ensemble of triangulations or quadrangulations, or their dual planar graphs. The partition function for the Ising model on a single graph  $G^n$  with  $n$  vertices

$$Z_{\text{single}}(G^n, \beta, h) = \sum_{\{\sigma\}} e^{\beta \sum_{(i,j)} \sigma_i \sigma_j + h \sum_i \sigma_i} \quad (1)$$

is promoted to a partition function which incorporates a sum over some class of graphs  $\{G^n\}$  by the coupling to gravity,

$$Z_n(\beta, h) = \sum_{\{G^n\}} Z_{\text{single}}(G^n, \beta, h). \quad (2)$$

The solution to the Ising model in [1] proceeded by first forming the grand canonical partition function

$$\mathcal{Z} = \sum_{n=1}^{\infty} \left( \frac{-4gc}{(1-c^2)^2} \right)^n Z_n(\beta, h) \quad (3)$$

and then noting that this could be expressed as the free energy

$$\mathcal{Z} = -\log \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp \left( -\text{Tr} \left[ \frac{1}{2} (\phi_1^2 + \phi_2^2) \right. \right.$$

$$\left. \left. - c\phi_1\phi_2 + \frac{g}{4} (e^h \phi_1^4 + e^{-h} \phi_2^4) \right] \right) \quad (4)$$

of a matrix model, where we have written the potential that generates  $\phi^4$  graphs. In the above  $\phi_{1,2}$  are  $N \times N$  Hermitian matrices,  $c = \exp(-2\beta)$  and the  $N \rightarrow \infty$  limit is performed in order to pick out planar graphs. The graphs of interest are generated as the Feynman diagrams of the “action” in equ. (4), which is constructed so as to weight each edge with the correct Boltzmann weights for nearest-neighbour interaction Ising spins. Since the edges carry matrix indices the graphs in question are “fat” or ribbon graphs.

The integral of equ. (4) can be evaluated using the results of [2] to give (when  $h = 0$ )

$$\mathcal{Z} = \frac{1}{2} \log \left( \frac{z}{g} \right) - \frac{1}{g} \int_0^z \frac{dt}{t} g(t) + \frac{1}{2g^2} \int_0^z \frac{dt}{t} g(t)^2, \quad (5)$$

where  $g$  is defined by

$$g(z) = 3c^2 z^3 + z \left[ \frac{1}{(1-3z)^2} - c^2 \right]. \quad (6)$$

### 2. ZEROES

The idea that the zeroes of the partition function could determine the phase structure of a spin model first appeared in Lee and Yang’s work [3]. They considered how the non-analyticity characteristic of a phase transition appeared from the partition function on finite lattices or graphs, which was a polynomial

$$Z = \sum D_{mn} c^m y^n \quad (7)$$

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for a lattice with  $m$  edges and  $n$  vertices, again with  $c = \exp(-2\beta)$ ,  $y = \exp(-2h)$ . They (and Fisher [4]) showed that the behaviour of the zeroes of this polynomial in the complex  $y$  or  $c$  plane, in particular the limiting locus as  $m, n \rightarrow \infty$ , determined the phase structure. For temperature driven transitions, in zero external field for simplicity, the thermodynamic limit of the free energy on some class of lattices or graphs  $\{G^n\}$  becomes

$$F(G^\infty, \beta) \sim - \int_L dc \rho(c) \ln(c - c(L)), \quad (8)$$

where  $L$  is some set of curves, or possibly regions, in the complex  $c$  plane on which the zeroes have support and  $\rho(c)$  is the density of the zeroes there.

The general question of how to extract the locus of zeroes analytically has been considered by various authors, notably Shrock and collaborators [5] for Ising and Potts models. It was first observed in [6] that such loci could be thought of as Stokes lines separating different regions of asymptotic behaviour of the partition function in the complex temperature or field planes. More recently, the case of models with first-order transitions has been investigated by Biskup *et al.* [7] who showed that the partition function of a statistical mechanical model defined in a periodic volume  $V$  which depends on some complex parameter such as  $c$  or  $y$  can be written in terms of complex functions  $F_l$  describing  $k$  different phases, where the various  $F_l$  are the metastable free energies per unit volume of the phases,  $\Re F_l = F$  characterises the free energy when phase  $l$  is stable. The zeroes of the partition function are then determined from the solutions of the equations

$$\Re F_l = \Re F_m < \Re F_k, \quad \forall k \neq l, m, \\ \beta V (\Im F_l - \Im F_m) = \pi \pmod{2\pi}. \quad (9)$$

The equations (9) are thus in perfect agreement with the idea that the loci of zeroes should be Stokes lines, since the zeroes of  $Z$  lie on the complex phase coexistence curves  $\Re F_l = \Re F_m$  in the complex parameter plane.

The specific Biskup *et al.* results apply to models with first-order transitions, but we are interested here in an Ising model with a third-order transition, so it might initially seem that these

results were inapplicable. We are saved by the fact that when considered in the complex temperature plane the transition is continuous only at the physical point itself (and possibly some other finite set of points).

### 3. FAT (AND THIN) ZEROES

The determination of the locus of Fisher zeroes for the Ising model on random graphs in the thermodynamic limit using the ideas of the previous section turns out to be rather straightforward, as we now describe. Since we wish to match  $\Re F$  between the various solution branches to obtain the loci of Fisher zeroes and  $F \sim \log(g(c))$  for the Ising model on planar graphs, the equation which determines the loci of zeroes in the thermodynamic limit is

$$\log |g_L(c)| = \log |g_H(c)|, \quad (10)$$

where the low-temperature solution  $g_L(c)$  and the various high-temperature solutions  $g_H(c)$  are given by solving  $g'(z) = 0$  in equ. (6) for  $z$ .

The resulting curve is shown in the  $c$  plane in Fig. 1 where it can be seen that in addition to the physical phase transition at  $c = 1/4$ , an unphysical transition with the same KPZ [8] exponents appears at  $c = -1/4$ . The interior of the curve is the ferromagnetic region and the exterior the paramagnetic and unphysical phases, separated by cuts on the imaginary axis which we have not plotted.

The points plotted in Fig. 1 are generated from a series expansion of  $Z$  in equ. (5), which is arrived at by reverting the expression for  $g(z)$  and substituting the resulting  $z(g)$  into equ. (5). Earlier work reported in [9] obtained similar results at lower orders. The form of the expression for  $Z$  means that the contributions from each of the terms in equ. (5) are proportional [10], so the full series for  $Z$  can be generated from  $\frac{1}{2} \log(z/g)$ .

The loci of Fisher zeroes are highly non-universal, and we also show the zeroes on “thin”, generic random  $\phi^3$  graphs for comparison in Fig. 2. These models can be thought of as the  $N \rightarrow 1$  limit of the matrix models, rather than the  $N \rightarrow \infty$  planar limit. Similar methods to those discussed above also serve in this case where

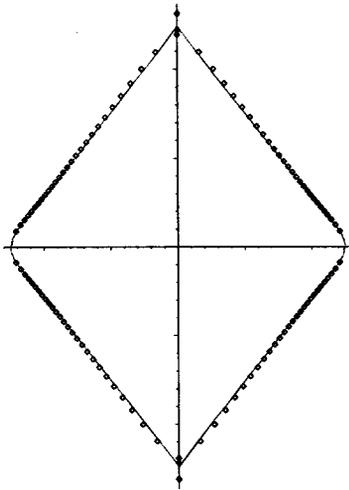


Figure 1. The Fisher zeroes on fat  $\phi^4$  graphs in the complex  $c$  plane.

one has mean-field behaviour [11]. For the Ising model on thin graphs  $F \sim \log \tilde{S}$ , where  $\tilde{S}$  is the saddle point action for either the low- or high-temperature phase. The equivalent of equ. (10) is then

$$|2(1-c)^3| = |(1+c)^2(1-2c)|. \quad (11)$$

Potts zeroes and chromatic zeroes are also accessible on the thin graphs.

In summary, we have seen that an analytic determination of Fisher zeroes for the Ising model on both fat and thin random graphs is possible, and that series expansions are easily obtainable. The general form of the solution also holds on (planar) random quadrangulations and  $\phi^3$  graphs, and in non-zero field, so all of these can also be investigated.

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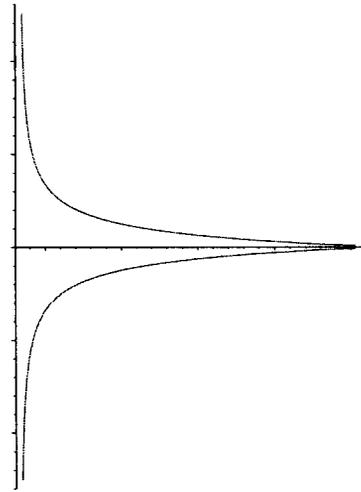


Figure 2. The Fisher zeroes on thin  $\phi^3$  graphs in the complex  $c$  plane.

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