Balls-in-boxes condensation on networks

L. Bogacz[a)]
Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Kraków, Poland and Institut für Theoretische Physik and Centre for Theoretical Sciences (NTZ), Universität Leipzig, Postfach 100 920, D-04009 Leipzig, Germany

Z. Burda[b)]
Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Kraków, Poland and Mark Kac Complex Systems Research Centre, Jagellonian University, Kraków, Poland

W. Janke[c)]
Institut für Theoretische Physik and Centre for Theoretical Sciences (NTZ), Universität Leipzig, Postfach 100 920, D-04009 Leipzig, Germany

B. Waclaw[d)]
Institut für Theoretische Physik and Centre for Theoretical Sciences (NTZ), Universität Leipzig, Postfach 100 920, D-04009 Leipzig, Germany and Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Kraków, Poland

(Received 3 January 2007; accepted 23 April 2007; published online 28 June 2007)

We discuss two different regimes of condensate formation in zero-range processes on networks: on a q-regular network, where the condensate is formed as a result of a spontaneous symmetry breaking, and on an irregular network, where the symmetry of the partition function is explicitly broken. In the latter case we consider a minimal irregularity of the q-regular network introduced by a single Q node with degree $Q>q$. The statics and dynamics of the condensation depend on the parameter $\alpha=\ln Q/q$, which controls the exponential falloff of the distribution of particles on regular nodes and the typical time scale for melting of the condensate on the Q node, which increases exponentially with the system size N. This behavior is different than that on a q-regular network, where $\alpha=0$ and where the condensation results from the spontaneous symmetry breaking of the partition function, which is invariant under a permutation of particle occupation numbers on the q nodes of the network. In this case the typical time scale for condensate melting is known to increase typically as a power of the system size. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2740571]

The formulation of the principles of nonequilibrium statistical mechanics is a challenge for theoretical physics. Nonequilibrium effects play an important role in many phenomena, but we do not have a consistent theory that would describe them. An exception may be systems close to equilibrium for which we can gain some insight into their dynamics using ideas of the linear response and of the fluctuation-dissipation theorem. The far-from-equilibrium dynamics is uncharted territory. It is therefore useful to look for solvable models belonging to the latter class, which could teach us about what happens in this case.

In this paper, we shall discuss zero-range processes on networks which belong to this class. On the one hand, they reveal an interesting, nontrivial, and very rich behavior including a condensation, far-from-equilibrium dynamics, and nonlinear effects as, for example, the formation or melting of the condensate. On the other hand, due to a relation to the balls-in-boxes model,[1,2] which is exactly solvable, also these nontrivial effects are analytically treatable.

The zero-range process is a stochastic process, which describes a gas of identical particles hopping between neighboring sites of a lattice or network on which the particles reside (for reviews, see Refs. 3–5). The transition rate for particles to hop from one site to an adjacent site depends only on the state of the node from which the particle hops and is independent either of the destination node or any other node. Since the hopping rate requires only ultralocal information, the corresponding process is called zero-range process.

When the density of particles exceeds a certain critical value, the system undergoes a condensation,[1] where a single node of the network which attracts a large number of particles. The condensation takes place in real space and not in momentum space as for the Bose-Einstein condensation. This type of condensation is called balls-in-boxes condensation; in short, "B-in-B" or "backgammon condensation," because it was first discovered in the balls-in-boxes (backgammon) model.[1,2]

The B-in-B condensation is observed in many systems. For example, in statistical models of random trees (also called branched polymers) one observes a phase transition between a phase of generic elongated trees to a phase where a typical tree looks rather like a bush with a singular node,

[a)]Electronic mail: bogacz@th.if.uj.edu.pl
[b)]Electronic mail: burda@th.if.uj.edu.pl
[c)]Electronic mail: wolfhard.janke@itp.uni-leipzig.de
[d)]Electronic mail: bwaclaw@th.if.uj.edu.pl

which has a finite fraction of all branches.[1,6] The statistics of the degree distribution in the tree model can be mapped onto the B-in-B model. The emergence of a singular node on a tree corresponds to the emergence of a condensate of many balls in one box in the B-in-B model. A similar geometrical phase transition is observed in models of quantum manifolds discussed in the context of quantum gravity.[7,8] At the phase transition the quantum manifolds collapse to a very singular geometry whose volume is almost fully concentrated in the closest neighborhood of a single point of the manifold. More precisely, the ratio v/V of the volume v of the neighborhood within a radius of order of an ultraviolet cutoff around this point to the total volume V of the whole manifold is finite in the limit $V\to\infty$ even if the radius is kept constant. Again, this phenomenon can be viewed as a particular realization of the B-in-B condensation.[7]

The B-in-B condensation explains many other phenomena such as, for instance, wealth condensation,[9] emergence of singular nodes in complex networks,[10,11] emergence of the Hagedorn fireball in hadron physics,[12,13] and some transitions observed in shaken granular gases.[14,15] Actually, the mathematics of the condensation is also almost identical to that of the Berlin-Kac phase transition in the spherical model.[16] In other words, the B-in-B mechanism is quite generic and common.

The state of the zero-range process is characterized by the distribution of the numbers of particles at all N nodes of the network: $\{m_i\}=\{m_1,\ldots,m_N\}$, where m_i denotes the occupation number of the ith node. The total number of particles $M=m_1+\cdots+m_N$ is constant during the process. Particles hop from nonempty sites with a rate $u(m_i)$ depending only on the site occupation number m_i. The outgoing current of particles from a site i is distributed equally among all q_i links emerging from the node, so the effective hopping rate per link is $u(m_i)/q_i$, where q_i is called the node degree. The function $u(m)$ is identical for all nodes, but the factor $1/q_i$ is not, since it explicitly depends on the node degree. In this paper, we assume that the network topology is fixed and so is the degree sequence $\{q_i\}$. The most fundamental question is whether the process has a steady state and, if so, whether it is unique. The answer to this question is affirmative if the network is connected. In this case the process has a unique steady state, which depends only on the node degrees $\{q_i\}$ and the numbers of nodes N and particles M. This steady state corresponds to the only equilibrium state, which is sooner or later reached by the process. In equilibrium, the partition function can be calculated analytically,[4]

$$Z=\sum_{m_1=0}^{M}\cdots\sum_{m_N=0}^{M}\delta_{m_1+\cdots+m_N,M}\prod_{i=1}^{N}p(m_i)q_i^{m_i}, \tag{1}$$

where

$$p(m)=\prod_{k=1}^{m}\frac{1}{u(k)},\quad p(0)=1. \tag{2}$$

The weight $p(m)$ is identical for every node, but the total node weights $p_i(m)\equiv p(m)q_i^m$ have an additional contribution q_i^m explicitly depending on the node degree. Because of the presence of the (Kronecker) delta function under the sum in Eq. (1), the partition function does not entirely factorize into a product of independent weights for individual nodes. The constraint on the total number of particles plays an important role as we shall see below, because the occupation numbers of individual nodes are not independent of each other.

The statics of the zero-range process is equivalent to a B-in-B model with the partition function (1) describing a system of M identical balls distributed in N boxes, each having a weight function $p_i(m)$. The probability that in equilibrium the system is in a state $\{m_i\}$ reads

$$P(m_1,\ldots,m_N)=\frac{1}{Z}\prod_{i=1}^{N}p(m_i)q_i^{m_i}=\frac{1}{Z}\prod_{i=1}^{N}p_i(m_i), \tag{3}$$

where $m_1+\cdots+m_N=M$ as before. It is interesting to notice that this probability is invariant with respect to the following change of the weights:

$$p(m)\to Ce^{\mu m}p(m). \tag{4}$$

Indeed under the change (4) the partition function (1) changes as $Z\to C^Ne^{\mu M}Z$. The multiplicative factor that appears in front of the partition function cancels out in the probability (3) and thus the statistical averages do not depend on the parameters C and μ. This invariance is an important property of the model. The parameter C has the meaning of a normalization and can be used, for example, to normalize the weights to a probability. One should notice that the hopping rate $u(m)=p(m-1)/p(m)$ is not affected by C. The parameter μ, or more specifically e^{μ}, rescales the hopping rate $u(m)\to e^{-\mu}u(m)$ or equivalently stretches the time scale $t\to te^{\mu}$.

The probability distribution (3) encodes the full information about the static properties of the system. For example, one can calculate statistical averages of any observable X,

$$\langle X\rangle=\sum_{\{m_i\}}P(m_1,\ldots,m_N)X(m_1,\ldots,m_N) \tag{5}$$

or correlations $\langle XY\rangle-\langle X\rangle\langle Y\rangle$, etc. A particularly interesting observable is the number of incidents that the ith node is occupied by m particles,

$$\pi_i(m)=\langle\delta_{m_i,m}\rangle=p_i(m)\frac{Z_i(m)}{Z}, \tag{6}$$

where, using the B-in-B analogy, $p_i(m)$ is the weight of the ith box, Z is the partition function for the total system of M balls in N boxes, and $Z_i(m)$ is the partition function for $M-m$ balls in the $N-1$ remaining boxes. In a similar manner one can count multinode distributions and average them over all configurations. For example, averaging the two-node incident function $\delta_{m_i,m}\delta_{m_j,n}$ over all configurations gives the probability $\pi_{ij}(m,n)$ that in equilibrium there are simultaneously m particles at node i and n particles at node j.

A zero-range process is said to be in the condensed phase if a finite fraction of all particles tends to occupy one node or, if one phrases it in terms of the underlying B-in-B model, if a finite fraction of all balls is concentrated in one box. This effect depends on the density of particles per node (or balls per box) $\rho=M/N$. If one keeps ρ constant and takes the limit $N\to\infty$, one sees that above a certain critical density ρ_c one box contains on the average a large number of balls $(\rho-\rho_c)N$, which grows with N, while any other box has only ρ_c balls.

The most thoroughly studied and probably most surprising example of the B-in-B condensation takes place in a system of identical boxes, that is, for which all the weight functions in Eq. (3) are identical: $p_1(m)=\cdots=p_N(m)$. The corresponding zero-range process is realized on a q-regular graph, which has identical degrees of all nodes. In this case the partition function (1) is symmetric with respect to the permutation of the box-occupation numbers $\{m_i\}$. The condensation appears there as a result of a spontaneous symmetry breaking, which selects one out of identical boxes for the location of the condensate. The criterion for the appearance of the condensation is controlled by the asymptotic value of the hopping rate $u(m)$ for $m\to\infty$,

$$\frac{p(m-1)}{p(m)} = u(m) \to u_\infty. \tag{7}$$

For $u_\infty=\infty$ [or if $p(m)=0$ for all m larger than a certain m_0] the corresponding critical density is infinite and there is no condensation in the model. The system is always in a fluid phase. This can be intuitively understood because if the hopping rate $u(m)\to\infty$, it amounts to an effective repulsion between particles which, in effect, avoids to occupy the same site. On the contrary, for $u_\infty=0$ there is no price to pay for a numerous occupation of the same site, so in effect the particles tend to condense. The critical density is zero in this case and therefore the system can be then only in the condensed phase. One can say that due to an effective attraction, particles tend to keep as close as possible to each other. The most interesting case is when u_∞ is finite: $0<u_\infty<\infty$. Actually it is sufficient to consider only the case $u_\infty=1$ since within this model all other values of u_∞ are equivalent to one as follows from the invariance with respect to the transformation (4). In the remaining part of the paper we shall therefore stick to the choice $u_\infty=1$. The quantitative behavior depends on the exact form of $u(m)$ but the critical properties, such as critical exponents, depend only on how $u(m)$ approaches unity when m goes to infinity. When it behaves as $u(m)=1+\beta/m+\cdots$ for large m, then $p(m)$ behaves asymptotically as $p(m)\sim m^{-\beta}$. In particular, one can choose $p(m)$ to be

$$p(m) = \frac{(\beta-1)\Gamma(\beta)m!}{\Gamma(\beta+m+1)} \sim \frac{(\beta-1)\Gamma(\beta)}{m^\beta}. \tag{8}$$

When the density $\rho=M/N$ exceeds the critical value ρ_c, a condensate with $N\Delta\rho$ particles is formed,[4,17] where $\Delta\rho=\rho-\rho_c$. The critical density is given by the formula[1,2]

$$\rho_c = \frac{\sum_{m=0}^{\infty} mp(m)}{\sum_{m=0}^{\infty} p(m)} \tag{9}$$

and can be concisely expressed in terms of the generating function $K(\mu)=\Sigma_m p(m)e^{-\mu m}$ as $\rho_c=-K'(0)/K(0)$. In particular, for $p(m)$ as in Eq. (8), the critical density is $\rho_c=1/(\beta-2)$. An interesting choice of the weights $p(m)$ is when one demands that every box has at least one particle and $p(m)=m^{-\beta}$. In this case the generating function has the following integral representation:

$$K(\mu) = \frac{1}{\Gamma(\beta)}\int_0^\infty \mathrm{d}t \frac{t^{\beta-1}}{e^{\mu+t}-1}, \tag{10}$$

which uncovers mathematical similarities between the B-in-B and the Bose-Einstein condensation. The statics of the model can be solved analytically,[1,2] with critical properties depending on β.

The dynamics of the condensation has also been studied analytically.[4,17,18] Two questions can be posed: What is the typical time scale for building the condensate from a homogeneous distribution of particles and what is its average life time? Here we shall focus on the latter. Once the condensate is formed, it moves across the system. It spends a long time at one particular node but sometimes melts and is rebuilt at another node. A typical time scale for melting the condensate has been derived using mean-field arguments.[17] In the mean-field approach one monitors only a single node of the network and derives effective equations balancing the inflow and outflow of particles for this node. One does not care about what happens in the remaining part of the system, which, in this approximation, is treated as a homogeneous reservoir of particles, where fluctuations are much faster than the dynamics of the condensate. The monitored node is characterized by the distribution of the number of particles $\pi_i(m)$, which, in an adiabatic approximation, is assumed to be that of the steady state. For a homogeneous system, $\pi_i(m)$ is identical for all nodes, so it is equal to the average overall nodes: $\pi_i(m)=\overline{\pi_i}(m)\equiv\pi(m)$, and it does not matter which node is monitored. The full information is encoded in $\pi(m)$. The occupation number of the monitored node may change in one step by one unit or stay constant. This sequence of changes can be viewed as a random walk in the effective one-dimensional potential $V(m)=-\ln\pi(m)$. The waiting time for a condensate to melt can be thus viewed as the time needed for a particle to randomly walk from m_*, which corresponds to the value of the condensate $m_*\equiv N\Delta\rho$, to some $m_0\ll m_*$. This time is related to going through the maximum of the potential, whose position corresponds to the position of the dip of the function $\ln\pi(m)$. This position is very close to $m_*/2$, because the excess of particles is shared mainly by two nodes.[17] In the condensed phase the shape of the function $\pi(m)$ can be approximated in the range of m from zero to the dip location (see Fig. 1) by $\pi(m)\approx p(m)$, where we assume that $\Sigma_m p(m)=1$ as follows from Eq. (8). So the value of the function $\pi(m)$ at the dip is roughly equal to $p(m_*/2)$, which gives the corresponding maximum of the effective potential $V_*=-\ln p(m_*/2)$. Thus, using the Arrhenius law, one can expect that the time needed for a random walk to go over this maximum is of order $\tau\sim e^{V_*}=1/p(m_*/2)$. For $p(m)$ asymptotically behaving as $\sim m^{-\beta}$ this yields $\tau\sim N^\beta$, i.e., a power law in the system size N, with a coefficient proportional to $(\Delta\rho)^\beta$. This crude argument gives already a good estimate. It can be polished if one implements all details of the zero-range dynamics into the mean-field analysis and works out the consequences of the detailed balance condition for the transition rates for a particle to hop into or from the monitored node or to stay at it.[3,17] One obtains an expression for a monitored node i for the waiting time $\tau^i_{m\to k}$, which tells us how long it takes to fall from m to $k<m$ particles, where only the first-passage time is taken into account,

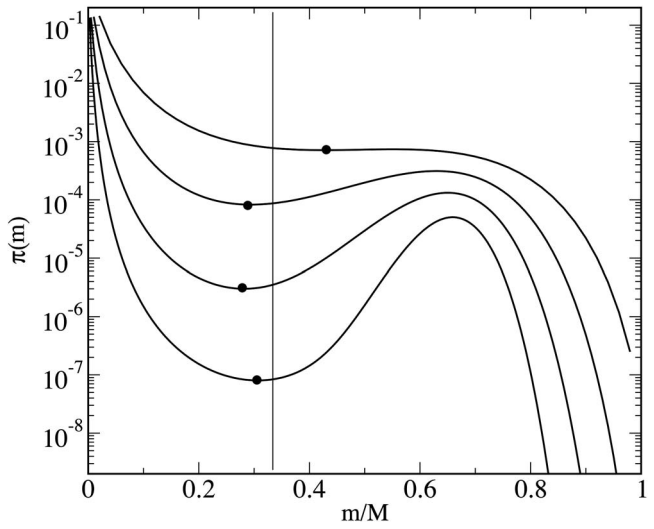

FIG. 1. Plots of the distribution $\pi(m)$ in the condensed phase, for $\beta=5$ and $\rho=1$. The critical density is $\rho_c=1/3$. From top to bottom: $N=M=50,100,200,400$. The position of the dip is marked by a filled circle on each curve. The vertical line denotes its asymptotic position as $N\to\infty$: $m_*/2M=1/3$. A similar picture can be found in Ref. 17 for $\beta=4$.

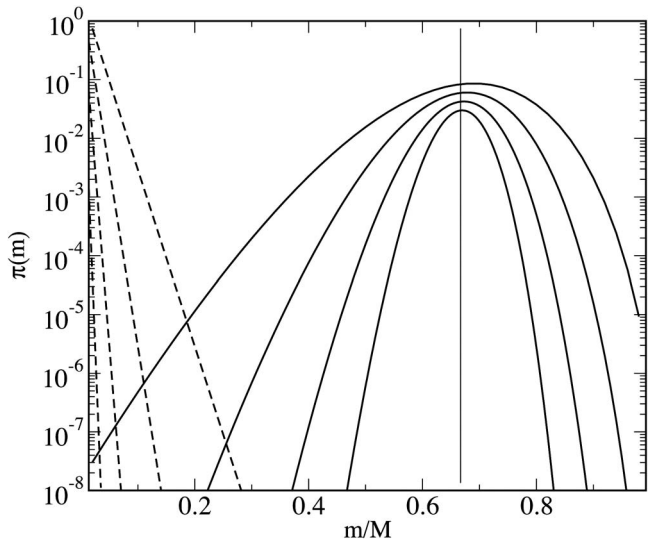

FIG. 2. Plots of $\pi_Q(m)$ (solid lines) and $\pi_q(m)$ (dashed lines) for the almost q-regular graph with one node $Q>q$. Here $q=4$, $Q=16$, $\rho=1$, and the critical density $\rho_c=1/3$. The curves from top to bottom show $N=M=50,100,200,400$. The vertical line marks the asymptotic position of $m_*/M=2/3$.

$$\tau^{i}_{m\to k}=\sum_{r=k+1}^{m}\frac{1}{u(r)\pi_i(r)}\sum_{l=r}^{M}\pi_i(l). \tag{11}$$

For a regular graph the distribution $\pi_i(m)$ is identical for each node, so the index i can be skipped. The typical melting time, defined to be $\tau=\tau_{m_*\to 0}$, grows as before as $\tau\sim N^{\beta}$, but with a slightly modified coefficient, which is now proportional to $(\Delta\rho)^{\beta+1}$. It is only a small correction to the previously derived result since the dependence on N is the same.

In contrast to homogeneous systems, much less is known about the zero-range process on inhomogeneous networks, where the symmetry of the partition function (1) resulting from identical node degrees is explicitly broken. Some attempts have been made for scale-free networks[19] to explore the phase diagram and the dynamics of condensate formation, but not the lifetime of the condensate.

We will now take advantage of Eq. (11) to estimate the typical time scale for the dynamics of condensation for inhomogeneous systems, assuming that (11) applies within the scope of the mean-field approximation also when $\pi_i(m)$ varies from node to node. We will study the effect of inhomogeneity by introducing to a q-regular graph a single node with degree $Q>q$. This type of irregularity with a single node being different from the others is well suited to study the effect of symmetry breaking in zero-range processes, which generates a condensate on a single node. The parameter $\alpha=\ln Q/q$ plays the role of an external field, which breaks this symmetry. Let us first determine the static properties of such a system. Because the node weight for the Q node, $p_Q(m)=Q^m p(m)$, differs from the weights for regular nodes, $p_q(m)=q^m p(m)$, by an exponential factor $(Q/q)^m$, it is clear that this node has a tendency to attract more particles than the others. We hence expect that $\pi_Q(m)$ increases fast with m, while $\pi_q(m)$ decreases. The exact form of these distributions depends on the particular form of the weight function $p(m)$ in Eq. (1), but this does not significantly change the generic behavior. The exponential effect coming from the factor $Q/q>1$ is dominant. In order to simplify the calculations we therefore assume that the transition rates do not depend on m: $u(m)=1$ or equivalently $p(m)=1$. One can show[20] that the distribution

$$\pi_Q(m)\propto\left(\frac{Q}{q}\right)^m\binom{M+N-m-2}{M-m}, \tag{12}$$

where we have skipped an overall normalization, develops a maximum for m close to the upper limit M. Let us shortly sketch the derivation of Eq. (12). The details can be found in Ref. 20. Applying Eq. (6) to the Q node we obtain $\pi_Q(m)=Q^m Z_q(m)/Z$, where Z is the partition function of the system and $Z_q(m)$ is the partition function for a q-regular graph with $N-1$ nodes and $M-m$ particles. Since we want to focus on the dependence of π_Q on m we can neglect the inessential normalization $1/Z$. The partition function $Z_q(m)$ for a q-regular graph can be easily calculated from Eq. (1). Assuming $q_1=\cdots=q_{N-1}=q$ and $p(m)=1$ we have

$$\begin{aligned}Z_q(m)&=q^{M-m}\sum_{m_1=0}^{M-m}\cdots\sum_{m_{N-1}=0}^{M-m}\delta_{m_1+\cdots+m_{N-1},M-m}\\&=q^{M-m}\binom{M+N-m-2}{M-m}.\end{aligned} \tag{13}$$

The sum over $m_1,\ldots,m_{N-1}$ gives the number of all possible partitions of $M-m$ balls in $N-1$ boxes. It yields the binomial factor in Eq. (12). The prefactor q^{M-m} combined with Q^M from the expression $\pi_Q(m)=Q^m Z_q(m)/Z$ gives an exponential m-dependent factor $(Q/q)^m$, which appears in Eq. (12). All other factors are independent of m.

As an example, we show in Fig. 2 the effective distribution, $\pi_Q(m)$, of particles at the Q node for different system sizes and for constant density of particles. Above the critical density $\rho_c=1/(Q/q-1)$, the distribution has a maximum for a number of particles, m_*, which linearly grows with the total system size, $m_*\sim N$. Clearly this means that particles con-

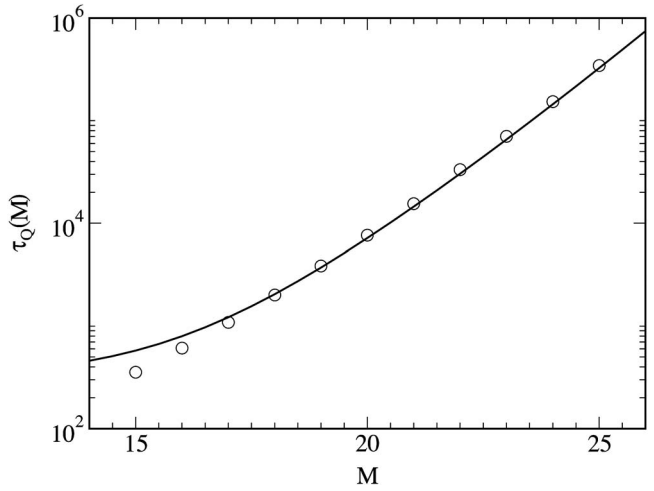

FIG. 3. The typical time scale τ_Q related to the average lifetime of the condensate, for $N=20$, $q=4$, $Q=16$, and different M. The circles denote experimental points.

dense at the Q node. In the same figure we also show the distribution of particles $\pi_q(m)$ on a regular node. It falls off exponentially,

$$\pi_q(m) \propto \left(\frac{q}{Q}\right)^m. \tag{14}$$

In comparison to Fig. 1, the minimum of the average distribution $\pi(m)$ is located roughly at the intersection point of the solid and dashed curves. The depth of this intersection grows much faster with the system size than the corresponding depth of the valley in Fig. 1, so we expect that the first-passage time τ also grows much faster with N. Indeed, we shall show that it now grows exponentially.

Having the distributions of balls we can calculate the typical time scales $\tau_Q=\tau^Q_{m_*\to 0}$ for the condensate to disappear for the first time from the Q node or $\tau_q=\tau^q_{m_*\to 0}$ for the same quantity at a regular q node, applying Eq. (11). Let us concentrate on τ_Q. First, we observe that because $\pi_Q(m)$ decreases for $m>m_*$, the condensate having much more than m_* balls decays fast until it reaches the equilibrium state with $m\approx m_*$. Therefore, the transition time $\tau^Q_{M\to 0}$ is close to τ_Q. We will use the former to approximate τ_Q since it is easier to calculate. Inserting the distribution of balls (12) into the formula (11) we obtain

$$\tau_Q \approx \sum_{r=1}^{M}\sum_{l=r}^{M} e^{\alpha(l-r)}\binom{M+N-l-2}{M-l} \Bigg/ \binom{M+N-r-2}{M-r}. \tag{15}$$

This equation can be approximately evaluated by changing variables and extending the range of summation to infinity. Finally, we arrive at a relatively closed formula for τ_Q. It is, however, quite complicated and we will not display it here. An interested reader is referred to Ref. 20. Here we will show only a graphical representation of the result in Fig. 3, where we plot this quantity for $N=20$ and various M. One can see that it agrees well with the Monte Carlo experiment.

In the thermodynamic limit for fixed density ρ, the time τ_Q simplifies to

$$\tau_Q \sim \exp\{N[\rho\log\rho-(1+\rho)\log(1+\rho)+\rho\alpha]\}. \tag{16}$$

For $\rho\gg 1$, we get $\tau_Q\propto e^{\alpha\rho N}$, which means that here the characteristic "melting time" grows exponentially with the system size, while τ_q is found to grow only linearly. Actually the time needed for the condensate to evaporate from the Q node can be estimated also using the Arrhenius law, as before, if one applies it to the effective potential $V(m)=-\ln\pi_Q(m)$. As follows from Eq. (12), this function has its maximum at $m=0$ and the value of this maximum, when one normalizes (12), can be estimated from the Arrhenius law: $\tau_*\approx e^{V(0)}=1/\pi_Q(0)\approx e^{\alpha\Delta\rho N}$. So, in contrast to the homogeneous case the time grows now exponentially with the system size.

Let us summarize differences between the condensation observed in zero-range processes on a q-regular network and an irregular network. In the first case the partition function is symmetric with respect to permutations of the occupation numbers $\{m_1,m_2,\ldots,m_N\}$, so that any permutation of particles is as probable as any other. This symmetry is spontaneously broken at the critical point where a single node containing the condensate becomes different from the others. The symmetry is reduced to the group of permutations of the remaining $N-1$ nodes. On an irregular network, on the other hand, the symmetry is explicitly broken. Because of the nature of the symmetry breaking, which produces a condensate on a single node, one can expect that already a model with an irregularity on a single node is sufficient to capture the main characteristics of this transition. We have studied such a minimal irregularity coming from a single node with degree $Q>q$. The parameter $\alpha=\ln Q/q$ plays the role of an external field. The situation is very similar as for standard phase transitions. For $\alpha>0$, one observes a characteristic exponential suppression $\pi_q(m)\sim e^{-\alpha m}$ of the particle distributions on q-regular nodes, which can be compared to the exponential falloff of the two-point correlation function in standard field theoretical models, while for the absence of the external field one observes long-range fluctuations: $\pi_q(m)\sim p(m)$, which, for the interesting class of weights $p(m)$, is of the power-law type: $\pi_q(m)\sim m^{-\beta}$. This change of behavior has also immediate implications for the dynamics: a typical time scale for the condensate melting for $\alpha>0$ grows exponentially with the system size and for $\alpha=0$ subexponentially, typically as a power.

A next step is to consider zero-range processes on complex networks with an arbitrary degree distribution. One can, in particular, address the question of self-averaging; that is, whether a zero-range process on a single typical network chosen at random from the given ensemble of networks with the given degree distribution behaves in the same way as the corresponding process averaged over many networks from this ensemble. This would be the first step towards the investigations of dynamics on networks coupled to network topology.

ACKNOWLEDGMENTS

We would like to thank P. Bialas for discussion. This work was supported in part by the EC-RTN Network "ENRAGE" under Grant No. MRTN-CT-2004-005616, an Institute Partnership grant of the Alexander von Humboldt Foundation, an EC Marie Curie Host Development Fellowship under Grant No. IHP-HPMD-CT-2001-00108 (L.B. and

W.J.), an EC ToK grant (No. MTKD-CT-2004-517186 "COCOS," and a Polish Ministry of Science and Information Society Technologies grant (No. 1P03B-04029) (2005-2008) (Z.B.). B.W. acknowledges support by a Fellowship of the German Academic Exchange Service (DAAD).

[1]P. Bialas, Z. Burda, and D. A. Johnston, Nucl. Phys. B **493**, 505 (1997).
[2]P. Bialas, Z. Burda, and D. A. Johnston, Nucl. Phys. B **542**, 413 (1999).
[3]C. Godréche, arXiv:cond-mat/0603249; arXiv:cond-mat/0604276.
[4]M. R. Evans and T. Hanney, J. Phys. A **38**, R195 (2005).
[5]M. R. Evans, S. N. Majumdar, and R. K. P. Zia, J. Stat. Phys. **123**, 357 (2006).
[6]P. Bialas and Z. Burda, Phys. Lett. B **384**, 75 (1996).
[7]P. Bialas, Z. Burda, B. Petersson, and J. Tabaczek, Nucl. Phys. B **495**, 463 (1997).
[8]B. V. de Bakker, Phys. Lett. B **389**, 238 (1996).
[9]Z. Burda, D. Johnston, J. Jurkiewicz, M. Kamiński, M. A. Nowak, G. Papp, and I. Zahed, Phys. Rev. E **65**, 026102 (2002).
[10]P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).
[11]P. Bialas, Z. Burda, and B. Waclaw, AIP Conf. Proc. **776**, 14 (2005).
[12]R. Hagedorn, Nuovo Cimento, Suppl. **3**, 147 (1965); Nuovo Cimento A **56**, 1027 (1968).
[13]Ph. Blanchard, S. Fortunato, and H. Satz, Eur. Phys. J. C **34**, 361 (2004).
[14]J. Eggers, Phys. Rev. Lett. **83**, 5322 (1999).
[15]A. Lipowski and M. Droz, Phys. Rev. E **65**, 031307 (2002).
[16]T. H. Berlin and M. Kac, Phys. Rev. **86**, 821 (1952).
[17]C. Godréche and J. M. Luck, J. Phys. A **38**, 7215 (2005).
[18]S. Grosskinsky and T. Hanney, Phys. Rev. E **72**, 016129 (2005).
[19]J. D. Noh, Phys. Rev. E **72**, 056123 (2005).
[20]B. Waclaw, L. Bogacz, Z. Burda, and W. Janke, arXiv:cond-mat/0703243.