Quantum Analytical Mechanics:

What is it and

what is it good for?

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	Classical analytical mechanic CAM	Quantum analytical mechanics QAM
model	differentiable paths on manifolds	continuous not differentiable paths on manifolds
kinematic and dynamic laws	$dx(t) = v(t)dt$ $m\,a(t) = F(t)$	$dx(t) = [v(x(t),t)+u(x(t),t)]dt+\sqrt{\frac{\hbar}{m}}dW_f(t)$ $dx(t) = [v(x(t),t)-u(x(t),t)]dt+\sqrt{\frac{\hbar}{m}}dW_b(t)$ $m\bar{a} = F$
Hamilton principle	$S[x] = \int_{t_i}^{t_f}\left\{\frac{1}{2}m\,v(t)^2-V(x,t)\right\}dt$	$J[x] = \int_{t_i}^{t_f}\{\frac{1}{2}m[v(x(t),t)-iu(x(t),t)]^2$ $-V(x(t),t)\}$
Hamilton - Jacobi - Bellmann equations	$\partial_t\rho(x,t)+\frac{1}{m}\nabla\cdot[\rho\nabla S(x,t)]=0$ $v(x,t)=\frac{1}{m}\nabla S(x,t)$ $\partial_t S(x,t)+\frac{1}{2}m[\nabla S(x,t)]^2+V(x,t) = 0$	$\partial_t\rho(x,t)+\frac{1}{m}\nabla\cdot[\rho\nabla S(x,t)]=0$ $v(x,t)=\frac{1}{m}\nabla S(x,t)$ $\partial_t S(x,t)+\frac{1}{2}m[\nabla S(x,t)]^2+V(x,t)-\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0$
Hamilton equations	$dx(t) = \frac{p(t)}{m}dt$ $dp(t) = -\partial_x V(x)dt$	$dx(t) = \frac{p(t)}{m}dt+\sqrt{\frac{\hbar}{m}}dW_f(t)$ $dp(t) = -\partial_x V(x)dt+\sqrt{\frac{\hbar}{m}}q(x(t))dW_b(t)$

Nelson 1966

Pavon 1995

Schrödinger / Madelung 1926

Köppe, Beyer, Grecksch, Paul 2017-2024

E. Nelson, *On the derivation of the Schrödinger equation from Newtonian mechanics,* Phys. Rev. **150**, 1079 (1966)

The double well potential (ground state)

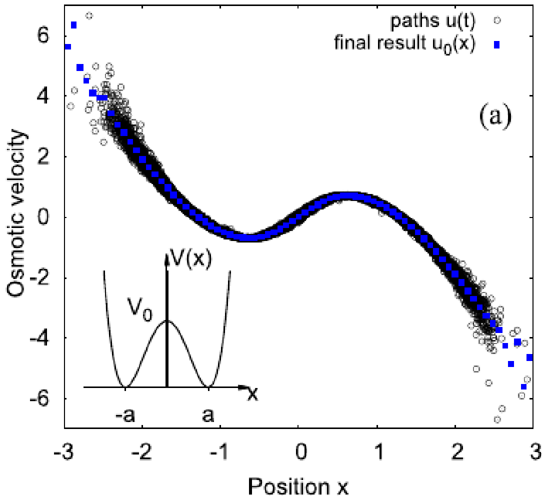
J. Köppe, M. Patzold, W. Grecksch, W. Paul, *Quantum Hamilton equations of motion for bound states of one-dimensional quantum systems*, J. Math. Phys. **59**, 062102 (2018)

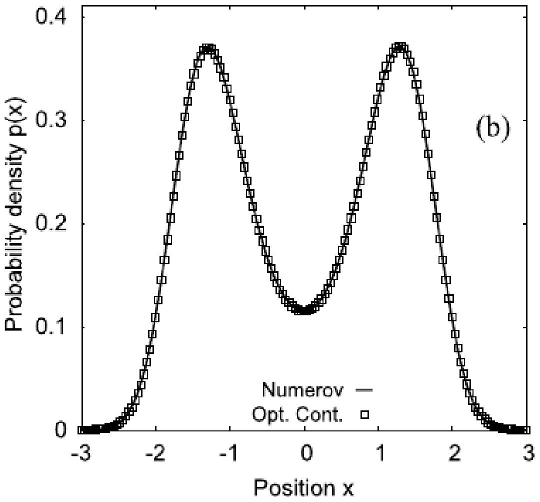
Ground state

$$dx(t) = u(x(t))\,dt + \sqrt{\frac{\hbar}{m}}\,dW_f(t)$$

$$du(x(t)) = -\frac{1}{m}\frac{dV}{dx}\,dt + \sqrt{\frac{\hbar}{m}}\,\partial_x u\,dW_b(t)$$

$$V(x) = V_0\left[\left(\frac{x}{a}\right)^2 - 1\right]^2$$





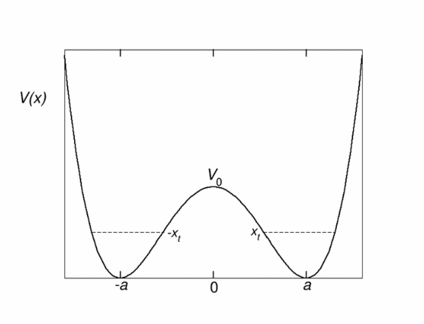
Determine u(x) iteratively and from that the probability density ρ(x)

$$\rho(x) = |\psi|^2 = \exp\left\{\frac{2m}{\hbar}\int^x u(x')\,dx'\right\}$$

Tunneling – a Kramers problem

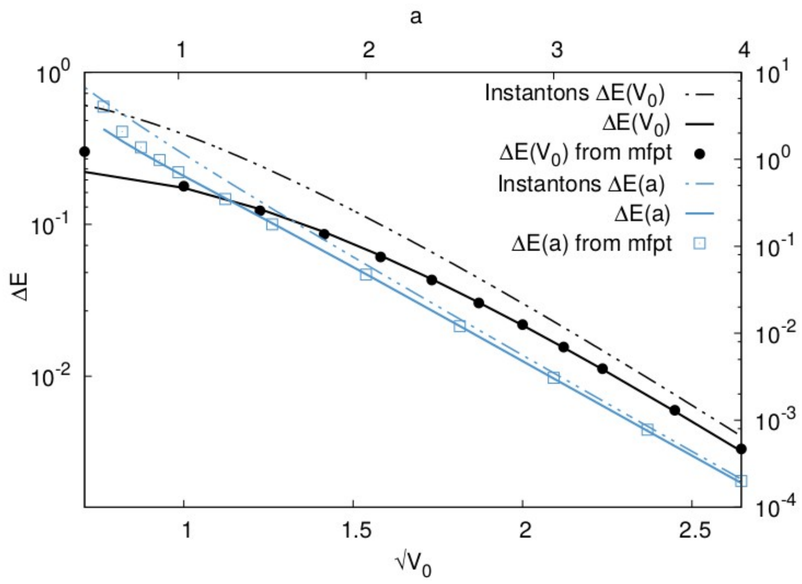
Itô SDE $$dx(t) = u(x,t)dt + \sqrt{\frac{\hbar}{m}}dW_f(t)$$

Focker-Planck equation $$\partial_t \rho(x,t) = -\partial_x\big(u(x,t)\rho(x,t)\big) + \frac{\hbar}{2m}\partial_x^2 \rho(x,t)$$



$$\tau_{MFPT} = \frac{2m}{\hbar}\int_{-\infty}^{-x_t}\tilde{\rho}_0(x)\int_x^{x_t}\frac{1}{\rho_0(x')}\int_{-\infty}^{x'}\rho_0(x'')dx''dx'dx$$

$$\Delta E = \frac{2\hbar}{\tau_{MFPT}}$$



Instanton-Gas

$$\Delta E = \frac{2^{17/4}V_0^{1/4}}{\sqrt{\pi a}} \exp\left(-\frac{2^{5/2}}{3}\sqrt{V_0}\,a\right)$$

A. I. Vaĭnshteĭn et al., *ABC of instantons*, Phys.-Usp. **25**, 195 (1982)

The Kepler problem, aka, the hydrogen atom

M. Beyer, M. Patzold, W. Grecksch, W. Paul, *Quantum Hamilton Equations for Multidimensional Systems*, J. Phys. A: Math. Theor. **52** 165301 (2019)

Metric tensor $g_{ij}(r,\vartheta,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\vartheta \end{pmatrix}$ innner product $\vec{u}\cdot\vec{u} = g_{ij}u^i u^j$

→ additional terms in the equations of motion for (r,ϑ,φ) and $(p_r, p_\vartheta, p_\varphi)$

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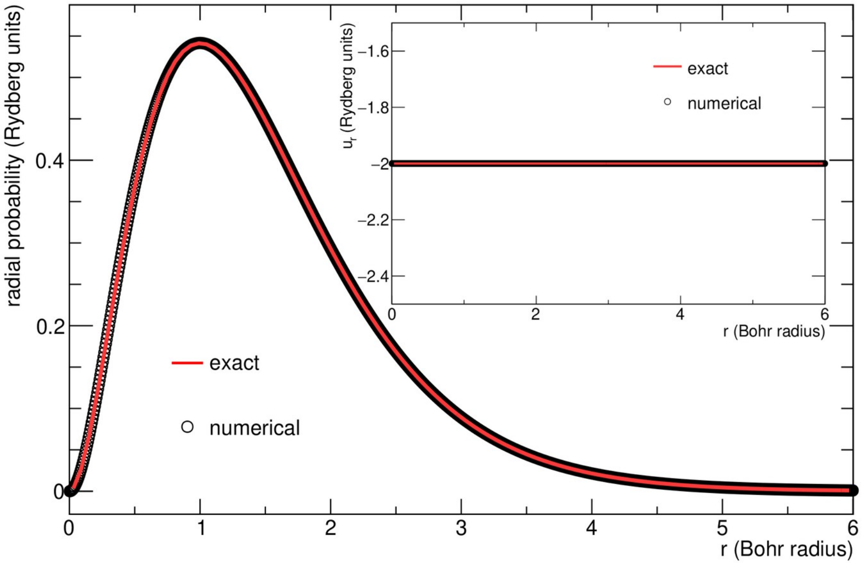
Equations of motion for the radial coordinate in s-state:

$$dr = \left(\vec{u}\cdot\hat{r} + \frac{\hbar}{\mu r}\right)dt + \sqrt{\frac{\hbar}{\mu}}\,dw_r$$

$$\mu du_r = \left[-\frac{e^2}{4\pi\varepsilon_0 r^2} + \frac{\hbar\,\vec{u}\cdot\hat{r}}{\mu r^2}\right]dt + \left[\underline{Q}\cdot d\vec{W}_b\right]_r$$

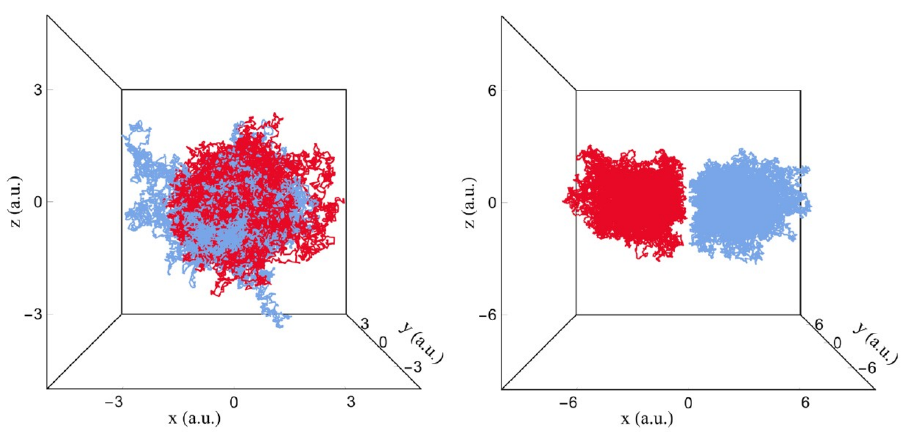
⟹ The H-atom is stable!

The Kepler problem, aka, the hydrogen atom



Osmotic velocity and probability density in the H ground state.

Using a SUSY factorization and Cartesian coordinates for the excited states.

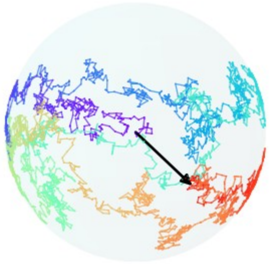


^1s ^2p_x

Spin as an orientation degree of freedom of a particle

Configuration manifold $\mathbb{R}^3 \times SO(3)$: $\vec{r} \in \mathbb{R}^3$ $\quad \vartheta \in [0, \pi], \varphi \in [-\pi, \pi], \chi \in [-\pi, \pi]$ Euler angles

ϑ and φ define the orientation of the magnetic moment in space, which defines the z-axis of the body reference frame, χ is the body rotation around that axis.



Riemannian metric of SO(3):

$$g_{ij}(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\vartheta \\ 0 & \cos\vartheta & 0 \end{pmatrix} \Longrightarrow g^{ij}(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sin^2\vartheta} & -\frac{\cos\vartheta}{\sin^2\vartheta} \\ 0 & -\frac{\cos\vartheta}{\sin^2\vartheta} & \frac{1}{\sin^2\vartheta} \end{pmatrix}$$

The SDEs for the Euler angles therefore are (including the curvature effects)

$$d\vartheta = \left[\omega_v^\vartheta \pm \omega_u^\vartheta \mp \frac{\sigma_I^2}{2}\cot\vartheta\right] dt + h_j^\vartheta dW_\pm^j$$

$$d\varphi = \left[\omega_v^\varphi \pm \omega_u^\varphi\right] dt + h_j^\varphi dW_\pm^j$$

$$d\chi = \left[\omega_v^\chi \pm \omega_u^\chi\right] dt + h_j^\chi dW_\pm^j$$

with

$$h_j^i = \left(h^\vartheta h^\varphi h^\chi\right) = \sigma_I \begin{pmatrix} \cos\varphi & -\cot\vartheta \sin\varphi & \frac{\sin\varphi}{\sin\vartheta} \\ \sin\varphi & \cot\vartheta\cos\varphi & -\frac{\cos\varphi}{\cos\vartheta} \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\sigma_I = \sqrt{\frac{\hbar}{I_m}}$$

G. E. Uhlenbeck, S. Goudsmit, Nature **117**, 264 (1926); F. Bopp, R. Haag, Z. Naturforsch. **5A**, 644, (1950)
M. Beyer, W. Paul, *Particle spin described by Quantum Hamilton Equations*, Annal. Phys. **535**, 200433 (2023)

Spin as an orientation degree of freedom of a particle

Dynamics of the spin components

$$dS_\vartheta = \left[\partial_\vartheta V_{eff}(\vartheta, S_\varphi, S_\chi) + \frac{\gamma}{\sin^2\vartheta}\left(S_\varphi - \cos\vartheta S_\chi\right)\left(B_x\cos\varphi - B_y\sin\varphi\right) + i\frac{\hbar S_\vartheta}{2 I_m \sin^2\vartheta}\right]dt + \sqrt{\frac{\hbar}{I_m}}\,\tilde{\Pi}_{\vartheta j}\, dW_-^j$$

$$dS_\varphi = \gamma\left[\vec{B}\, x\, \vec{S}\right]_z dt + \sqrt{\frac{\hbar}{I_m}}\,\tilde{\Pi}_{\varphi j}\, dW_-^j$$

$$dS_\chi = \sqrt{\frac{\hbar}{I_m}}\,\tilde{\Pi}_{\chi j}\, dW_-^j$$

With the effective rotation potential $V_{eff}(\vartheta, S_\varphi, S_\chi) = \frac{1}{2 I_m \sin^2\vartheta}\left(S_\varphi^2 + S_\chi^2 - 2\cos\vartheta S_\varphi S_\chi\right)$

⇒

- The expectation value of the canonical angular momentum for rotation around the body fixed axis S_χ is constant.
- Let the magnetic field define the z-axis: then the expectation value for the angular momentum for rotation around that axis S_φ (precession) is constant.
- The nutation motion S_ϑ is complex.

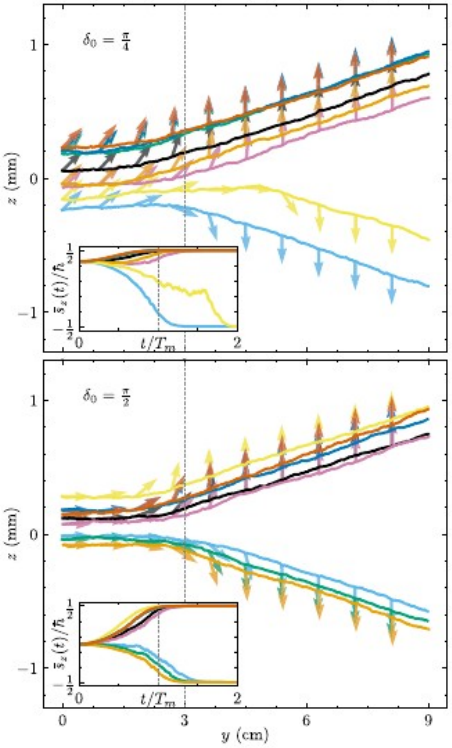
Stern-Gerlach Experiment

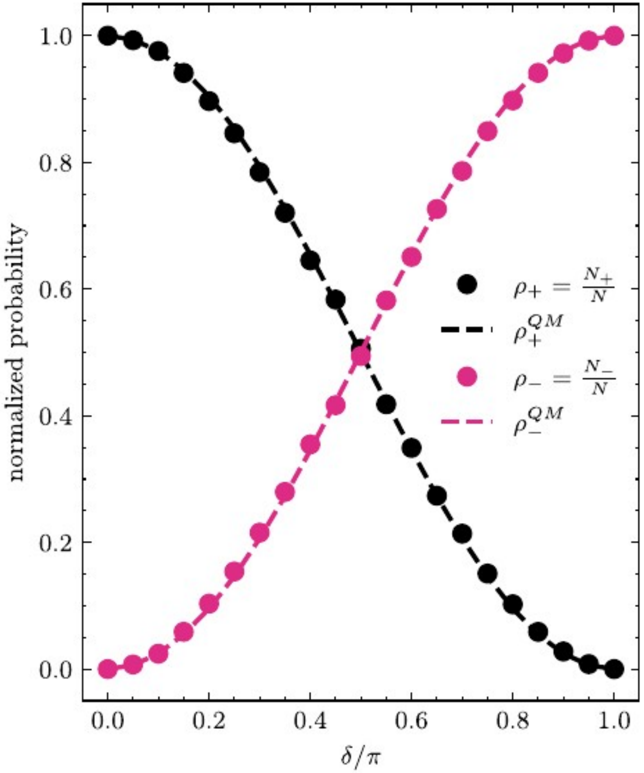
Consecutive SG experiments with angle δ between them: polarized beams

10^6 paths for each setting

$$\tau_{rot} = \frac{ma^2}{\hbar} \ll \tau_{trans} = \frac{m(\Delta x)^2}{\hbar}$$

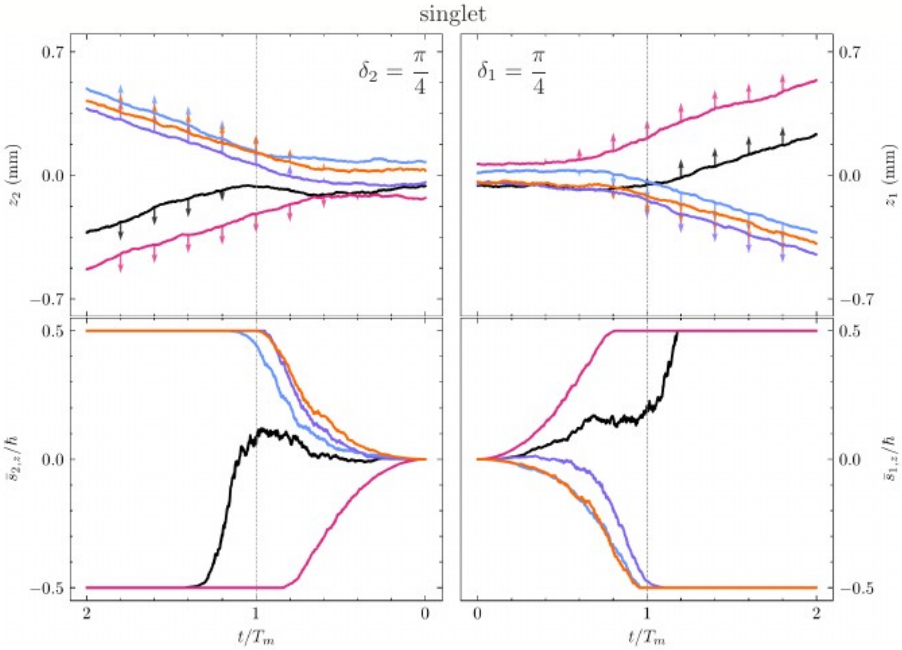
Dashed lines are predictions from the Pauli equation



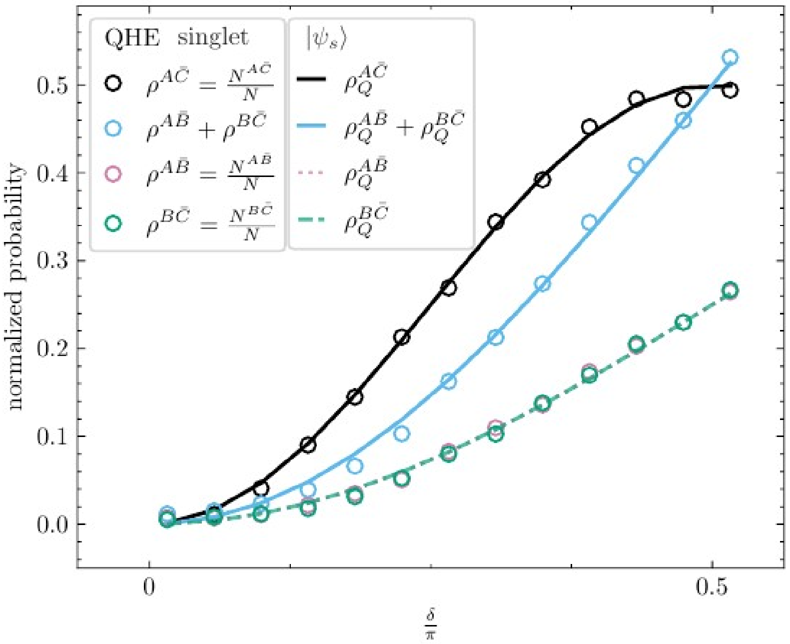


M. Beyer, W. Paul, *Stern–Gerlach, EPRB and Bell Inequalities: An Analysis Using the Quantum Hamilton Equations of Stochastic Mechanics*, Foundations of Physics **54**, 20 (2024)

Bell inequalities and EPRB thought experiment



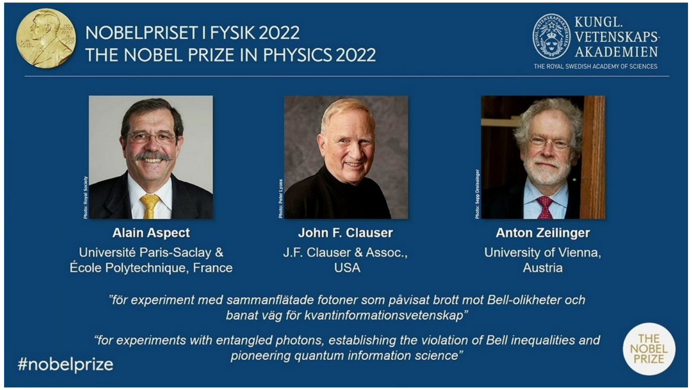
Simultaneous paths of singlet spins in two separated SG setups



Violation of Bell inequality for δ<π/2 as predicted for the singlet state

Bell inequalities and EPRB thought experiment

Nobel prize 2022



From the citation of the nobel committee

For a long time, the question was whether the correlation was because the particles in an entangled pair contained hidden variables, instructions that tell them which result they should give in an experiment.

In the 1960s, John Stewart Bell developed the mathematical inequality that is named after him. **This states that if there are hidden variables, the correlation between the results of a large number of measurements will never exceed a certain value**. However, quantum mechanics predicts that a certain type of experiment will violate Bell's inequality, thus resulting in a stronger correlation than would otherwise be possible.

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This means that quantum mechanics cannot be replaced by a theory that uses hidden variables.

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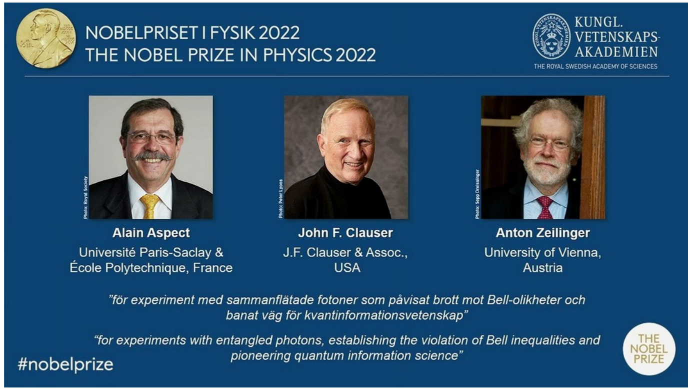

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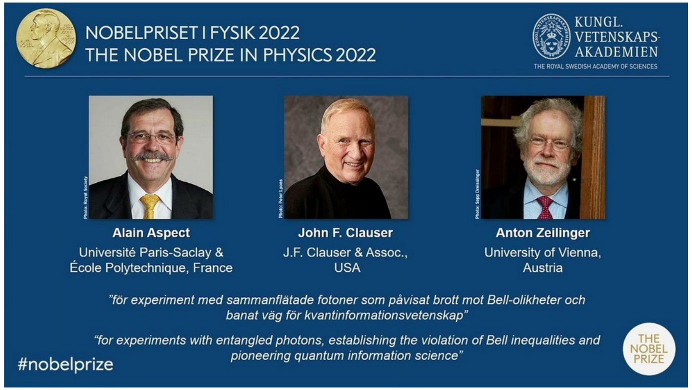

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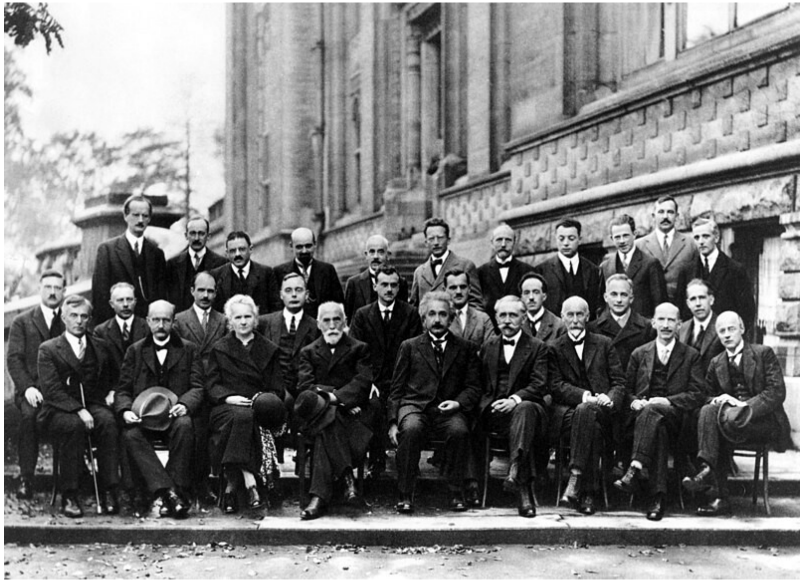
Contrary to what the Nobel committee formulates, Bell's theorem states that QM correlations can not be reproduced by a **Bell-separable (local) hidden variable theory**.

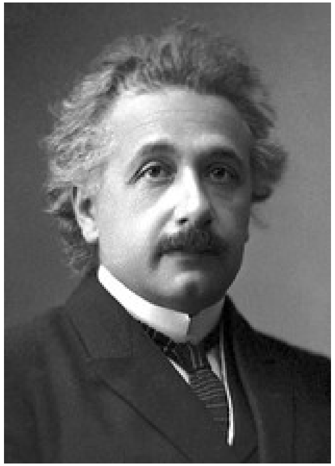
Our description contains non-separable probability distributions due to angular momentum conservation.

Solvay Conference 1927

Niels Bohr



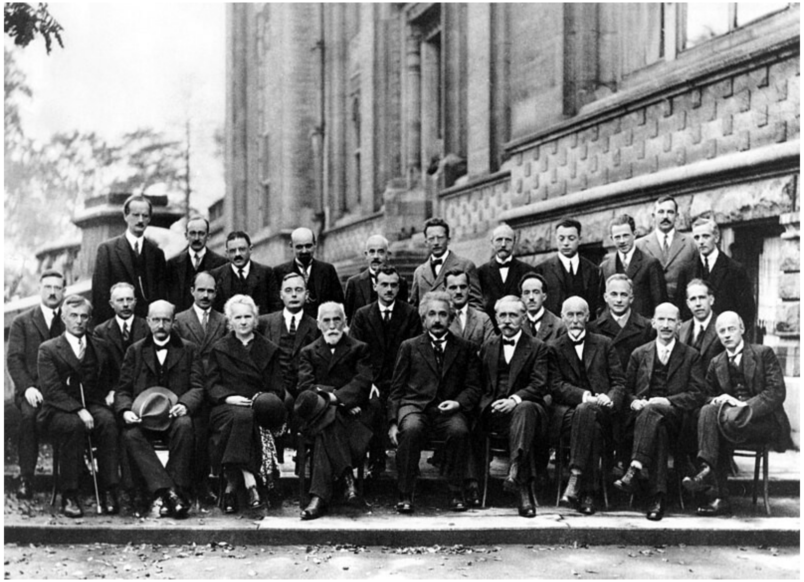
The wave function is epistemic
but is there a reality below it?



Albert Einstein

Solvay Conference 1927

Niels Bohr



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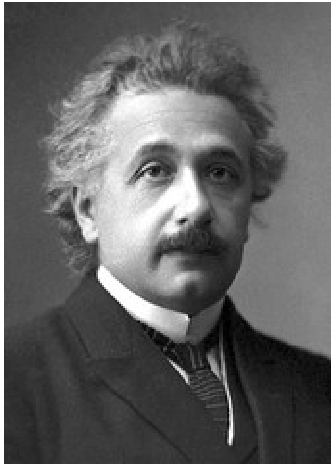

Albert Einstein

QAM: Yes, that of an open system; individual paths are accessible; duration of a process is an observable, the use of hidden variables removes the measurement problem ...

Statement by Roger Penrose

The one thing that one can say against quantum mechanics is that it makes absolutely no sense.