Open extended quantum systems

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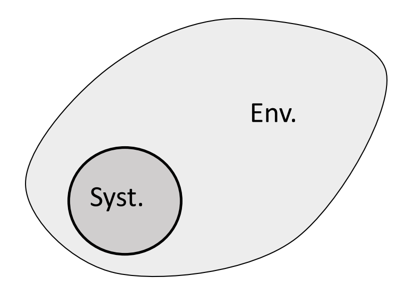
Open Quantum Dynamics

Open Quantum Dynamics: microscopic derivation



System-Environment description

$$H = H_S + H_E + V$$

with $V = V(X_S, Y_E)$ and where
$X_S = X_e \otimes \mathbb{1}_e$ and $Y_E = \mathbb{1}_s \otimes Y_e$.

Unitary dynamics of the full system: $U(t) = e^{-i(H_S + H_E + V)t}$
System's density matrix

$$\rho^S(t) = \mathrm{tr}_E\{\rho(t)\} = \mathrm{tr}_E\{U(t)\rho(0)U^\dagger(t)\} = \Lambda_t(\rho^S(0)) \ .$$

Open Quantum Dynamics: microscopic derivation

Weak coupling limit $V_I(t) = \sum_i X_I^i(t) \otimes Y_I^i(t) = \mathcal{O}(\epsilon)$ with $\rho(0) = \rho_S \otimes \omega$

Spectral decomposition[1]

$$X^i(t) = \sum_\omega e^{-i\omega t} \tilde{X}^i(\omega)\,, \qquad \tilde{X}^i(\omega) = \sum_{\epsilon_m - \epsilon_n = \hbar\omega} \langle n|X^i|m\rangle |n\rangle\langle m|$$

Under weak coupling + Markovian + secular approximations

$$\frac{d}{dt}\rho^S(t) = -i\left[H_s + \Delta_I, \rho^S(t)\right] + \sum_{i,j,\omega} \gamma^{ij}(\omega)\left(\tilde{X}^j(\omega)\rho^S(t)\tilde{X}^i(\omega)^\dagger - \frac{1}{2}\left\{\tilde{X}^i(\omega)^\dagger \tilde{X}^j(\omega), \rho^S(t)\right\}\right)\,.$$

with dissipation coefficients

$$\gamma^{ij}(\omega) \equiv \int_{-\infty}^{\infty} ds\; e^{i\omega s}\langle Y^i(s) Y^j\rangle^0\,.$$

[1] D. K. 2022 *Physique quantique des champs et des transitions de phase* (Paris: Ellipses Références sciences)

Open Quantum Dynamics: microscopic derivation

Dissipation matrix γ positive definite $\Rightarrow$ Diagonal form

$$\frac{d}{dt}\rho^S = -i\left[H_s + \Delta_I, \rho^S\right] + \sum_{\omega,k} L^k(\omega)\rho^S L^k(\omega)^\dagger - \frac{1}{2}\left\{L^k(\omega)^\dagger L^k(\omega), \rho^S\right\}$$

with $L^k(\omega) = \sum_i \sqrt{\lambda_k(\omega)} v_k^*(i,\omega)\tilde{X}^i(\omega)$ the so-called jump operators.

Open Quantum Dynamics: abstract derivation

Dynamical point of view: **Completely Positive** map[2]

A dynamical map Λ_t is a Kraus map

$$\rho(t) = \Lambda_t \rho_0 = \sum_\alpha K^\alpha \rho_0 K^{\alpha\dagger}$$

where

$$\sum_\alpha K^{\alpha\dagger} K^\alpha = \mathbb{1}_s \ .$$

This condition garanties trace (probability) conservation:

$$\mathrm{tr}\left\{\rho(t)\right\} = \mathrm{tr}\left\{\sum_\alpha K^\alpha \rho_0 K^{\alpha\dagger}\right\} = \mathrm{tr}\left\{\sum_\alpha K^{\alpha\dagger} K^\alpha \rho_0\right\} = \mathrm{tr}\left\{\rho_0\right\} \ .$$

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[2]Alicki R and Lendi K 1987 *Quantum dynamical semigroups and applications* LNP0717 (Springer); Weimer H, Kshetrimayum A and Orús R 2021 *Rev. Mod. Phys.* **93** 015008 ; Attal S 2006 *Open Quantum Systems II: The Markovian Approach* (Lecture Notes in Mathematics) ed S Attal, A Joye and C-A Pillet (Berlin: Springer)

Open Quantum Dynamics: abstract derivation

Under the conditions

i) Λ_t is a dynamical map

ii) $\Lambda_t \Lambda_s = \Lambda_{t+s}$ semi-group property (Markov)

iii) $\mathrm{tr}\{A\Lambda_t\rho\}$ is a continuous function of t for all $A \in \mathfrak{B}(\mathfrak{H}_s)$ and ρ

there exists a generator $\mathcal{L}$ such that $\Lambda_t = e^{t\mathcal{L}}$ with the Lindblad form[3]

$$\mathcal{L}\rho \equiv -i[H,\rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

[3] Lindblad G 1976 *Comm. in Math. Phys.* **48** 119; Gorini V, Kossakowski A and Sudarshan E C G 1976 *J. Math. Phys.* **17** 821

Open Quantum Dynamics: physical interpretation

The Lindblad equation $\dot{\rho} = \mathcal{L}(\rho)$ may also be written as

$$\dot{\rho} = -i\left(H_{eff}\,\rho - \rho H_{eff}^{\dagger}\right) + \sum_{j=1}^{n} L_j \rho L_j^{\dagger}\ ,$$

with a non Hermitian Hamiltonian

$$H_{eff} \equiv \tilde{H} - \frac{i}{2}\sum_{j=1}^{n} L_j^{\dagger} L_j = iL_0\ .$$

i) $-i\left(H_{eff}\,\rho - \rho H_{eff}^{\dagger}\right)$ leads to a non-unitary damping process $\Rightarrow$ superposition of damped states $e^{-iE_n t} e^{-\gamma_n t} |\,\psi_n\,\rangle$ where $H_{eff}|\,\psi_n\,\rangle = (E_n - i\gamma_n)|\,\psi_n\,\rangle$.

ii) the L_j operators, for $j = 1, \ldots, n$ are leading to a non differentiable evolution of the state vector.

Vectorisation (Finite Hilbert space)

Thanks to the isomorphism $\mathfrak{B}(\mathfrak{H}_s) \sim \mathfrak{H}_s^2$, with the mapping

$$\begin{cases} \mathfrak{B} & \longrightarrow & \mathfrak{H}^2 \\ |j\rangle\langle i| & \longmapsto & |i\rangle \otimes |j\rangle \end{cases} \qquad \Rightarrow \qquad i\frac{d}{dt}|\,\rho\,\rangle = \mathbf{H}|\,\rho\,\rangle$$

with a super-Hamiltonian

$$\mathbf{H} = \mathbb{1} \otimes H - H^T \otimes \mathbb{1} + i\sum_k W_k$$

where

$$W_k = L_k^* \otimes L_k - \frac{1}{2}\left(\mathbb{1} \otimes L_k^\dagger L_k + (L_k^\dagger L_k)^T \otimes \mathbb{1}\right)$$

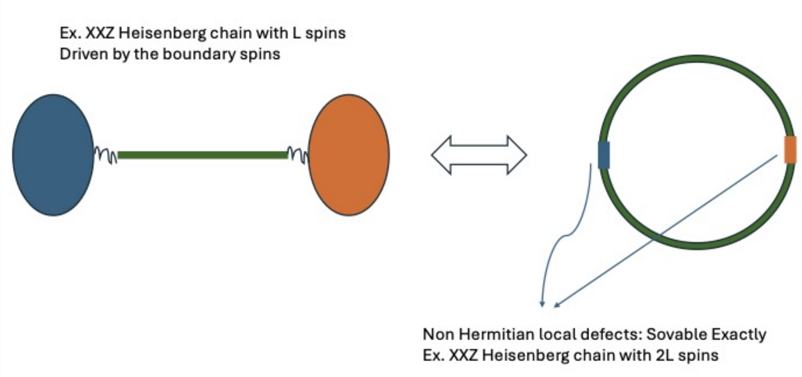
Probability conservation $\Rightarrow$ zero eigenvalue

$$\langle\, 1\,|\mathbf{H} = 0 \qquad \langle\, 1\,| = \sum_i \langle\, i, i\,|\,.$$

The steady state ρ^* is given by the corresponding right eigenvector:

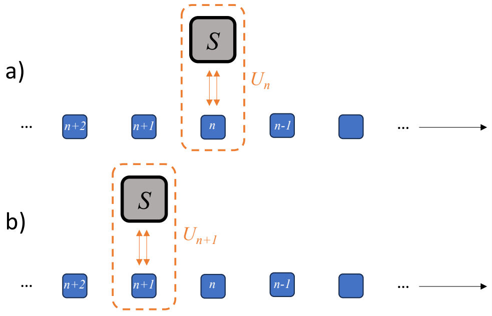
$$\mathbf{H}|\,\rho^*\,\rangle = 0$$

Vectorisation (Finite Hilbert space)[4]



[4] Prosen T 2011 *Phys. Rev. Lett.* **107** 137201; D. K. , Popkov V and Schütz G M 2013 *Phys. Rev. Lett.* **110** 047201

Unraveling by collision models



a) after the first $n-1$ ancillae have interacted with the system S, the nth ancilla arrives and interacts with the system via the unitary U_n for a finite duration of time.
b) Subsequently, the nth ancilla is replaced by the $n+1$th new ancilla, and the interaction process is repeated again and again.

Unraveling by collision models

Dynamical map $\mathcal{K}_\tau$ is such that

$$\rho \longmapsto \mathcal{K}_\tau[\rho] = \mathrm{tr}_{\mathfrak{H}} \left\{ K(\tau)\, \rho \otimes \eta\, K^\dagger(\tau) \right\}$$

with

$$K(\tau) = e^{-i\tau(H_0+V)} = \mathbb{1} - i\tau(H_0 + V) - \frac{\tau^2}{2} V^2 + o(\tau)$$

Here, we assume that while H_0 is of order one, V scales as $\mathcal{O}(1/\sqrt{\tau})$ ensuring that τV^2 remains of order one. This scaling is essential to obtain the non-trivial limit.

Some concrete situations

Free fermionic systems[5]

System made of a finite number L_S of fermions (we may take the limit L_S to infinity if necessary) described by the Fermi-Dirac algebra

$$\{c_i, c_j^\dagger\} = \delta_{ij} \ , \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \quad \forall i, j \in L_S \ .$$

Free dynamics generated by the Hamiltonian

$$H_S = \sum_{i,j=1}^{L_S} T_{ij}^S \, c_i^\dagger c_j = c^\dagger T^S c$$

Interaction with a bath (in the collision model) made of identical ancillae which are each of them made of L_A fermionic modes.

$$V = g \sum_{i=1}^{L_S} \sum_{j=1}^{L_A} \Theta_{ij} c_i^\dagger a_j + \Theta_{ij}^* a_j^\dagger c_i = g \left(c^\dagger \Theta a + a^\dagger \Theta^\dagger c \right)$$

[5] D. K. 2024 JPA 57, 315004 *Free fermions in the repeated interactions scheme*

Free fermionic systems

Dynamical equation for the two-point function

$$C_{ij}(t) = \mathrm{tr}_{\mathfrak{H}_S}\{c_j^\dagger c_i \rho(t)\}$$

Assuming that the initial state of system+ancilla is gaussian then the dynamics is govern by the Lyapunov equation[6]

$$\frac{d}{dt}C(t) = PC(t) + C(t)P^\dagger + F$$

$$P \equiv -iT^S - \frac{g^2}{2}\Theta\Theta^\dagger \ .$$

($H_S = c^\dagger T^S c$ and $V = g(c^\dagger \Theta a + h.c.)$)

$F = g^2 \Theta C^A \Theta^\dagger$ depends on the bath through C^A.

[6]Coppola M, Landi G T and D. K. 2023 Phys. Rev. A 107 052213; D. K. 2024 JPA 57 (31), 315004; Coppola M, D. K. and Landi G T 2024 Phys. Rev. B

Free fermionic systems

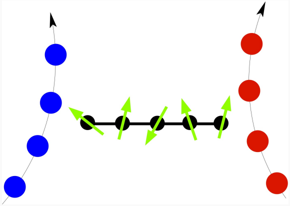
Steady state solution of the Lyapunov equation

$$PC^* + C^*P^\dagger + F = 0\ , \quad \text{with} \quad P \equiv -iT^S - \frac{g^2}{2}\Theta\Theta^\dagger$$

The solution C^* is unique if and only if P and $-P^\dagger$ have no eigenvalues in common (A sufficient condition is that $\Theta\Theta^\dagger$ is invertible)[7].

[7] Evans D E *Comm. Math. Phys.* **54** 293

Boundary driven chains



Fermions hopping on a 1d lattice (XX spin chain) with boundary sites that are coupled to a right and a left bath described by fermions

$H_S = c^\dagger T^S c$ with

$$T^S = \begin{pmatrix} 0 & J & & & \\ J & 0 & J & & \\ & \ddots & \ddots & \ddots & \\ & & J & 0 & J \\ & & & J & 0 \end{pmatrix} .$$

The boundary sites are coupled to a right and a left bath described by the 2×2 matrix

$$C^A = \begin{pmatrix} n_{left} & 0 \\ 0 & n_{right} \end{pmatrix} .$$

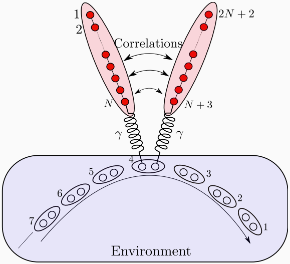
Unique steady state reached with a relaxation time $\tau_L \sim L^3$.
Ballistic motion with current

$$\frac{j}{2J} = \frac{\delta n}{\Gamma_1 + \Gamma_L}$$

and the bulk density

$$n_{bulk} = \bar{n} - (\Gamma_1 - \Gamma_L)\frac{j}{2J} .$$

– Rainbow state [8]



The correlation matrix associated to thermal Bell states is given by

$$C^A = \frac{1}{2}\begin{pmatrix} 1 & \tanh\frac{\beta}{2} \\ \tanh\frac{\beta}{2} & 1 \end{pmatrix}$$

For $\beta \to \pm\infty$ it describes the pure Bell states $\frac{1}{\sqrt{2}}(|\,a,0\,\rangle \pm |\,0,b\,\rangle)$.
Steady state solution

$$C^* = \mathbb{1}_N \otimes C^A(\beta_{eff})\ ,$$

with β_{eff} defined through

$$\tanh\frac{\beta_{eff}}{2} = \frac{2g_a g_b}{g_a^2 + g_b^2}\tanh\frac{\beta}{2}\ .$$

[8] Zippilli S, Li J and Vitali D 2015 *Phys. Rev.* A **92** 032319; Wendenbaum P, Platini T and D. K. 2015 *Phys. Rev. A* **91** 040303; D. K. 2024 *J. Phys. A* **57** 315004.

Boundary driven chains: XXZ spin chain

Anisotropic Heisenberg XXZ spin chain with boundary fields

The Hamiltonian

$$H = \sum_{k=1}^{N-1} \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \quad + (f^L \sigma_u +^R \sigma_v)$$

Non-unitary dynamics governed by the Boundary Lindblad equation

$$\frac{d}{dt}\rho = -i\,[H, \rho] + \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho)$$

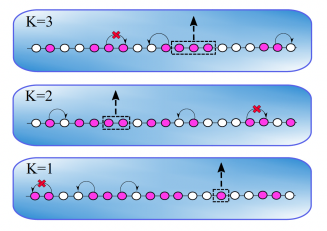
with left and right boundary dissipators forcing polarization in two directions.

Exact MPA steady state with $U_q(SU(2))$ quantum algebra[9].

[9] T. Prosen, Phys. Rev. Lett. 107, 137201 (2011).
D. K.i, V. Popkov and G. Schuetz, Phys. Rev. Lett. 110, 047201 (2013). G. Landi and D. K., Phys. Rev. B 91, 174422 (2015)

Bulk driven: Atom losses on one-dimensional atomic gas[10]

Lattice Tonks-Girardeau gas with K-bodies losses



Lattice Tonks-Girardeau gas described by the Hamiltonian

$$H_{HCB} = -\frac{1}{2}\sum_{j\in\mathbb{Z}}\left(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right)$$

where

$$[\sigma_j^+, \sigma_j^-] = \delta_{ij}\sigma_j^z\ , \quad (\sigma_j^+)^2 = (\sigma_j^-)^2 = 0\ .$$

$$\dot{\rho}(t) = -i[H_{HCB}, \rho(t)] + \Gamma\sum_{j\in\mathbb{Z}}\left(L_j\rho(t)L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho(t)\}\right)$$

with jump operators

$$L_j = \prod_{l=0}^{K-1}\sigma_{j+l}^-$$

[10] F. Riggio, L. Rosso, D. K. J. Dubail, Phys. Rev. A 109, 023311 (2024); L. Rosso, A. Biella and L. Mazza, SciPost Phys. 12, 044 (2022)

Adiabatic K-bodies losses

Small subsystems relaxes to a GGE state

- We expect that, under unitary evolution, the density matrix of a small subsystem quickly relaxes to a **GGE** state.
- Expectation values of local observables can then be evaluated with respect to the GGE density matrix

$$\rho_{GGE}(\{\langle Q_a\rangle\}) \propto e^{-\sum_a \beta_a Q_a}$$

- The exact dynamical equation $\langle \dot{Q}_a\rangle(t) = \Gamma \sum_j \Re\left\{\langle L_j^\dagger [Q_a, L_j]\rangle(t)\right\}$ is replaced by

$$\langle \dot{Q}_a\rangle(t) = \Gamma \sum_j \Re\left\{\langle L_j^\dagger [Q_a, L_j]\rangle_{\rho_{GGE}(\{\langle Q_b\rangle\})}\right\} .$$

Slow evolution of the rapidity distribution

Expectation of generic conserved quantities

$$\langle Q[f]\rangle = \sum_k f(k)\langle c^\dagger(k)c(k)\rangle$$

with rapidities (occupation number)

$$\rho(k) = \langle c^\dagger(k)c(k)\rangle\ .$$

Slow evolution of the rapidity distribution

$$\dot{\rho}(k) = -\Gamma F[\rho](k)$$

with the loss functional

$$F[\rho](k) = \sum_j \Re\langle L_j^\dagger[L_j, c^\dagger(k)c(k)]\rangle_{\rho_{GGE}(\rho(k))}$$

where the GGE density matrix is Gaussian and characterized by its two-point function $\langle c^\dagger(k)c(k')\rangle_{\rho_{GGE}(\rho(k))} = \rho(k)\delta_{kk'}$.

One-body losses

For one body losses the functional is given by

$$F[\rho](k) = L\langle\sigma_1^+\sigma_1^- c^\dagger(k)c(k)\rangle_{GGE,\rho} - L\langle\sigma_1^+ c^\dagger(k)c(k)\sigma_1^-\rangle_{GGE,\rho}$$

Conserves the parity of the state $\Rightarrow$ Wick's theorem

$$\begin{aligned}
&L\langle\sigma_1^+\sigma_1^- c^\dagger(k)c(k)\rangle_{GGE,\rho} = L\langle c_1^\dagger c_1 c^\dagger(k)c(k)\rangle_{GGE,\rho} \\
&= \sum_{qq'} e^{i(q-q')}\langle c^\dagger(q')c(q)c^\dagger(k)c(k)\rangle_{GGE,\rho} \\
&= \sum_{qq'} e^{i(q-q')}\left[\langle c^\dagger(q')c(q)\rangle\langle c^\dagger(k)c(k)\rangle + \langle c^\dagger(q')c(k)\rangle\langle c(q)c^\dagger(k)\rangle\right] \\
&= \langle N\rangle\rho(k) + \rho(k)[1-\rho(k)]
\end{aligned}$$

Changes the parity of the state thus we need to relate the Fourier modes of the fermions with periodic boundary conditions to the ones with antiperiodic boundary conditions. $\Rightarrow$ Wick's theorem

One-body losses

For one body losses the functional is given by

$$F[\rho](k) = \rho(k) - \rho^2(k) + \left(\fint_{-\pi}^{\pi} \frac{dp}{2\pi} \cot\left(\frac{k-p}{2}\right) \rho(p) \right)^2 + n\left(n + \fint_{-\pi}^{\pi} \frac{dp}{2\pi} \frac{\rho(k) - \rho(q)}{\sin^2\left(\frac{k-q}{2}\right)} \right)$$

In terms of the Hilbert Transform $\mathscr{H}[f](x) = \fint_{-\pi}^{\pi} \frac{dy}{2\pi} \frac{f(y)}{\tan \frac{x-y}{2}}$ one has

$$\dot{\rho}(k) = -\Gamma\Big(\rho(k) - \Big\{\rho^2(k) - n^2 - \left(\mathscr{H}[\rho](k)\right)^2 - 2n\mathscr{H}[\rho'](k)\Big\}\Big)$$

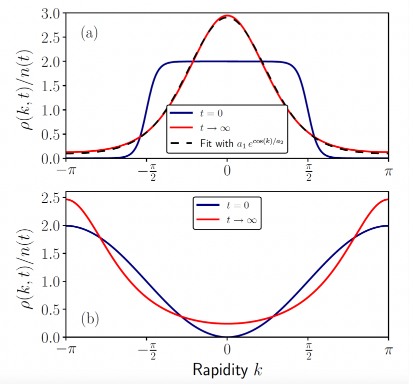
One-body losses

$\Rightarrow$ Rapidity distribution never becomes thermal ($n(t) = n(0)e^{-\Gamma t}$)

$$\rho(k,t) = n(0)e^{-\Gamma t} g(k,t)$$

with some non-trivial function $g(k,t)$

see F. Riggio et al, 2024 Phys. Rev. A 109, 023311, Bouchoule et al, 2020, SciPost Phys. 9, 044



Two-bodies losses

Two bodies losses $L_j = \sigma_j^- \sigma_{j+1}^- = c_{j+1} c_j$

Loss functional

$$F[\rho](k) = \sum_j \langle L_j^\dagger [L_j, c^\dagger(k) c(k)] \rangle_{GGE,\rho} = L \langle c_2^\dagger c_1^\dagger [c_2 c_1, c^\dagger(k) c(k)] \rangle_{GGE,\rho}$$

No parity changes $\Rightarrow$ Wick's theorem

$$\begin{aligned} \dot{\rho}(k) &= -\Gamma F[\rho](k) = -\Gamma \frac{2}{\pi} \int_{-\pi}^{\pi} dq \sin^2\left(\frac{k-q}{2}\right) \rho(q)\rho(k) \\ &= \underbrace{-2\Gamma \rho(k) n}_{\text{mean field}} + \underbrace{\frac{\Gamma}{\pi} \int_{-\pi}^{\pi} dq \cos(k-q) \rho(q) \rho(k)}_{\text{Quantum correlations}} \end{aligned}$$

see also Rossini et al, 2021, Phys. Rev. A 103, L060201,

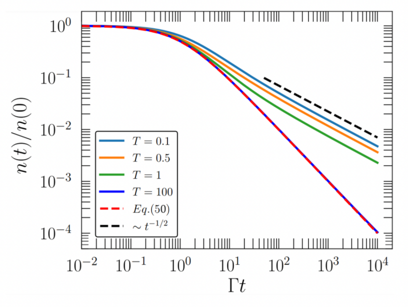
Rosso et al, 2022, SciPost Phys. 12, 044

Two-bodies losses

For reflexion symmetric initial distributions $\rho_0(k) = \rho_0(-k)$

$$\dot{\rho}(k) = \underbrace{-2\Gamma\rho(k)n}_{\text{mean field}} + \underbrace{2\Gamma\rho(k)\cos(k)\int_{-\pi}^{\pi}\frac{dq}{2\pi}\cos(q)\rho(q)}_{\text{Quantum correlations}}$$

- For $\dot{\rho}(k) = -2\Gamma\rho(k)n \Rightarrow n(t) = \frac{1}{1+2n(0)\Gamma t} \sim t^{-1}$
- Generic initial distribution $n(t) \sim t^{-1/2}$



Two-bodies losses

- For $\dot{\rho}(k) = -2\Gamma\rho(k)n$ one has $\rho(k,t)/n(t) \simeq \rho(k,0)/n(0)$
  No relaxation to a thermal state
- Generic initial distribution $\Rightarrow \rho(k,t)/n(t) \simeq e^{\beta(t)\cos k}$ distribution goes to a low-density low temperature thermal state.

Summary

Summary

- Knowledge of steady state properties in (some) one dimensional extended systems
- K-bodies losses in 1d atomic gas (integrable and non-integrable)
- Relaxation properties: Understand the spreading of entanglement in extended systems
- Time-dependent Lindblad equation $\Rightarrow$ Quantum thermodynamics

Time-dependent Lindblad dynamics

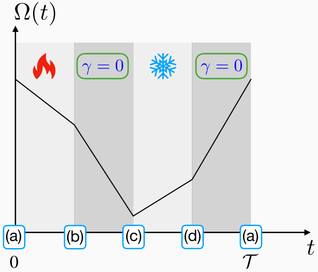
Time-dependent Lindblad equation

$$\mathcal{L}_t\rho = -i[H(t),\rho] + \sum_j \gamma_j(t)\left(L_j\rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j,\rho\}\right)$$

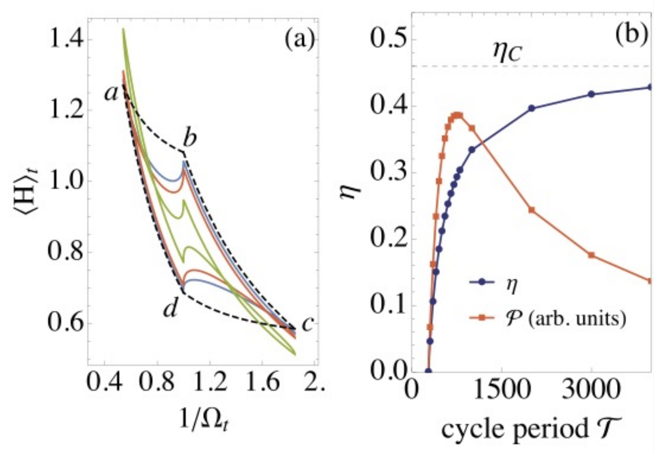
time dependence in both unitary and dissipative parts.

Carnot cycle:

- Isentropic expansion/compression: $\gamma = 0$ and variation of $H(t)$.
- hot expansion/cold compression: $\gamma \neq 0$ with thermal bath properties and variation of $H(t)$.



Quantum Carnot Cycle[11]



[11]S Scopa, GT Landi, D K *Lindblad-Floquet description of finite-time quantum heat engines* 2018 Phys. Rev. A 97 (6), 062121; S Scopa, GT Landi, A Hammoumi, D K 2019 *Exact solution of time-dependent Lindblad equations with closed algebras* Physical Review A 99 (2), 022105