

Generalised time-translation-invariance in physical ageing

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Resumé:

- I. What is the question ?
- II. Symmetries in non-equilibrium dynamics
- III. More on dynamical symmetries
- IV. Modified time-translation symmetry
- V. Description of phase-ordering kinetics
- VI. Conclusions

“Whenever you have to do with a structure-endowed entity, try to determine its group of automorphisms. You can expect to gain a deep insight.”

H. WEYL, *Symmetry* (1952)

I. What is the question ?

phase transitions \iff diverging length scale ξ

CARDY 1985

👉 look for physical systems with time-dependent length $L(t) \rightarrow \infty$ for $t \gg 1$
 \Rightarrow systems undergoing **physical ageing** are natural candidates

Defining properties of **physical ageing**:

- 1 slow dynamics (relaxations slower than exponential)
- 2 breaking of time-translation-invariance (system out of equilibrium)
- 3 dynamical scaling

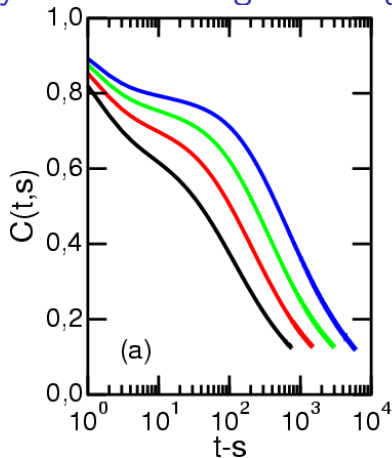
* physically obtained after **quench** from high-temperature initial state
* realise dynamic growing length $L(t) \sim t^{1/z}$, with z : dynamical exponent
* first realised in **glasses** quenched from the liquid phase to below the glass-transition temperature

STRUİK 1978

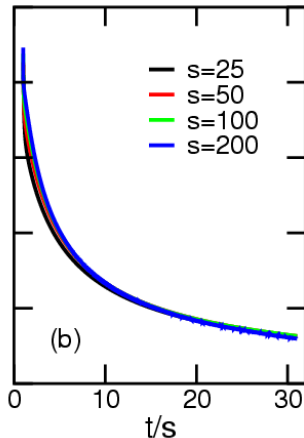
- **phase-ordering kinetics** of simple magnets quenched into temperatures $T < T_c$
- **non-equilibrium critical dynamics** after a quench onto criticality $T = T_c$

⚠ all what is said here is implicitly about the continuum limit only

Dynamical scaling in the ageing 3D Ising model, $T < T_c$



no time-translation invariance



dynamical scaling

$C(t, s)$: autocorrelation function, quenched to $T < T_c$

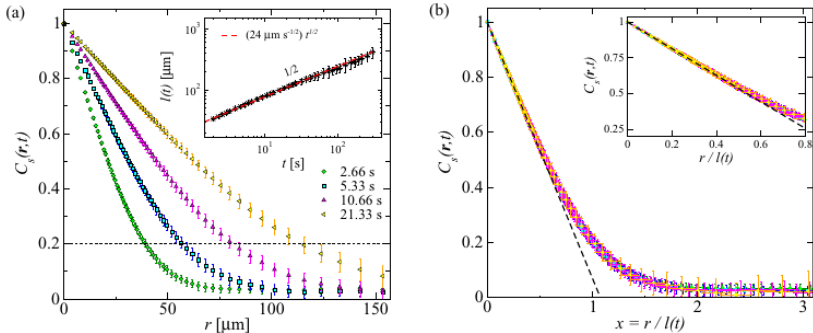
PLEIMLING \sim 2010

scaling regime: $t, s \gg \tau_{\text{micro}}$ and $t - s \gg \tau_{\text{micro}}$

? derive scaling function in a model-independent way ?

Phase-ordering kinetics has been measured recently in liquid crystals

ALMEIDA, TAKEUCHI, Phys. Rev. **E104**, 054103 (2021)



- find single-time correlator $C(t, r)$ for several times t after quench
⇒ time-translation-invariance is broken
- measure growth law $L(t)$, find $L(t) \sim t^{1/2}$
- do observe **data collapse** as expected from dynamical scaling
- for small arguments, scaling function linear ⇒ **sharp interfaces**

$$\Rightarrow \boxed{z = 2}$$

👉 results as expected for 2D Ising model

II. Symmetries in non-equilibrium dynamics

? what can be said on non-equilibrium dynamics through its symmetries ?

ageing: systems obey naturally **dynamical scaling**

? what does this imply for single-time and two-time correlators ?

$$C(t, s; \mathbf{r}) := \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{0}) \rangle - \underbrace{\langle \phi(t, \mathbf{r}) \rangle}_{=0} \underbrace{\langle \phi(s, \mathbf{0}) \rangle}_{=0}$$

with $\phi = \phi(t, \mathbf{r})$ coarse-grained order-parameter (continuum field)

and t : observation time, s : waiting time

initial condition: un-magnetised state, hence $\langle \phi(0, \mathbf{r}) \rangle = 0 \Rightarrow \langle \phi(t, \mathbf{r}) \rangle = 0 \quad \forall t > 0$

also admit spatial translation-invariance, throughout

$$C(t, s; \mathbf{r}) = \kappa^{-\varphi} C(\kappa^z t, \kappa^z s; \kappa \mathbf{r})$$

$\kappa > 0$ rescaling factor

the dynamical exponent z distinguishes the behaviour of time and space

choose $\kappa = s^{-1/z}$

$$C(t, s; \mathbf{r}) = s^{\varphi/z} C\left(\frac{t}{s}, 1; \frac{\mathbf{r}}{s^{1/z}}\right) = s^{\varphi/z} F_C\left(\frac{t}{s}; \frac{\mathbf{r}}{s^{1/z}}\right)$$

$$C(t, s; \mathbf{r}) = s^{\varphi/z} C\left(\frac{t}{s}, 1; \frac{\mathbf{r}}{s^{1/z}}\right) = s^{\varphi/z} F_C\left(\frac{t}{s}; \frac{\mathbf{r}}{s^{1/z}}\right)$$

combine with **boundary conditions**, for two types of non-equilibrium dynamics

- **non-equilibrium critical dynamics**, quenches to $T = T_c$

$$C(t, t; \mathbf{r}) \stackrel{t \rightarrow \infty}{\sim} |\mathbf{r}|^{-(d-2+\eta)}$$

- **phase-ordering kinetics**, quenches to $0 < T < T_c$

$$C(t, t; \mathbf{0}) = \begin{cases} 1 & \text{discrete symmetry} \\ M_{\text{eq}}^2 & \text{continuous symmetry} \end{cases} = \text{cste.}$$

N.B.: leave out distinct behaviour of Ising models etc. quenched exactly to $T = 0$

VADDAKAYIL *et al.* 19; JANKE *et al.* 21, 23

\Rightarrow fixes the exponent $\varphi = -bz$:

if $T = T_c$: $C(t, s; \mathbf{r}) = s^{-(d-2+\eta)/z} F_C\left(\frac{t}{s}; \frac{\mathbf{r}}{s^{1/z}}\right) \Rightarrow b = \frac{d-2+\eta}{z}$

if $T < T_c$: $C(t, s; \mathbf{r}) = F_C\left(\frac{t}{s}; \frac{\mathbf{r}}{s^{1/2}}\right) \Rightarrow b = 0$

also use $z = 2$ for phase-ordering kinetics at $T < T_c$ BRAY, RUTENBERG 1994

dynamical scaling establishes the existence of the scaling function F_C , but does **not** state anything about its form

III. More on dynamical symmetries

? Which dynamical symmetries could enhance dynamical scaling ?

obvious candidates: spatial translations and rotations

⇒ do not seem to lead to very remarkable consequences

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(a) at equilibrium, at criticality $T = T_c$: system is conformally invariant, one spatial direction is (arbitrarily) called 'time', generator $X_{-1} = -\partial_t$

(b) equilibrium dynamics, at criticality $T = T_c$, dynamics is stationary in physical time t , generator $X_{-1} = -\partial_t$

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(b) equilibrium dynamics, at criticality $T = T_c$, dynamics is stationary in physical time t , generator $X_{-1} = -\partial_t$

(c) non-equilibrium critical dynamics at $T = T_c$, is **not** time-translation-invariant, use **modified** generator $X_{-1} = -\partial_t + \xi/t$

(d) phase-ordering kinetics at $T < T_c$, is **not** time-translation-invariant but does obey dynamical scaling, use **modified** generator $X_{-1} = -\partial_t + \xi/t$

IV. Modified time-translation symmetry

phase-ordering kinetics is out of equilibrium \Rightarrow **no** time-translation-invariance

Proposal: make a *gauge transformation*, write a modified time-translation

$$\bar{X}_{-1} = -\partial_t + \frac{\xi}{t}$$

where ξ is a dimensionless constant. Generator follows from *similarity transformation* of standard time-translations $X_{-1} = -\partial_t$

$$\bar{X}_{-1} = e^{\xi \ln t} X_{-1} e^{-\xi \ln t} = -\partial_t + \frac{\xi}{t}$$

Can also gauge-transform dilatations $X_0 = -t\partial_t - \delta$, as follows

$$\bar{X}_0 = e^{\xi \ln t} X_0 e^{-\xi \ln t} = -t\partial_t - (\delta - \xi)$$

where δ is a scaling dimension.

N.B. the effective scaling dimension of the operator ϕ on which \bar{X}_0 acts is $\delta - \xi$.

Postulate: *Out of equilibrium, the dynamical symmetry generators are the (modified) time-translations \bar{X}_{-1} and the dilatations \bar{X}_0 .*

☞ then a scaling operator ϕ is characterised by two scaling dimensions δ, ξ .

? what is the form of a co-variant two-time correlator $C = \langle \phi\phi \rangle$?

For a function $C = C(t, s)$, have two conditions

$$\bar{X}_{-1}C = \left(-\partial_t - \partial_s + \frac{\xi_1}{t} + \frac{\xi_2}{s} \right) C = 0$$

$$\bar{X}_0C = \left(-t\partial_t - s\partial_s - \delta_1 + \xi_1 - \delta_2 + \xi_2 \right) C = 0$$

Go to new variables $u = t/s, v = t - s$, hence $C(t, s) = \mathcal{C}(u, v)$ and

$$\left((u-1)\partial_u + \xi_1 \frac{1}{u} + \xi_2 \right) \mathcal{C}(u, v) = 0$$

$$\left(-v\partial_v - (\delta_1 + \delta_2 - \xi_1 - \xi_2) \right) \mathcal{C}(u, v) = 0$$

which can be integrated separately to give (up to normalisation)

$$C(t, s) = s^{-\delta_1 - \delta_2 + \xi_1 + \xi_2} \left(\frac{t}{s} \right)^{-\xi_1} \left(\frac{t}{s} - 1 \right)^{-\delta_1 - \delta_2 + 2\xi_1}$$

Since $C = C(t, s; \mathbf{r})$, must add spatial part and write generators

$$\bar{X}_{-1}C = \left(-\partial_t - \partial_s + \frac{\xi_1}{t} + \frac{\xi_2}{s} \right) C = 0$$

$$\bar{X}_0C = \left(-t\partial_t - s\partial_s - \frac{1}{z}r\partial_r - \delta_1 + \xi_1 - \delta_2 + \xi_2 \right) C = 0$$

N.B.: use spatial rotation-invariance such that $\mathbf{r} \cdot \partial_{\mathbf{r}} = r\partial_r$ with $r = |\mathbf{r}|$

which is solved by (\mathcal{F}_C is an undetermined function)

$$C(t, s; \mathbf{r}) = s^{-\delta_1 - \delta_2 + \xi_1 + \xi_2} \left(\frac{t}{s} \right)^{-\xi_1} \left(\frac{t}{s} - 1 \right)^{-\delta_1 - \delta_2 + 2\xi_1} \mathcal{F}_C \left(\frac{r}{(t-s)^{1/z}} \right)$$

• correlators $C = \langle \phi\phi \rangle$ are built from two **identical** scaling operators, hence $\delta = \delta_1 = \delta_2$ and $\xi = \xi_1 = \xi_2$. Then asymptotically, for $t \gg s$

$$C(t, s; \mathbf{r}) \simeq s^{-2(\delta - \xi)} \left(\frac{t}{s} \right)^{-2\delta + \xi} \mathcal{F}_C \left(\frac{r}{t^{1/z}} \right) \quad (C_{\text{sym}})$$

Six consequences of combined time-translation and dynamical scale-invariance

in direct space, have the two-time time-space correlator $C = \langle \phi \phi \rangle$, for $t \gg s$

$$C(t, s; \mathbf{r}) = s^{-2(\delta-\xi)} \left(\frac{t}{s}\right)^{-2\delta+\xi} \mathcal{F}_C \left(\frac{\mathbf{r}}{t^{1/z}}\right) \quad (C_{\text{sym}})$$

(A) for $\mathbf{r} = \mathbf{0}$ find the two-time autocorrelator $C(t, s) = s^{-b} f_C(t/s)$, with expected asymptotics $f_C(y) \sim y^{-\lambda_C/z}$ as $y \gg 1$

☞ can identify the ageing exponents b and λ_C/z

$$\text{if } T = T_c: \quad b = 2(\delta - \xi) = \frac{d - 2 + \eta}{z}, \quad \frac{\lambda_C}{z} = 2\delta - \xi = \xi + \frac{d - 2 + \eta}{z}$$

$$\text{if } T < T_c: \quad b = 2(\delta - \xi) = 0, \quad \frac{\lambda_C}{2} = 2\delta - \xi = \xi$$

N.B.: for $T < T_c$, the order-parameter scaling operator ϕ is dimensionless since $\delta - \xi = 0$

⇒ have understood the asymptotic power-law of scaling function $f_C(y)$

in direct space, have the two-time time-space correlator $C = \langle \phi \phi \rangle$, for $t \gg s$

$$C(t, s; \mathbf{r}) = s^{-2(\delta-\xi)} \left(\frac{t}{s}\right)^{-2\delta+\xi} \mathcal{F}_C \left(\frac{\mathbf{r}}{t^{1/z}}\right) \quad (C_{\text{sym}})$$

(B) The linear response function $R = \left. \frac{\delta \langle \phi \rangle}{\delta h} \right|_{h=0} = \langle \phi \tilde{\phi} \rangle$ can be fixed similarly. From 'special conformal' invariance, it follows $\delta = \tilde{\delta}$, but ξ and $\tilde{\xi}$ remain independent. For $t \gg s$, this gives from previous results

$$R(t, s; \mathbf{r}) = s^{-2\delta+\xi+\tilde{\xi}} \left(\frac{t}{s}\right)^{-2\delta+\xi} \mathcal{F}_R \left(\frac{\mathbf{r}}{t^{1/z}}\right) \quad (R_{\text{sym}})$$

compare with the expected asymptotics $R(t, s; \mathbf{0}) = s^{-1-a} f_R(t/s)$, where $f_R(y) \sim y^{-\lambda_R/z}$ as $y \gg 1$. **Identify**, for either $T = T_c$ or $T < T_c$

$$\text{for } T = T_c \text{ or } T < T_c: \quad 1 + a = 2\delta - \xi - \tilde{\xi}, \quad \frac{\lambda_R}{z} = 2\delta - \xi$$

\Rightarrow equality $\lambda_C = \lambda_R$ between auto-correlation and auto-response exponents

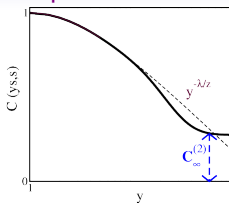
N.B.: for $T = T_c$ checked in models A & C (wrong in model B SIRE 04) CALABRESE, GAMBASSI 05
for $T < T_c$ proven in model A BRAY, HUMAYUN, NEWMAN 1991; PICONE, MH 04

(C) Finite-size effects: system in hyper-cubic box, of size N^d

New phenomenology: auto-correlator $C(y, s)$ relaxes to plateau value

$$C_{\infty}^{(2)} = \lim_{y \rightarrow \infty} C\left(y, s; \frac{1}{N}\right)$$

with new 'coordinate' $1/N$ and $y = t/s > 1$.



Now write the two co-variance conditions for $C = \langle \phi \phi \rangle$

$$\bar{X}_{-1} C = \left(-\partial_t - \partial_s + \frac{\xi}{t} + \frac{\xi}{s}\right) C = 0, \quad \bar{X}_0 C = \left(-t\partial_t - s\partial_s - \frac{1}{z} \frac{1}{N} \partial_{1/N} - 2(\delta - \xi)\right) C = 0$$

which give the simple finite-size scaling form

$$t > s$$

$$C\left(t, s; \frac{1}{N}\right) = s^{-2(\delta - \xi)} \left(\frac{t}{s}\right)^{-2\delta + \xi} \mathcal{F}_C\left(\frac{N}{t^{1/z}}\right) = s^{-b} \left(\frac{t}{s}\right)^{-\lambda/z} \mathcal{F}_C\left(\frac{N}{t^{1/z}}\right)$$

N.B.: * scaling alone would give a scaling functions of *two* variables

* generalised time-translations needed for function of single variable $u = N t^{-\frac{1}{z}}$

* scaling function \mathcal{F}_C should depend on boundary conditions

$$C\left(t, s; \frac{1}{N}\right) = s^{-b} \left(\frac{t}{s}\right)^{-\lambda/z} \mathcal{F}_C\left(\frac{N}{t^{1/z}}\right) \quad (C_{\text{fin}})$$

Expected limit behaviour:

$$\begin{cases} \mathcal{F}_C(u) = \text{cste.} & u \rightarrow \infty & \text{recover infinite-size system} \\ \mathcal{F}_C(u) \sim u^{-\lambda} & u \rightarrow 0 & \text{make the plateau } y\text{-independent} \end{cases}$$

Deduce (finite-size) **scaling of plateau height** $C_\infty^{(2)}$:

$$\begin{cases} \text{if waiting time } s \text{ fixed} & C_\infty^{(2)} \sim N^{-\lambda} \\ \text{if system size } N \text{ fixed} & C_\infty^{(2)} \sim s^{\lambda/z-b} \end{cases}$$

N.B.: there exist also heuristic deductions, which suggest the physical correctness

Tests of this prediction:

periodic boundary conditions

- ① spherical model, $2 < d < 4$ dimensions, $T < T_c$ MH 23
- ② Glauber-Ising model 1D, $T = 0$ MH 25
- ③ Glauber-Ising model 2D, $T < T_c$ see talk **D. Warkotsch**

can also deduce finite-size scaling of cluster size $L^2(t) = t^{2/z} \mathcal{F}_L(Nt^{-1/z})$

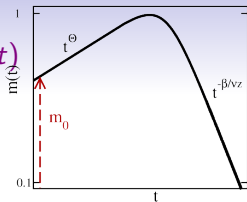
(D) Magnetised initial states at $T = T_c$

initial magnetisation $m_0 \ll 1$: two distinct regimes of $m(t)$

JANSSEN-SCHAUB-SCHMITTMANN (1989):

$$\lambda = d - \Theta z$$

field-theory & short-time OPE



Exponent Θ independent of equilibrium exponents

From non-equilibrium field-theory, derive

e.g. TÄUBER 14, MH & PLEIMLING 10

$$m(t) = \langle \phi(t, 0) \rangle = m_0 \int_{\mathbb{R}^d} d\mathbf{r} R(t, 0; \mathbf{r}) = m_0 \widehat{R}(t, 0; \mathbf{0})$$

where $\widehat{R}(t, s; \mathbf{q}) = \int_{\mathbb{R}^d} d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} R(t, s; \mathbf{r})$ is Fourier-transformed response

Here m_0 acts as a further dimensionful variable. Derive that

$$\widehat{R}(t, s; \mathbf{0}; m_0) = s^{-2\delta+d/z+\xi+\tilde{\xi}} \left(\frac{t}{s}\right)^{-2\delta+d/z+\xi} \mathcal{F}_R \left(m_0 t^{-x_0/z}\right) \quad (\mathbf{R}_{\text{mag}})$$

for initial times $s \rightarrow s_{\text{micro}}$ becomes a constant, absorbed into the scaling function

find:

$$m(t) = m_0 t^\Theta \mathcal{F}_R \left(m_0 t^{\Theta+\beta/(\nu z)}\right), \quad \Theta = \frac{d-\lambda}{z} \quad \text{recovered}$$

expected limits: (i) $\mathcal{F}_R(0) = \text{cste.}$ recover short-time scaling

(ii) $\mathcal{F}(u) \stackrel{u \gg 1}{\sim} u^{-1}$ late-time regime m_0 -independent

in direct space, have the two-time time-space correlator $C = \langle \phi \phi \rangle$, for $t \gg s$

$$C(t, s; \mathbf{r}) = s^{-2(\delta-\xi)} \left(\frac{t}{s}\right)^{-2\delta+\xi} \mathcal{F}_C \left(\frac{\mathbf{r}}{t^{1/z}}\right) \quad (C_{\text{sym}})$$

(E) consider correlator in momentum space $\widehat{C}(t, s; \mathbf{q}) = \int_{\mathbb{R}^d} d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} C(t, s; \mathbf{r})$ such that the global correlator $Q(t, s) = \frac{1}{N} \langle \sum_{n,m} \sigma_n(t) \sigma_m(s) \rangle$ becomes

$$\begin{aligned} Q(t, s) = \widehat{C}(t, s; \mathbf{0}) &= \int_{\mathbb{R}^d} d\mathbf{r} C(t, s; \mathbf{r}) = s^{-2(\delta-\xi)} \left(\frac{t}{s}\right)^{-2\delta+\xi} \int_{\mathbb{R}^d} d\mathbf{r} \mathcal{F}_C \left(\frac{|\mathbf{r}|}{t^{1/z}}\right) \\ &= s^{-b+\lambda/z} t^{-\lambda/z+d/z} \underbrace{\int_{\mathbb{R}^d} d\mathbf{u} \mathcal{F}_C(|\mathbf{u}|)}_{=\text{cte.}} \end{aligned}$$

such that for $t \gg s$ one reads off

$$\text{if } T = T_c: \quad \widehat{C}(t, s; \mathbf{0}) \sim s^{-\Theta+(2-\eta)/z} t^\Theta$$

$$\text{if } T < T_c: \quad \widehat{C}(t, s; \mathbf{0}) \sim s^{-\Theta+d/z} t^\Theta$$

TOMÉ, DE OLIVEIRA 1988

DA SILVA, TOMÉ, DE OLIVEIRA 23

and recover the JANSSEN-SCHAUB-SCHMITTMANN relation $\lambda = d - z\Theta$,

where $\Theta = \theta'$ is the initial slip exponent.

JANSSEN, SCHAUB, SCHMITTMANN 1989

N.B.: no explicit discussion of operator product expansion etc. required

N.B.': global correlator $Q(t, 0) \sim t^\Theta$ measures Θ without initial magnetisation $m_0 \ll 1$

? How to find the value of the dynamical exponent z ?

Extract from second moment of single-time correlator: computationally tedious

An alternative: study short-time dynamics of squared magnetisation $\langle m^2(t) \rangle$.
in direct space, have single-time time-space correlator $C = \langle \phi \phi \rangle$, for $t \gg s$

$$C(t, t; \mathbf{r}) = t^{-2(\delta-\xi)} \mathcal{F}_C \left(\frac{\mathbf{r}}{t^{1/z}} \right) \quad (C_{\text{sym}'})$$

find by repeating previous steps, now for $Q(t, t)$ (Fourier transform)

$$\langle m^2(t) \rangle \sim \hat{C}(t, t; \mathbf{0}) \sim t^{d/z-b} \sim \begin{cases} t^{d/z} & \text{if } T < T_c \\ t^{(2-\eta)/z} & \text{if } T = T_c \end{cases}$$

JANKE, CHRISTIANSEN, MAJUMDER 23
TOMÉ, DE OLIVEIRA 1998
DA SILVA, TOMÉ, DE OLIVEIRA 23

N.B: had been argued for heuristically in the past

\Rightarrow computationally efficient technique, competitive with other methods

Besides the input of dynamical scaling (expected to hold by definition), the **single hypothesis of modified time-transition-invariance** with a 'gauged' generator $X_{-1} = -\partial_t + \xi/t$ and a dimensionless constant ξ (albeit operator-dependent) leads to the following general results, for **both $T = T_c$ and $T < T_c$** :

- 1 power-law asymptotics of scaling functions $f_C(y) \sim y^{-\lambda_C/z}$ and $f_R(y) \sim y^{-\lambda_R/z}$ for $y \gg 1$, of both auto-correlations and auto-responses
- 2 the equality $\lambda = \lambda_C = \lambda_R$ between auto-correlation and auto-response exponents
- 3 finite-size scaling of auto-correlator plateau $C_\infty^{(2)}$ with λ
- 4 the Janssen-Schaub-Schmittmann scaling relation $\lambda = d - z\Theta$ with the initial slip exponent Θ recovered, at $T = T_c$
- 5 scaling of global correlator: Janssen-Schaub-Schmittmann relation extended to $T < T_c$, and also scaling of $\langle m^2(t) \rangle$

these results are all confirmed in many different models, both at $T = T_c$ and for $T < T_c$
there are heuristic arguments for most relations discussed here

Find form of F_C or f_C from the solutions of the kinetic equation

$$\left(\partial_t - \frac{1}{2\mathcal{M}} \Delta_r \right) \phi(t, \mathbf{r}) = \phi^3(t, \mathbf{r})$$

with **noisy initial conditions** and $\langle \phi(t, \mathbf{r}) \rangle = 0$.

N.B.: a linear term in ϕ would break dynamical scaling; thermal noise gives corrections to scaling

Must transform the Schrödinger operator $\mathcal{S} = \partial_t - \frac{1}{2\mathcal{M}} \Delta_r$ as follows

$$\overline{\mathcal{S}} = e^{\xi \ln t} \mathcal{S} e^{-\xi \ln t} = \partial_t - \frac{\xi}{t} - \frac{1}{2\mathcal{M}} \Delta_r$$

\Rightarrow find an additional $1/t$ potential, and transformed order-parameter $\overline{\phi} = t^\xi \phi$

Transform $\mathcal{S} \phi = \phi^3$: $(t^\xi \mathcal{S} t^{-\xi}) (t^\xi \phi) = t^\xi (t^{-\xi} \overline{\phi})^3$

N.B.: $\overline{\phi}$ dimensionless $\Rightarrow \left(\partial_t - \frac{\xi}{t} - \frac{1}{2\mathcal{M}} \Delta_r \right) \overline{\phi} = t^{-2\xi} \overline{\phi}^3$

\Rightarrow **non-linear term irrelevant for $2\xi > 1 \iff \lambda_C > 1$.**

• Because of YRD-inequality $\lambda_C \geq \frac{d}{2} > 1$ satisfied for $d > 2$.

For 2D models, $\lambda_C \simeq 1.25 > 1$ also satisfied (i.e. Ising & Potts models).



dynamical symmetry of phase-ordering kinetics from linear equation $\overline{\mathcal{S}} \phi = 0$.

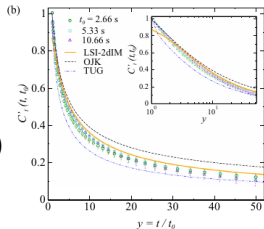
VI. Conclusions

- **dynamical scaling** is natural postulate in **phase-ordering kinetics** & $z = 2$
 - through similarity transformation: **modified time-translation-invariance**
 derive: (1) power-law asymptotics, (2) equality $\lambda_C = \lambda_R$, (3) JSS scaling relation,
 (4) global correlators purely from dynamical symmetry, for $T \leq T_c$
 - in modified representation, order parameter $\bar{\phi}$ becomes dimensionless
 and **non-linear term** in kinetic equation **irrelevant**
- \Rightarrow study dynamical symmetries of **linear** modified Schrödinger operator $\overline{\mathcal{L}}$

aim: derive form of single-time correlator etc.
 two-time autocorrelators

'best available description' of liquid crystal experiments for $C(t, s)$

ALMEIDA, TAKEUCHI 21



👉 unified set-up for description of ageing phenomenology

👉 much better justification for applicability of Schrödinger-invariance

rappel: first preliminary publication on Schrödinger-invariance in 1992 HERRMANN, JANKE, KARSCH

Perspective: use non-equilibrium field theory

generating functional $Z = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} e^{-\mathcal{J}[\phi, \tilde{\phi}]}$,

with action $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \int d\mathbf{r}' d\mathbf{r}'' \tilde{\phi}(0, \mathbf{r}') \underbrace{a(\mathbf{r}' - \mathbf{r}'')}_{\text{initial correlator}} \tilde{\phi}(0, \mathbf{r}'')$

can find responses and correlators from **dynamical Schrödinger-symmetry** of \mathcal{J}_0 & BARGMAN rule

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(α) response $\underbrace{R(t, s; \mathbf{r})}_{\text{computed in noisy model}} = \langle \phi(t, \mathbf{r}) \tilde{\phi}(s, \mathbf{0}) \rangle_0 = \underbrace{R_0(t, s; \mathbf{r})}_{\text{from symmetry of deterministic } \mathcal{J}_0}$

(β) single-time correlator $C(t; \mathbf{r}) = \langle \phi(t, \mathbf{r}) \phi(t, \mathbf{0}) \rangle = \int d\mathbf{r}' d\mathbf{r}'' \underbrace{a(\mathbf{r}' - \mathbf{r}'') \langle \phi(t, \mathbf{r}) \phi(t, \mathbf{0}) \tilde{\phi}(0, \mathbf{r}') \tilde{\phi}(0, \mathbf{r}'') \rangle_0}_{\text{four-point response}}$

four-point function to be found from all (non-local) symmetries of \mathcal{J}

• if initially $a(\mathbf{r}) = a_0 \delta(\mathbf{r})$, then

$$C(t, r) = C_0 \exp\left(-\frac{M}{4} \frac{|r|^2}{t}\right) = f_C\left(\frac{|r|^2}{t}\right)$$

reproduces exact (scaling) result in **spherical model**

• for **Ising model**, need detailed discussion of $a(\mathbf{r})$

work in progress with S. STOIMENOV (Sofia)

