Critical behavior of the quantum Potts chain with aperiodic perturbation

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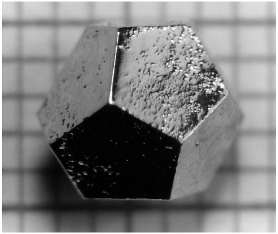
Overview

Aperiodicity in critical phenomena

- Condensed Matter Physics → discovery of quasicrystals



(a) quasicrystal

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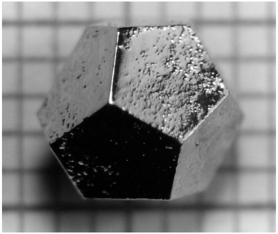
- Mathematics $\rightarrow$ Aperiodic tilling $\rightarrow$ Penrose tiling



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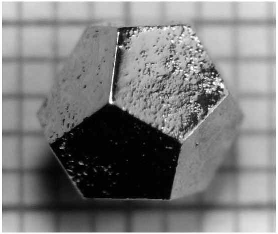

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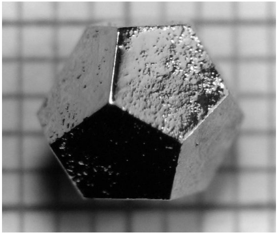

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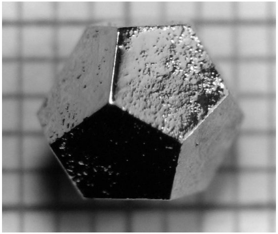
- The most famous example is the Fibonacci sequence: $0 \to 01$ and $1 \to 0$.



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Substitution rules and Geometric fluctuations

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  - The sequence after the n first iterations of the substitution rules:

$$\begin{array}{lc} n = 0 & 0, \\ n = 1 & 01, \\ n = 2 & 0110, \\ n = 3 & 01101001, \\ n = 4 & 0110100110010110, \\ \ldots\ldots\ldots\ldots & \end{array}$$

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- To compute fluctuations$\to$ Substitution matrix:

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\mathbb{M} = \begin{pmatrix} n_A^{S(A)} & n_A^{S(B)} & \cdots \\ n_B^{S(A)} & n_B^{S(B)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \xrightarrow{Fibonacci} \quad \mathbb{M} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad (1)
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- wandering exponent: $\omega \equiv \frac{\ln|\zeta_2|}{\ln \zeta_1}$, $\zeta_1 > \zeta_2$

1 Introduction

2 Strong-Disorder Renormalization Group approach

- Strong-Disorder RG approach

3 The quantum aperiodic Potts chain

- Quantum Potts chain
- SDRG results on quantum aperiodic Potts chain

4 Conclusion

SDRG (S.K. Ma and C. Dasgupta (1979))

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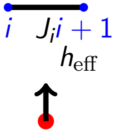
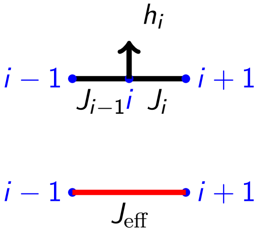
- Diagonalize the Hamiltonian of this coupling
- Treat the system perturbativly

SDRG for RTFIM

- Renormalization of exchange coupling $J_i \sigma_i^z \sigma_{i+1}^z$
  - The two spins acts as a *macrospin* in a field

$$h_{eff} = \frac{h_i h_{i+1}}{J_i} \tag{3}$$





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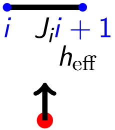
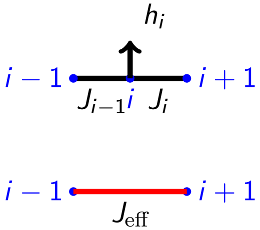
- Renormalization of a strong transverse field $h_i \sigma_i^x$
  - The spin is frozen and the neighbors are coupled by

$$J_{eff} = \frac{J_{i-1} J_i}{h_i} \qquad (4)$$





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  - FP independent of q? $\rightarrow$ Monte-Carlo simulations: q is relevant [Chatelain *et al.*].

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- Generalization of Ising model: q possible states, $\sigma_i = 1, 2, ..., q$
- The Pure chain ($J_i = J$ and $h_i = h$):
  - For $q = 2 - 4$: 2nd order PT $\neq$ Universality class
  - For $q \geq 4$: 1st order PT

Quantum q-state Potts chain

- 2D classical Potts model $\rightarrow$ Anisotropic limit: $J_u \rightarrow 0$ and $J_v \rightarrow +\infty$

$$-\beta H = \sum_{x,y} J_u \delta_{\sigma_{x,y},\sigma_{x+1,y}} + \sum_{x,y} J_v [\delta_{\sigma_{x,y},\sigma_{x,y+1}} - 1] \quad (5)$$

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- Potts operators:

$$\hat{\Omega}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix}$$

for $q = 4$ for example, with $\omega = e^{\frac{2i\pi}{q}}$.

Quantum q-state Potts chain

- Ladder operators $N_i \left|\sigma_i\right\rangle = \left|\sigma_{i+1}\right\rangle$

$$N_i = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

SDRG rules [Senthil and Majumdar]

- For strong bond J_i:

$$h_{eff} = \frac{2h_i h_{i+1}}{qJ_i} \tag{7}$$

- For strong field h_i:

$$J_{eff} = \frac{2J_{i-1}J_i}{qh_i} \tag{8}$$

SDRG for a family of aperiodic sequences

- substitution rules:

$$a \to ab^k, \quad b \to a \qquad \text{for} \quad b^k \equiv \underbrace{bb\dots b}_{k\times\text{letters}} \tag{9}$$

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$$r^{(j)} = \frac{h^{(j)}}{J_b^{(j)}}, \quad s^{(j)} = \frac{J_a^{(j)}}{h^{(j)}}, \tag{10}$$

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- For an IDFP: $\ln s$ and $\ln r$ diverges $\to C_k$ remains finite.

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Conclusion $\rightarrow$ Luck criterion

- $k = 1 \rightarrow$ Irrelevant for $q = 2$

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- To observe this, we study PF, PD and TF → first for Ising and then for q-state Potts chain.
- We study the relevant Rudin-Shapiro sequence

Magnetic scaling dimension

- For IDFP, the energy cutoff scales like:

$$\Omega_j \sim e^{-L^{-\psi}} \;\Leftrightarrow\; L \sim \left(\ln\frac{\Omega_I}{\Omega_j}\right)^{-1/\psi} \tag{12}$$

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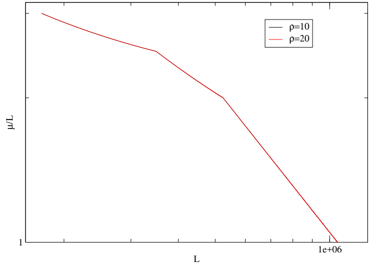
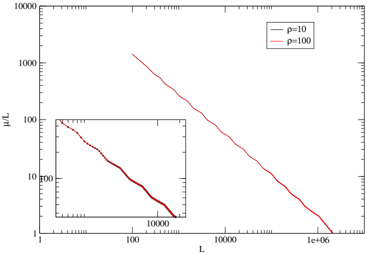
- The coupling ratio is defined: $\rho = J_1/J_0$

Magnetic scaling dimension for $q = 2$

- Thue-Morse: Irrelevant $\rightarrow$ no algebraic regime.



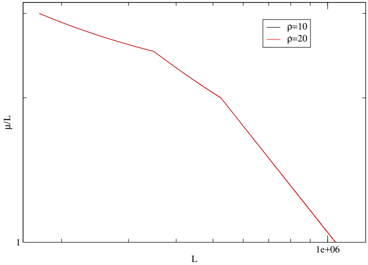
(a) Thue-Morse



(b) Paper-Folding

Magnetic scaling dimension for $q = 2$

- Thue-Morse: Irrelevant $\rightarrow$ no algebraic regime.
- Paper-Folding: $\beta/\nu = 0.252(3)$ $\rightarrow$ no dependence on ρ.



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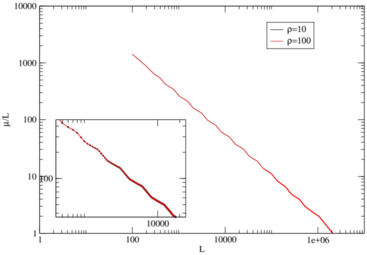

(b) Paper-Folding

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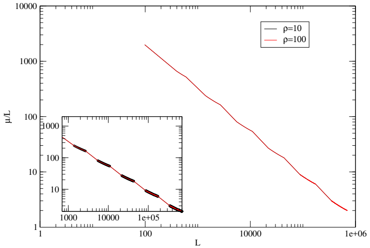
- Period-Doubling: $\beta/\nu = 0.254(6) \rightarrow$ close with numerical result of F.J. Oliveira Filho *et al.*



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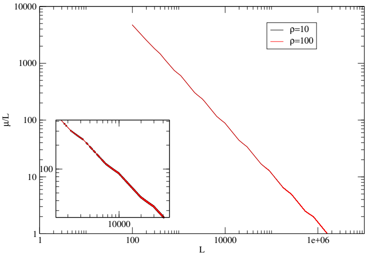

(b) Three-Folding

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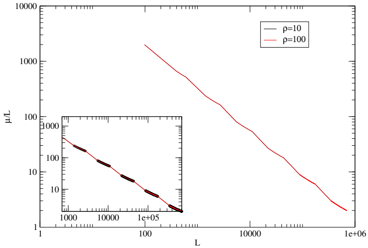
- Period-Doubling: $\beta/\nu = 0.254(6) \rightarrow$ close with numerical result of F.J. Oliveira Filho *et al.*
- Three-Folding: $\beta/\nu = 0.139(5) \rightarrow$ no dependence on ρ



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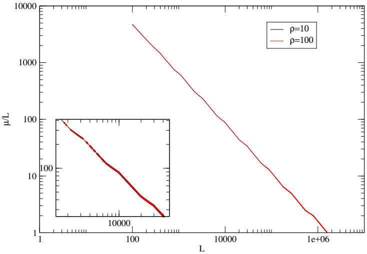


(b) Three-Folding

Magnetic scaling dimension for $q = 2$

- Rubin-Shapiro: $\beta/\nu = 0.175(6) \rightarrow$ is not compatible with the analytical prediction by F.J. Oliveira Filho *et al.*

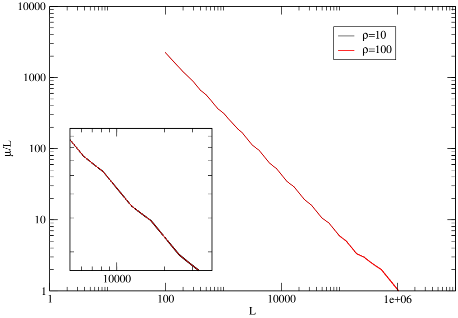
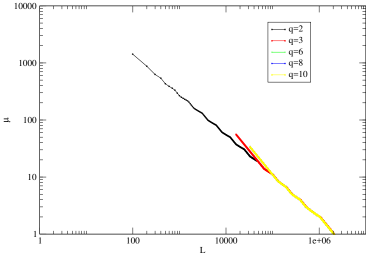


Figure: Rubin-Shapiro

Magnetic scaling dimension for $q > 2$

- Paper-Folding: no-dependence on q, exponent stable



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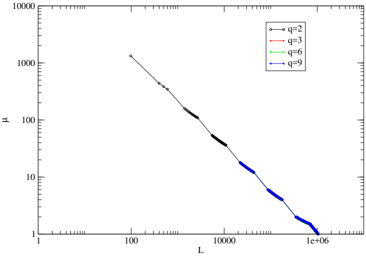

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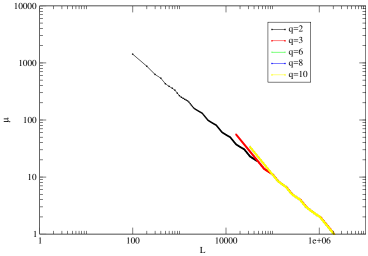

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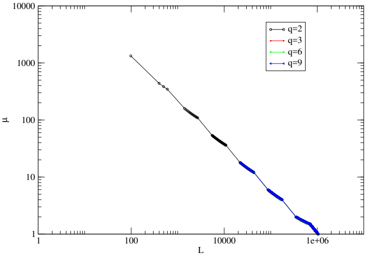

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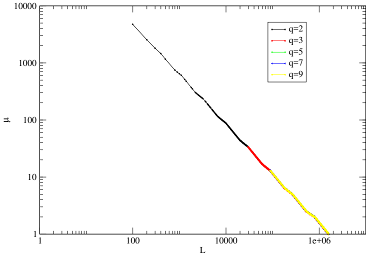
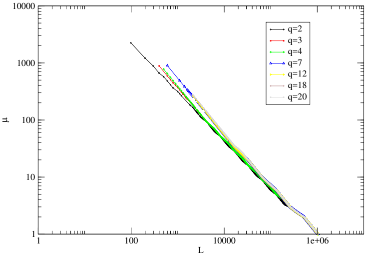
Magnetic scaling dimension for $q > 2$

- Three-Folding: no-dependence on q, exponent stable



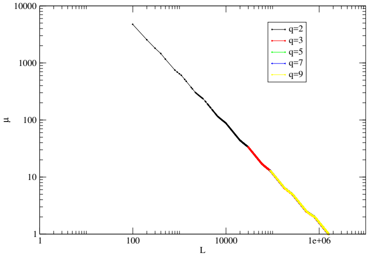
(a) Three-Folding



(b) Rubin-Shapiro

Magnetic scaling dimension for $q > 2$

- Three-Folding: no-dependence on q, exponent stable
- Rubin-Shapiro: small dependence on q.



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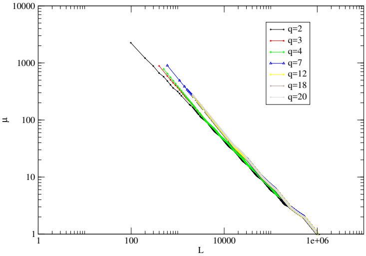
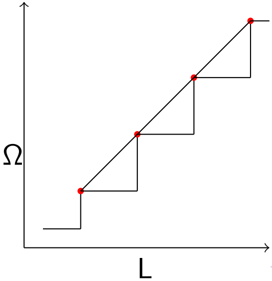


(b) Rubin-Shapiro

Estimation of dynamical exponent (Method)

- $\Omega(L)$ is not monotonous but displays steps.



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- $\Omega(L)$ is not monotonous but displays steps.
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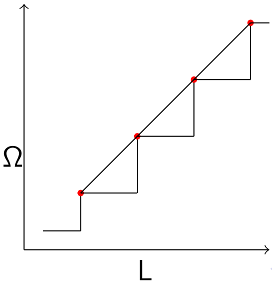

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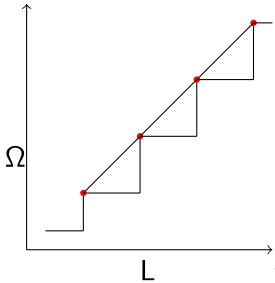

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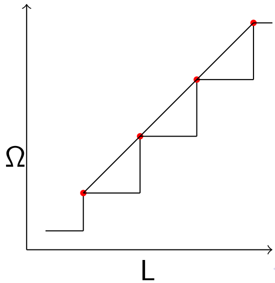

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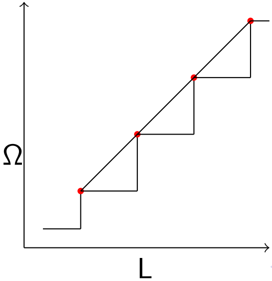
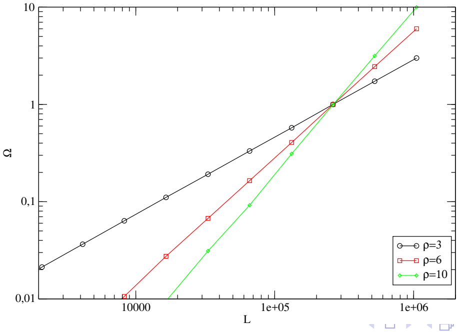
- The fit is performed into the scaling by the eq. $\Omega \sim L^z$.



Paper-Folding

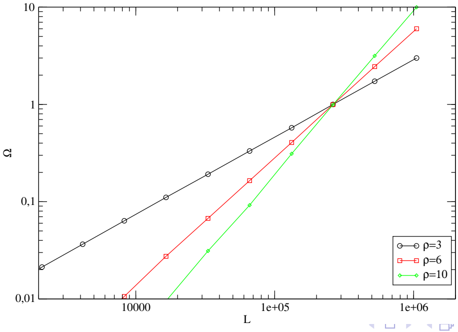
- Exact expression[Iglói *et al.*]: $z = \frac{\ln(\rho^{1/2}+\rho^{-1/2})}{\ln 2}$



Paper-Folding

- Exact expression[Iglói *et al.*]: $z = \frac{\ln(\rho^{1/2}+\rho^{-1/2})}{\ln 2}$
- Large-coupling limit: $z \simeq -\frac{\ln\rho}{\ln 4}$

coupling ratio	z (SDRG theory)	z (SDRG simul.)
$\rho = 3$	0.79248	0.79(1)
$\rho = 6$	1.2925	1.27(1)
$\rho = 10$	1.6610	1.66(6)



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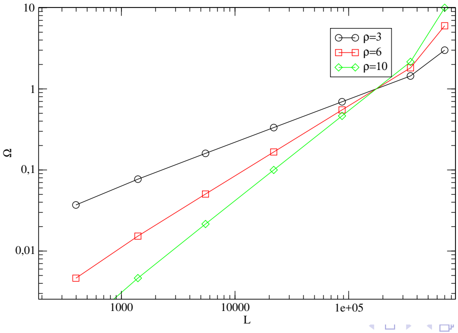

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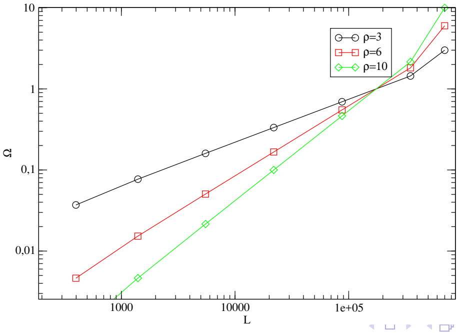
coupling ratio	z (exact)	z (asymp.)	z (SDRG)
$\rho = 3$	1.094	0.528	0.56(1)
$\rho = 6$	1.243	0.861	0.91(2)
$\rho = 10$	1.388	1.107	1.15(5)



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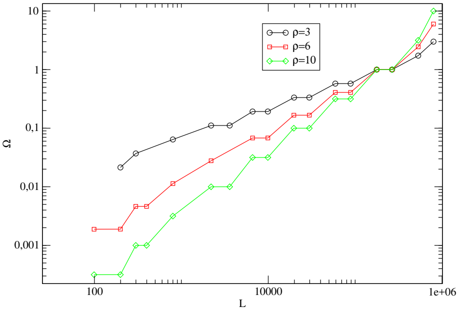

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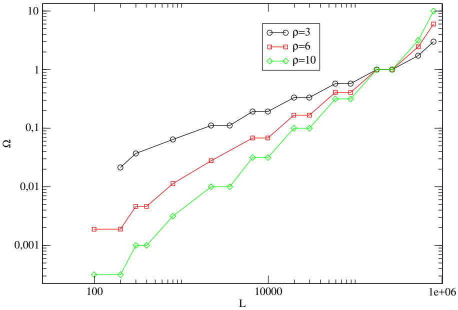
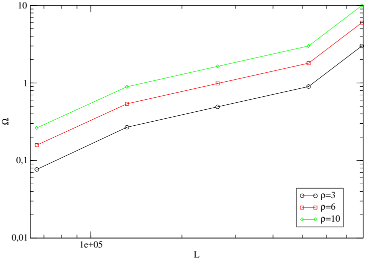
coupling ratio	z (exact)	z (asymp.)	z (SDRG)
$\rho = 3$	1.118	0.500	0.53(1)
$\rho = 6$	1.298	0.815	0.86(1)
$\rho = 10$	1.468	1.048	1.10(6)

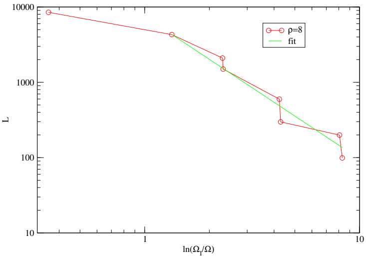


Rubin-Shapiro

- For relevant aperiodic sequence[F.J. Oliveira Filho *et al.*]: z is infinite and $\psi = \omega = 1/2$.



(a) Rubin-Shapiro:$z \to \infty$



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- For relevant aperiodic sequence[F.J. Oliveira Filho *et al.*]: z is infinite and $\psi = \omega = 1/2$.
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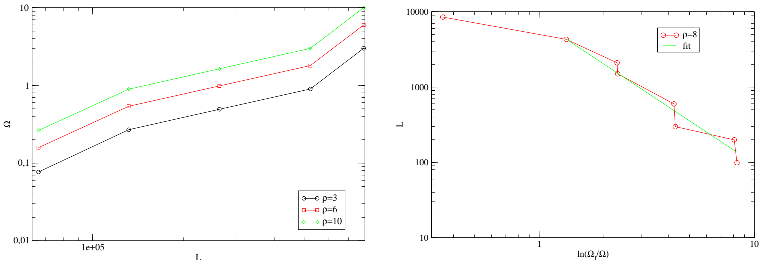

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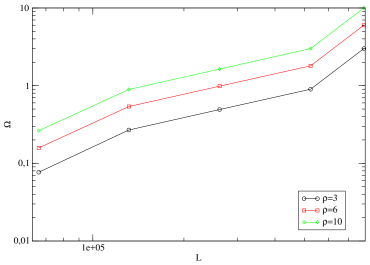
- Numericaly result: $\psi = 0.5(3)$



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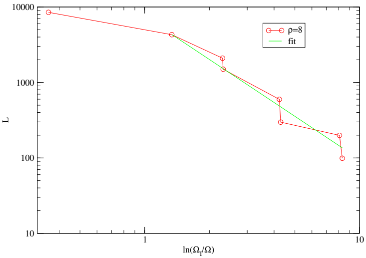
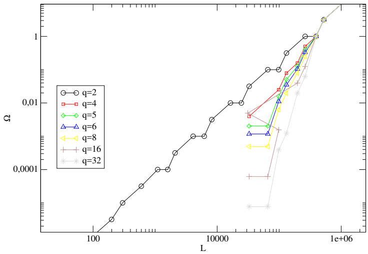
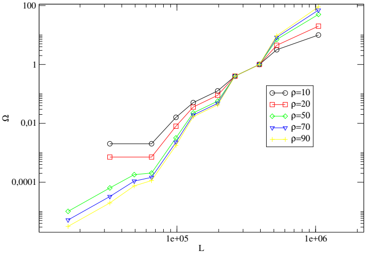


(b) Rubin-Shapiro

- The first three renormalizations are the same for all the ρ and q!



(a) $\rho = 10$



(b) $q = 5$

Paper-Folding ($q > 2$)

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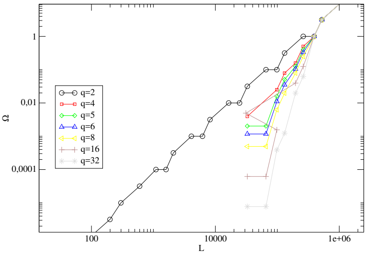

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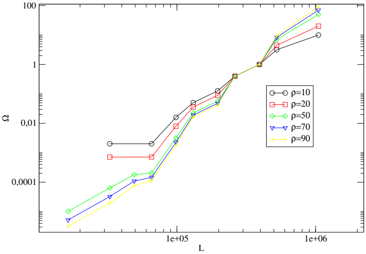

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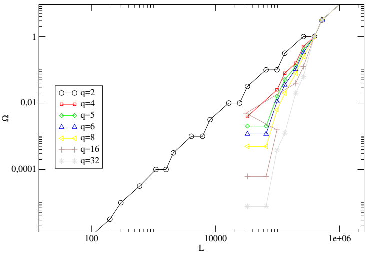
- For a fixed ρ the scaling is depends on q.
- For $q = 5$, as expected depends on ρ!



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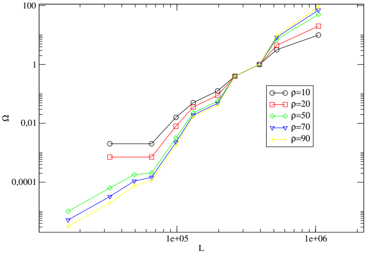

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Paper-Folding ($q > 2$)

- The dynamical exponent z increases with the number of states q!
- while for $q = 2$ is following the exact expression.

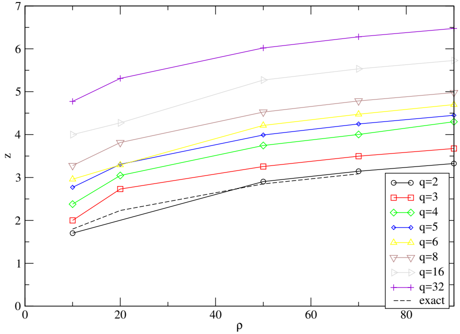

Figure: Dynamical exponent z with the coupling ratio ρ for several numbers of state q for Potts chains with the Paper-Folding sequence. The different curves correspond to different numbers of states q.

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