UNIVERSITÄT DUISBURG ESSEN

Offen im Denken

The square lattice Ising model on the rectangle

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- Introduction
- Connection to Casimir forces
- Free energy contributions
- Analytical finite-size scaling limit
- Hyperbolic parametrization
- Exact mapping onto effective spin model
- Conclusions & Outlook

Collaboration: Hendrik Hobrecht (see next talk), Felix M. Schmidt (UDE)

AH, *The square lattice Ising model on the rectangle I: finite systems*, J. Phys. A: Math. Theor. 50, 065201 (2017), arXiv:1609.01963
AH, *The square lattice Ising model on the rectangle II: finite-size scaling limit*, J. Phys. A: Math. Theor. 50, 265205 (2017), arXiv:1701.08722

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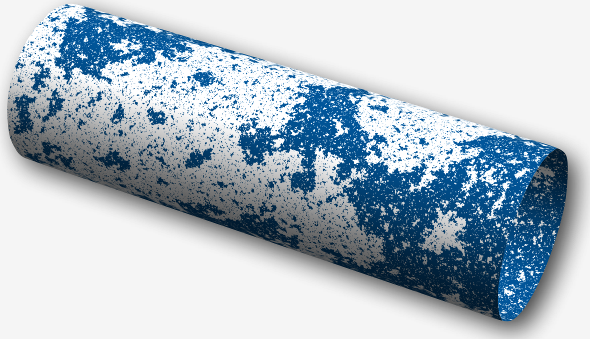

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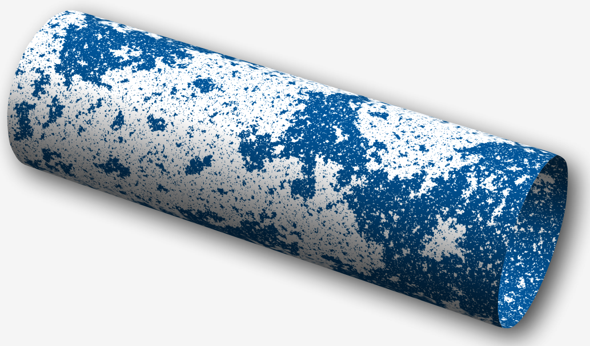

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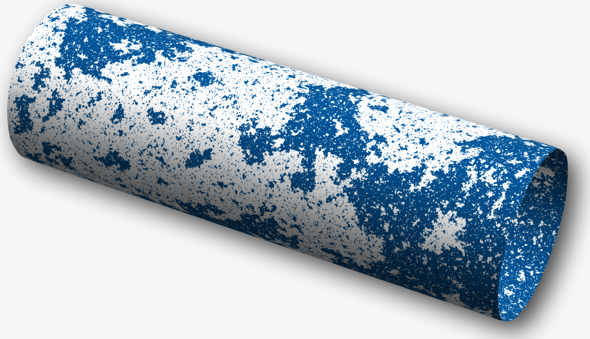

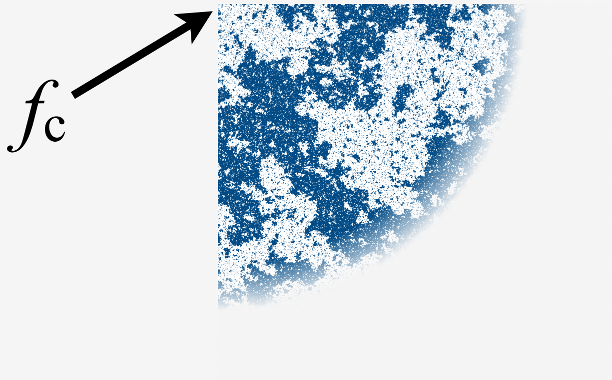

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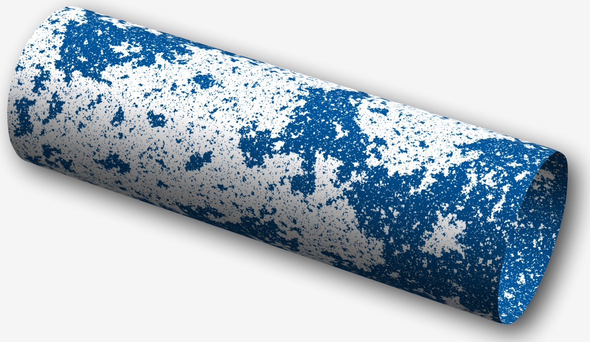
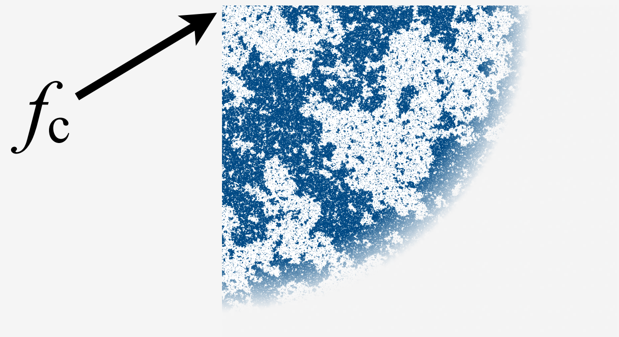
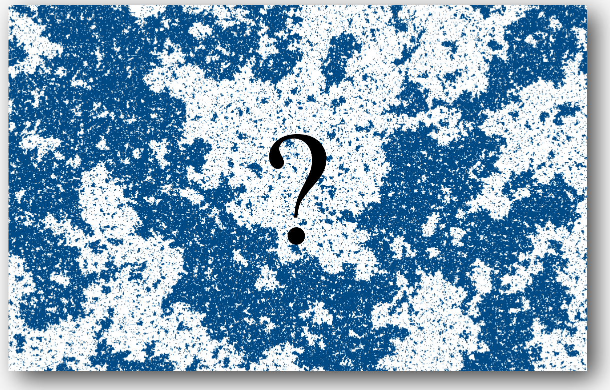
- **Problem**: exact solution on open rectangle <u>not</u> known (yet)







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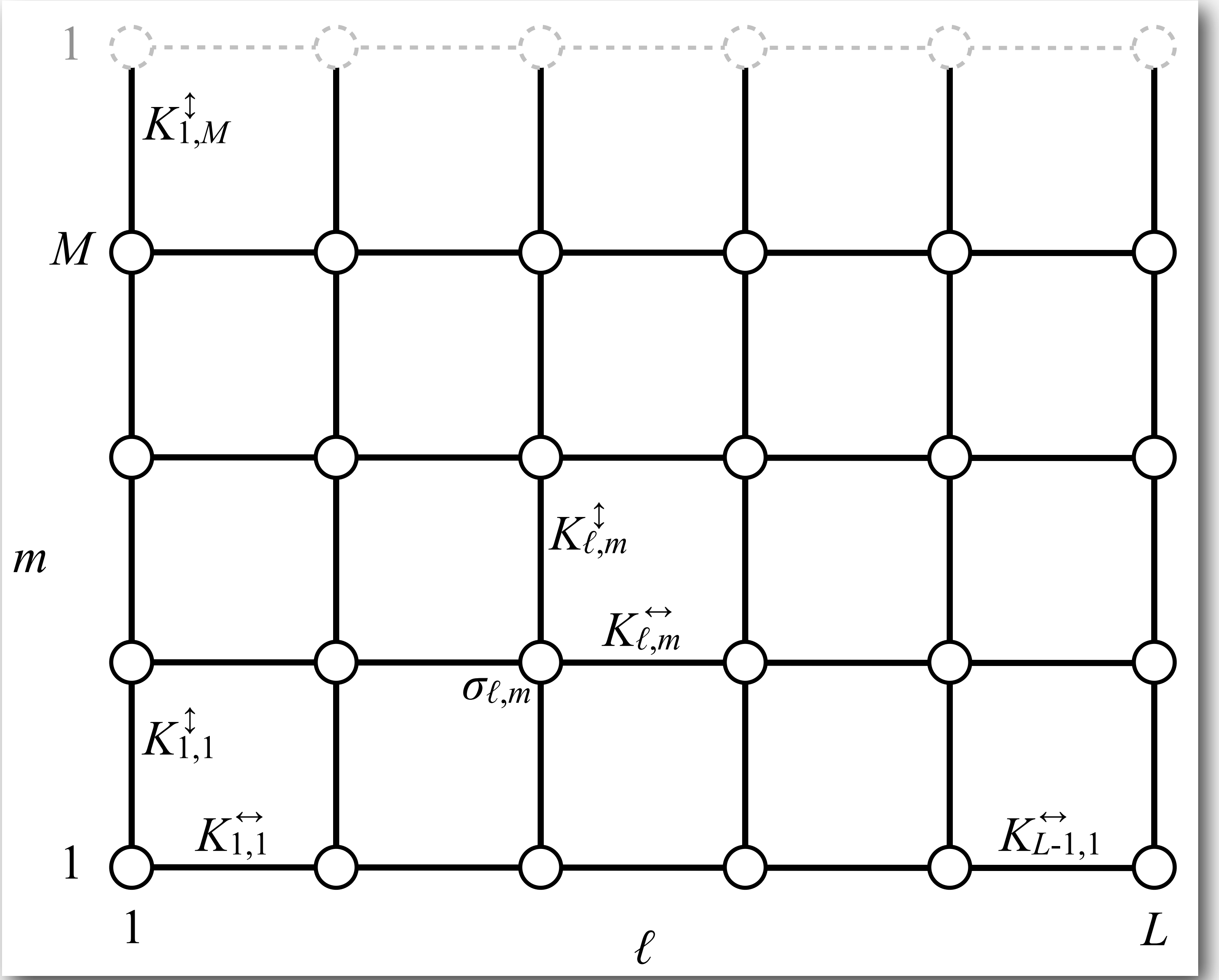
Inhomogeneous 2*d* Ising model on the cylinder

- partition function

$$Z = \operatorname{Tr} \exp \sum_{\ell=1}^{L} \sum_{m=1}^{M} \left(K^{\leftrightarrow}_{\ell,m} \sigma_{\ell,m} \sigma_{\ell+1,m} + K^{\updownarrow}_{\ell,m} \sigma_{\ell,m} \sigma_{\ell,m+1} \right)$$

- $\sigma_{\ell,m} = \pm 1$
- $K^{\leftrightarrow}_{L,m} = 0$
- allow <u>all</u> $K^{\leftrightarrow,\updownarrow}_{\ell,m}$ different!
- special case $K^{\updownarrow}_{\ell,M} = 0$: open rectangle



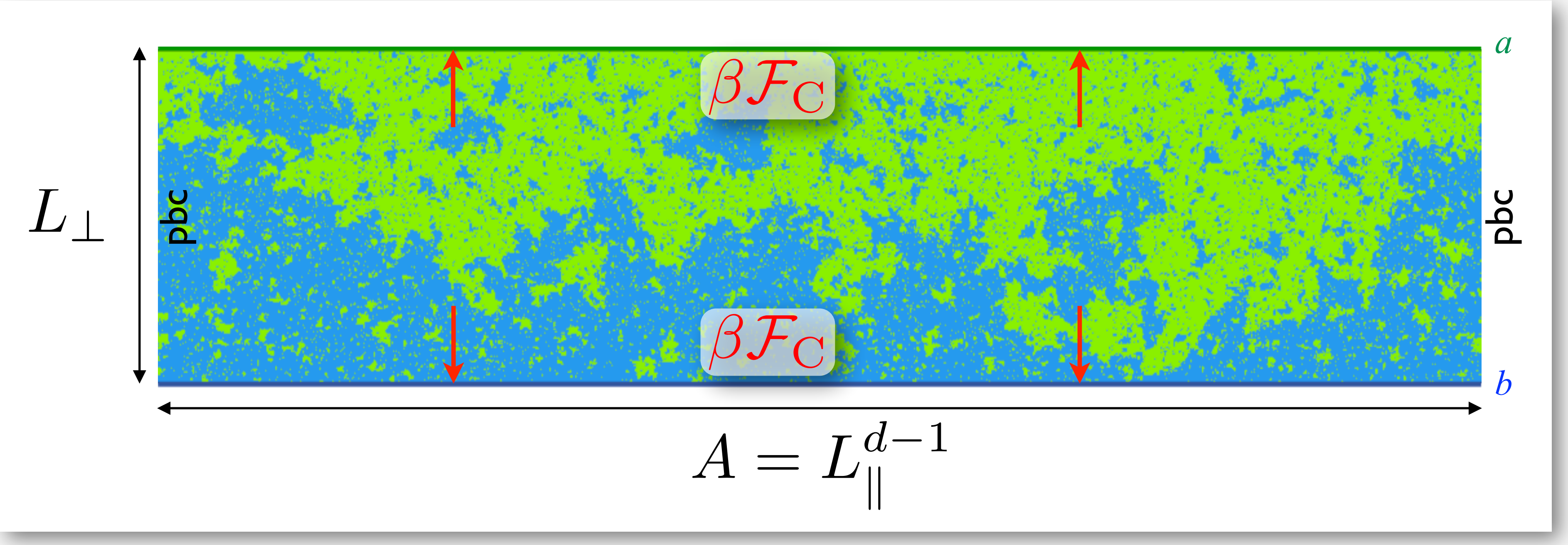
Critical Casimir forces

Critical Casimir forces (without corners yet)

- universal *finite-size* effect near criticality
- depends only on universality class, boundary conditions, shape
- reduced residual free energy for cylinder geometry

$$t = \frac{T}{T_\mathrm{c}} - 1$$

$$F_\mathrm{res}(t, L_\perp, L_\parallel) \equiv F(t, L_\perp, L_\parallel) - V f_\mathrm{b}(t) - A\left[f_\mathrm{s}^a(t) + f_\mathrm{s}^b(t)\right]$$

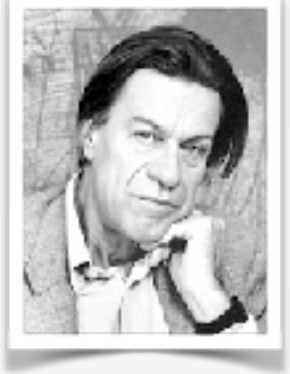


- critical Casimir force per area

$$\beta\mathcal{F}_\mathrm{C}(t, L_\perp, L_\parallel) \equiv -\frac{1}{A}\frac{\partial}{\partial L_\perp} F_\mathrm{res}(t, L_\perp, L_\parallel)$$

Hendrik B. G. Casimir (1909-2000)

Michael E. Fisher (*1931)



Pierre-Gilles de Gennes (1932-2007)

[8] M. E. Fisher and P.-G. de Gennes, C. R. Acad. Sci. Paris Ser. B 287, 209 (1978)

Universal scaling functions

- both F_{res} and $\beta\mathcal{F}_{\mathrm{C}}$ follow universal finite-size scaling functions

$$F_{\mathrm{res}}(t, L_\perp, L_\|) \simeq \Theta(x, \rho)$$

$$L_\perp^d \, \beta\mathcal{F}_{\mathrm{C}}(t, L_\perp, L_\|) \simeq \vartheta(x, \rho)$$

with scaling variables

$$x \equiv t \left(\frac{L_\perp}{\xi_+}\right)^{1/\nu} \underset{t>0}{\simeq} \left(\frac{L_\perp}{\xi_{\mathrm{b}}(t)}\right)^{1/\nu}$$

$$\rho \equiv L_\perp / L_\|$$

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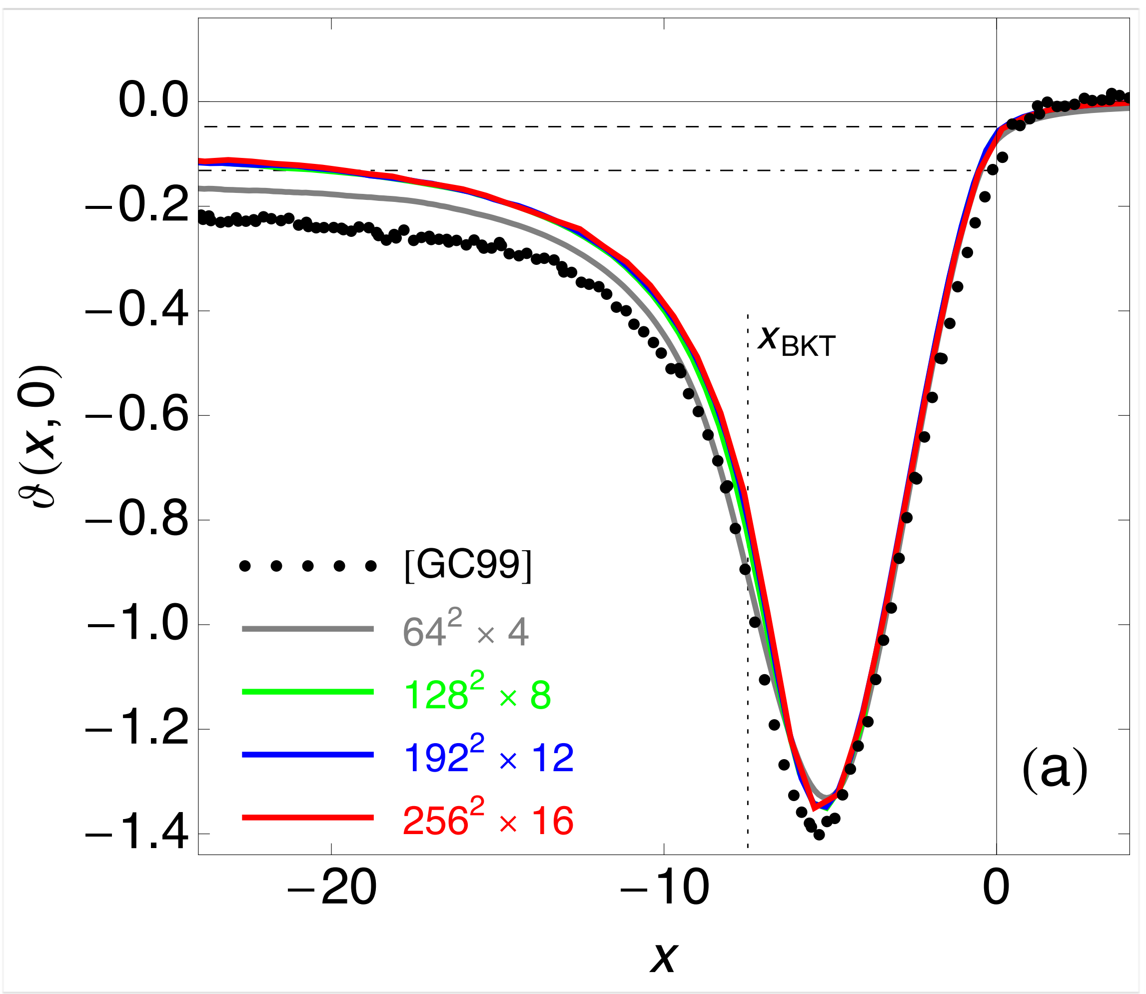
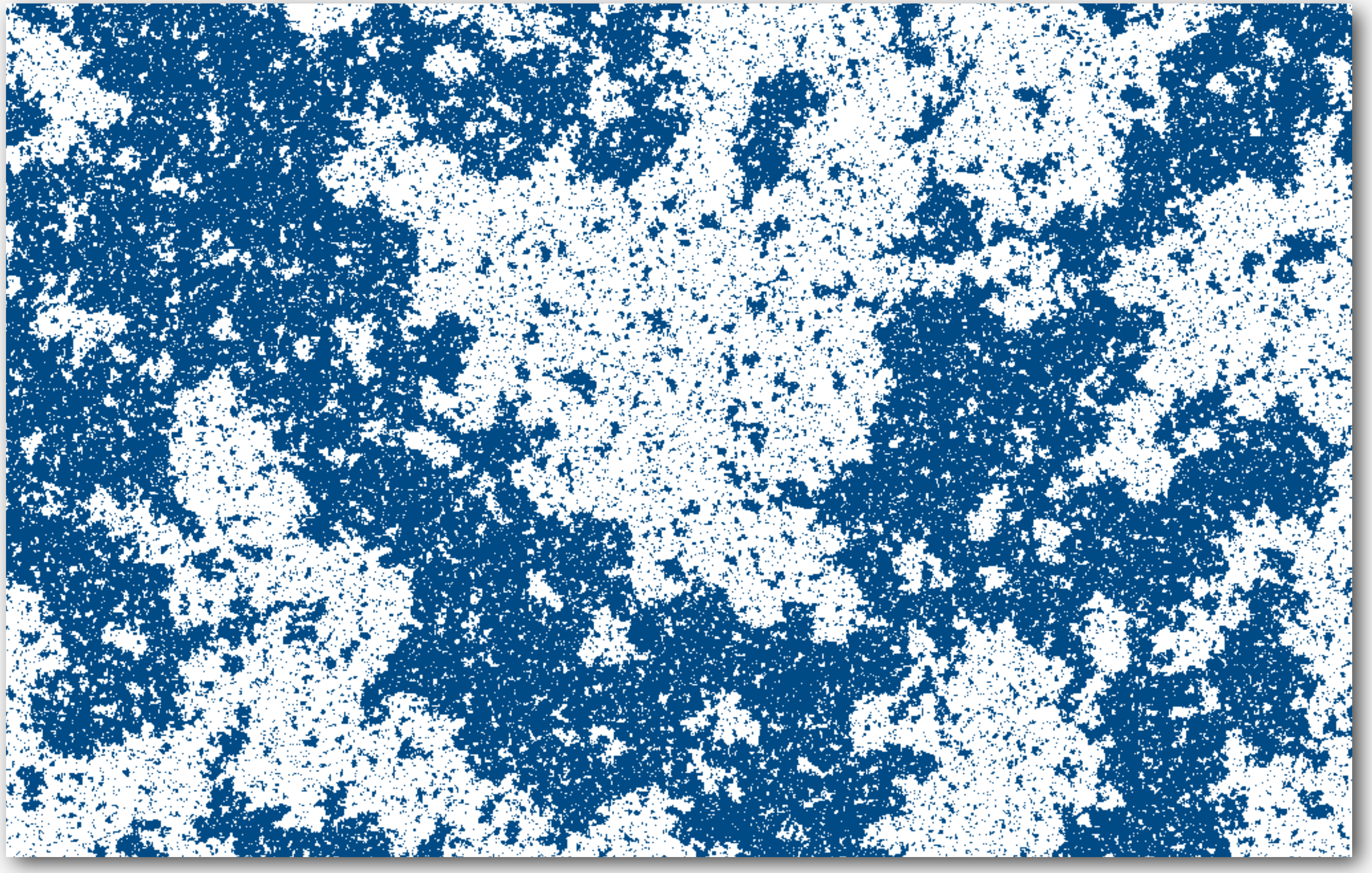
- Experiment: critical thinning of ^{4}He films near the λ transition
  Garcia & Chan, Phys. Rev. Lett. 83, 1187 (1999)

- Theory: MC simulation of XY model with open BCs
  AH, Phys. Rev. Lett. 99, 185301 (2007)



no free parameters!

Open rectangle (with corners): free energy contributions & analytic finite-size scaling limit



Free energy contributions on the rectangle

- we find

  from decomposition

- compare with

- decomposition of residuals (!)

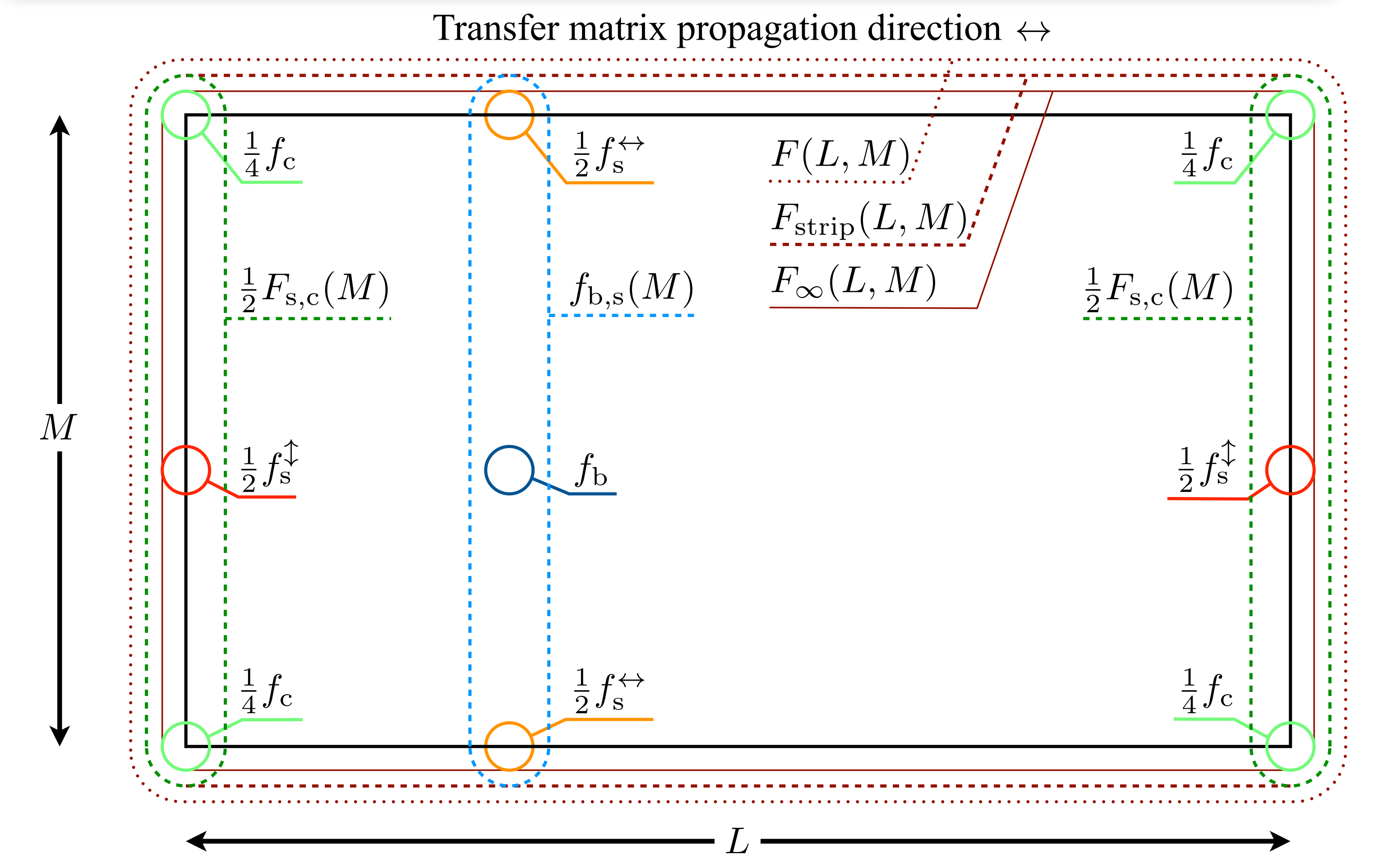
$$F^{\rm res}_{\rm strip}(L,M) = -\log\det(\mathbf{1}+\mathbf{Y})$$

$$F(L,M) = L f_{\rm b,s}(M) + F_{\rm s,c}(M) + F^{\rm res}_{\rm strip}(L,M)$$

$$F(L,M) = LM f_{\rm b} + L f_{\rm s}^{\leftrightarrow} + M f_{\rm s}^{\updownarrow} + f_{\rm c} + F^{\rm res}_{\infty}(L,M)$$

$$F^{\rm res}_{\infty}(L,M) = L f^{\rm res}_{\rm b,s}(M) + F^{\rm res}_{\rm s,c}(M) + F^{\rm res}_{\rm strip}(L,M)$$

- sketch of contributions



Finite-size scaling limit

- scaling variables

$$x = 2\tau M \qquad \rho = \frac{L}{M}$$

reduced temperature

$\tau = 1 - z/z_\mathrm{c}$

- scaling functions
  - Casimir potential
  - Casimir force

$$F_\infty^\mathrm{res}(L, M) \simeq \rho^{-1}\Theta(x, \rho)$$
$$\mathcal{F}(L, M) \simeq M^{-2}\vartheta(x, \rho)$$

- our model
  - Casimir potential

$$F_\infty^\mathrm{res}(L, M) = L f_\mathrm{b,s}^\mathrm{res}(M) + F_\mathrm{s,c}^\mathrm{res}(M) + F_\mathrm{strip}^\mathrm{res}(L, M)$$
$$\Rightarrow\ \Theta(x, \rho) = \Theta^{(\mathrm{oo})}(x) + \rho^{-1}\Theta_\mathrm{s,c}(x) + \Psi(x, \rho)$$

  - Casimir force

$$\vartheta(x, \rho \gtrsim 1) = -\Theta^{(\mathrm{oo})}(x) + \psi(x, \rho)$$
$$\vartheta(x, \rho \lesssim 1) = \vartheta^{(\mathrm{oo})}(x) - \rho x \Theta'_\mathrm{s,c}(x) - \frac{x\partial}{\partial x}\Psi(x, \rho^{-1}) - \psi(x, \rho^{-1})$$

$$\Theta^{(\mathrm{oo})}(x) = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}\omega\ \log\left(1 + \frac{\sqrt{x^2+\omega^2} - x}{\sqrt{x^2+\omega^2} + x}\mathrm{e}^{-2\sqrt{x^2+\omega^2}}\right)$$

$$\vartheta^{(\mathrm{oo})}(x) = -\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\ \sqrt{x^2+\omega^2}\left(1 + \frac{\sqrt{x^2+\omega^2} + x}{\sqrt{x^2+\omega^2} - x}\mathrm{e}^{2\sqrt{x^2+\omega^2}}\right)^{-1}$$

Universal finite-size scaling functions

- Casimir potential [8]

$$\Theta(x,\rho) = \Theta^{(oo)}(x) + \rho^{-1}\Theta_{s,c}(x) + \Psi(x,\rho)$$

- log. divergence from corner free energy

$$\Theta_{s,c}(x) = -\frac{1}{8}\log|x| - \frac{3}{4}\log 2\,\frac{|x|}{x} + \mathit{reg.}$$

- at $x = 0$ we confirm CFT predictions, e.g.,

$$\Psi(0,\rho) = \frac{\pi}{48} + \frac{1}{4\rho}\log\eta(\mathrm{i}\rho)$$

- Casimir amplitude $\neq$ $x = 0$ Casimir potential!

$$\Delta_C(\rho) = \frac{1}{4\rho}\log\eta(\mathrm{i}\rho) \qquad \Theta(0,\rho) = \infty$$

- Casimir force changes sign with x and ρ

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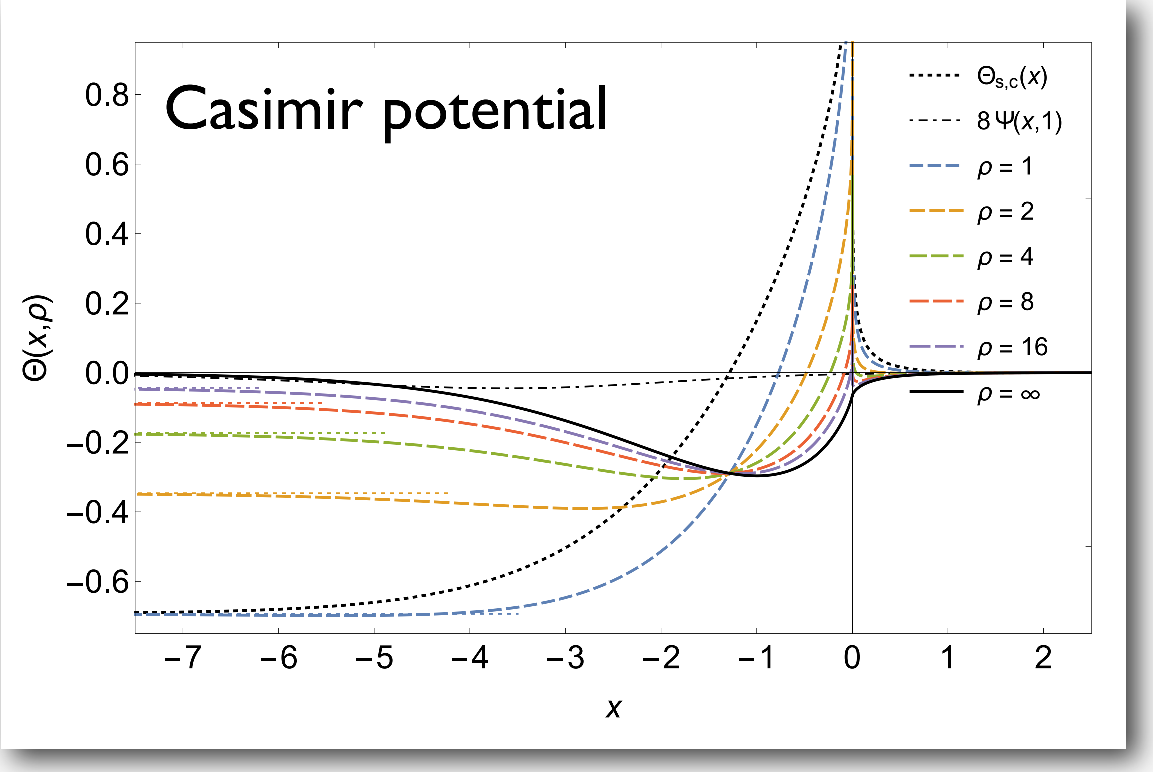
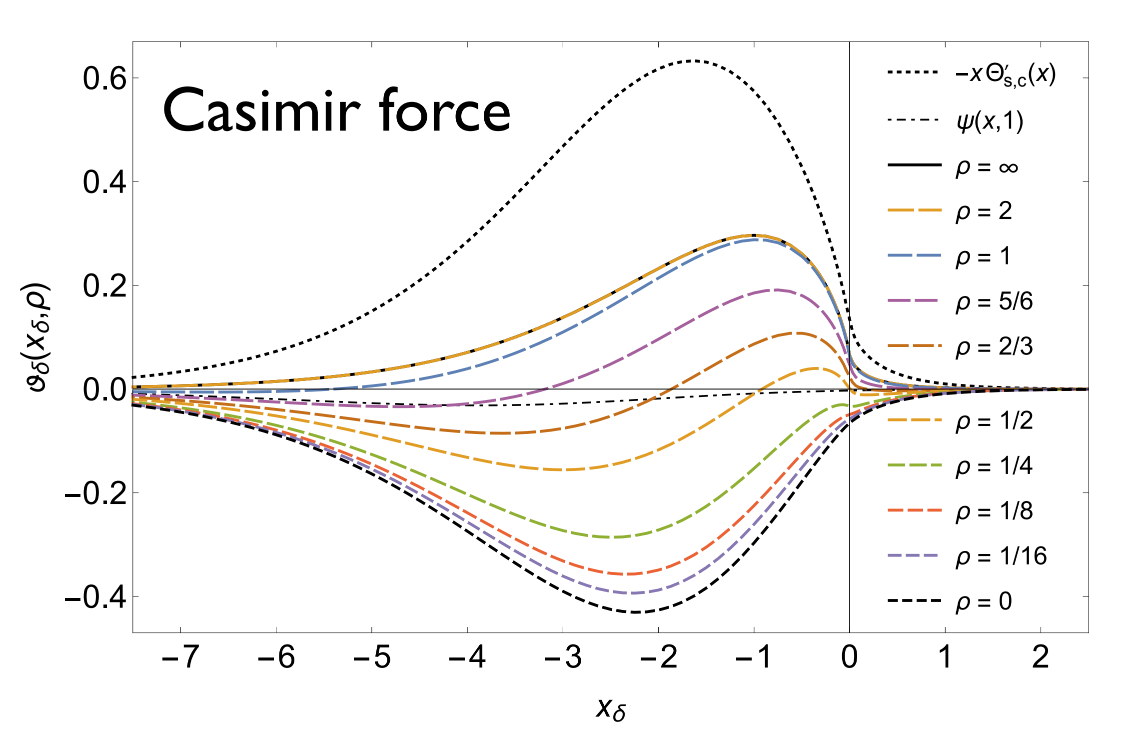




Hyperbolic Parametrization

- (isotropic) FSS limit

$$z \mapsto z_c \left(1 - \frac{x}{2M}\right) \quad L \mapsto \rho M \quad \varphi \mapsto \frac{\Phi}{M} \quad \gamma \mapsto \frac{\Gamma}{M} \quad M \to \infty$$

- of Onsager dispersion

$$\cosh\gamma = t_+ z_+ - t_- z_- \cos\varphi$$

- gives

$$\Gamma = \sqrt{x^2 + \Phi^2}$$

- characteristic polynomial

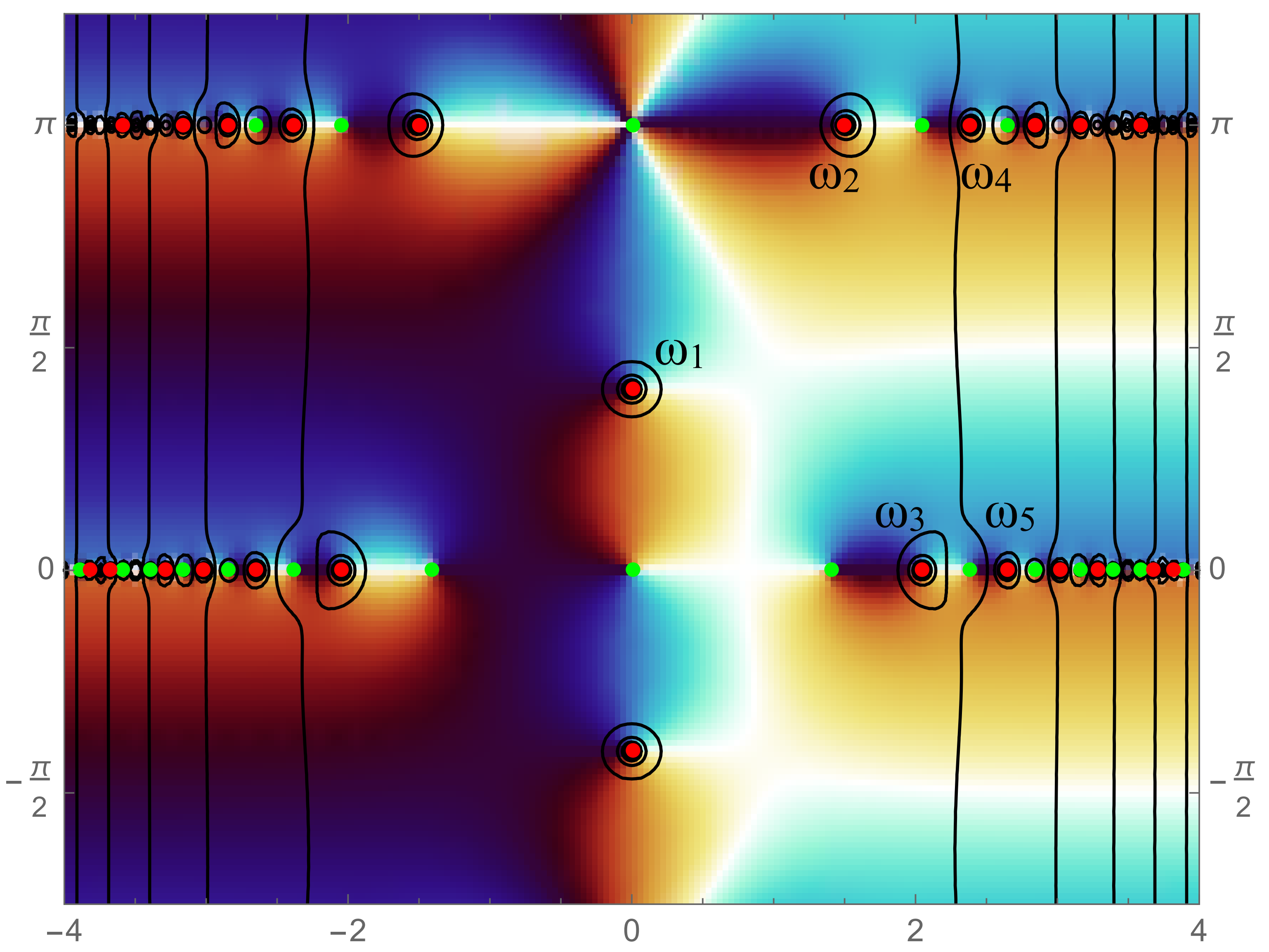
$$P(\Phi) \equiv \cos\Phi + \frac{x}{\Phi}\sin\Phi$$

- complex integrals in Φ plane: branch cuts
- solution: hyperbolic parametrization

$$\Phi = x \sinh\omega, \quad \Gamma = x \cosh\omega$$

- separation of odd & even zeroes!

ω plane for $x = -2$



Calculation of $\Theta_{\mathrm{s,c}}(x)$

- $\Theta_{\mathrm{s,c}}(x)$ was calculated numerically in [8] using a scaling relation
- new result: direct calculation of regularized infinite double product

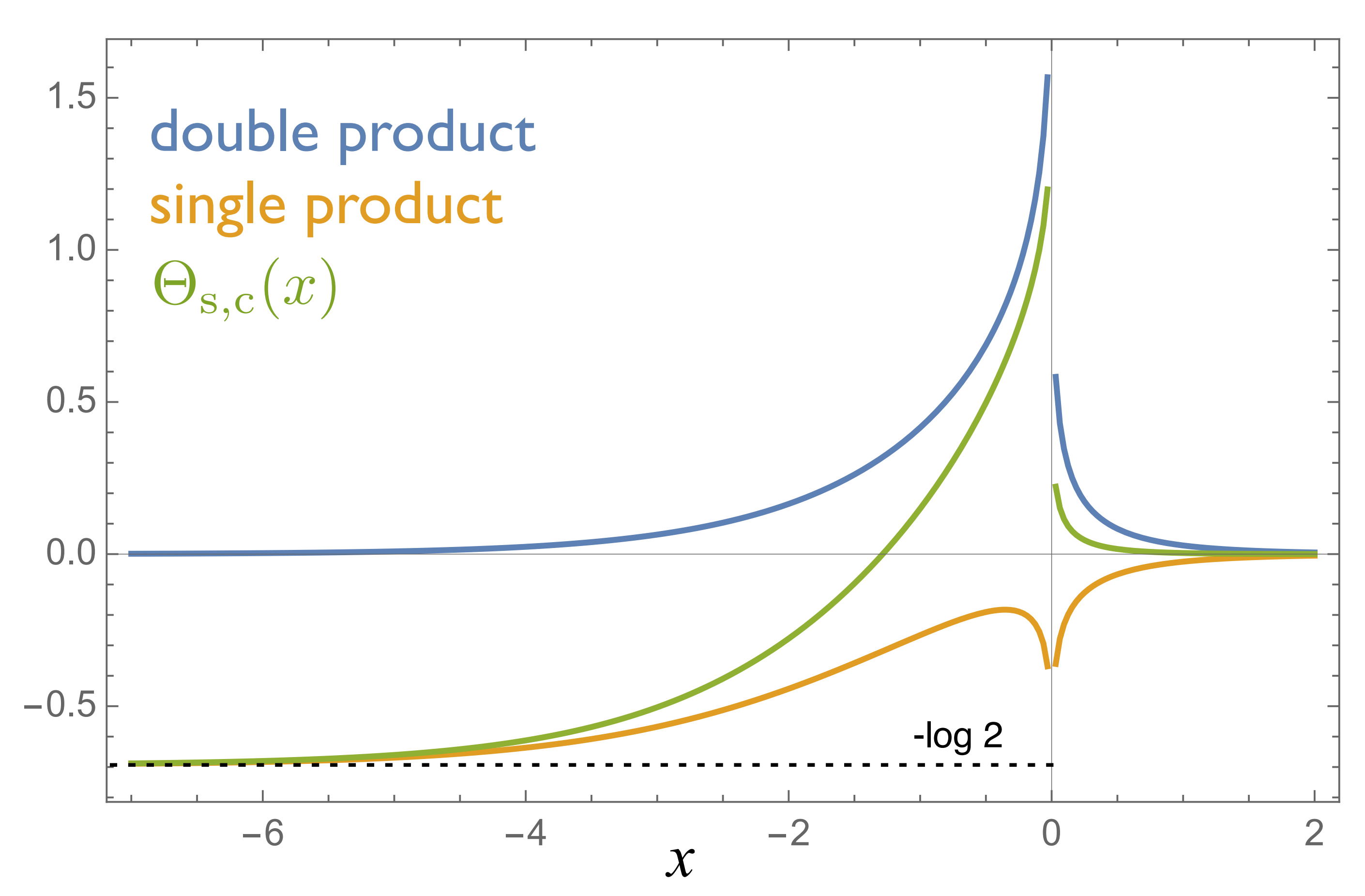
$$\frac{1}{2}\log \prod_{\substack{\mu=1\\ \mu\,\mathrm{odd}}}^{M-1} \prod_{\substack{\nu=2\\ \nu\,\mathrm{even}}}^{M} 2(\lambda_{\mu,+}-\lambda_{\nu,+})$$

$\lambda_{\mu,+}$: zeroes of a certain TM characteristic polynomial

- result after hyperbolic parametrization:

$$\frac{1}{2\pi^2}\int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}w \log(\cosh^2 t - \cosh^2 w)\frac{1-S(t)S(w)}{\cosh t \cosh w}$$

- second contribution simpler (single Cauchy integral)



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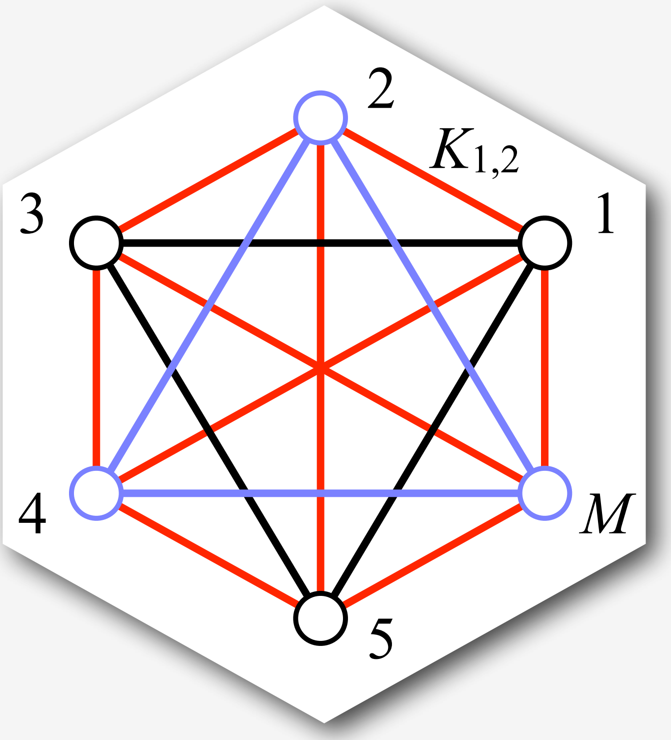
Effective spin model

- effective Hamiltonian ($s_\mu = \{0,1\}$, $b \to \infty$)

$$\mathcal{H}_{\text{eff}} = -\sum_{\mu<\nu=1}^{M} K_{\mu\nu} s_\mu s_\nu + L \sum_{\mu=1}^{M} \hat{\gamma}_\mu s_\mu + b \Big[\sum_{\mu=1}^{M} \sigma_\mu s_\mu \Big]^2$$



- two sublattices odd/even ($\sigma_\mu = \pm 1$) with equal sub. magnetization
- long-range couplings $K_{\mu\nu}$, intra-sub FM, inter-sub AFM
- magnetic moments $\hat{\gamma}_\mu > 0$, magnetic field L
- free energies: $F^{\text{res}}_{\text{strip}}(L, M) = F_{\text{eff}}(L, M)$
- critical Casimir force ≙ magnetization in magnetic field L

$$\mathcal{F}_{\text{strip}}(L, M) = -\frac{1}{M} \frac{\partial}{\partial L} F^{\text{res}}_{\text{strip}}(L, M) = \frac{1}{M} \frac{\partial}{\partial L} \log \text{Tr}\, e^{-\mathcal{H}_{\text{eff}}} = -m_{\text{eff}}(L)$$

- model might be used as starting point for further analysis

Conclusions & Outlook

- big step towards exact solution of 2d Ising model on (open) rectangle
- problem reduced from $4LM \times 4LM$ to $M/2 \times M/2$
- FSS functions calculated analytically
- corner free energy ⇒ log divergence of Casimir potential
- Casimir force changes sign
- hyperbolic parametrization lead to great simplifications (see also next talk by Hendrik)
- residual matrix can be mapped exactly onto effective spin model with M spins