Approximate ground states of the random-field Potts model from graph cuts and parallel tempering

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Abstract

While the ground-state problem for the random-field Ising model is polynomial, and can be solved using a number of well known algorithms for maximum flow, the analogue random-field Potts model corresponds to a multi-terminal flow problem that is known to be NP hard. Hence an efficient exact algorithm is extremely unlikely to exist. Still, it is possible to employ embedding of binary degrees of freedom into the Potts spins to use graph-cut methods to solve the corresponding ground-state problem approximately with polynomial methods. It is shown here that this works relatively well. We compare results produced by this heuristic algorithm to energy minima found by an appropriately tuned parallel tempering method that is configured to find ground states for the considered system sizes with high probability. The method based on graph cuts finds the same states in a fraction of the time. The new method is used for a first exploratory study of the random-field Potts model in two dimensions.

Introduction

- The RFPM is a system with disorder and frustration [1].
- The Hamiltonian is

$$\mathcal{H} = -J\sum_{\langle ij\rangle}\delta_{s_i,s_j} - \sum_i\sum_{\alpha=0}^{q-1} h_i^\alpha \delta_{s_i,\alpha} \qquad (1)$$

- q is the total number of Potts states and $s_i = \{0, 1, ..., q-1\}$. The fields h_i^α are quenched random variables at site i acting on state α, drawn here from a Gaussian distribution of standard deviation Δ.

Graph Cut (GC) Method

A specialized graph for the energy function is constructed such that the minimum cut on the graph minimizes the energy either globally or locally [2, 3].

The most general energy function treated is

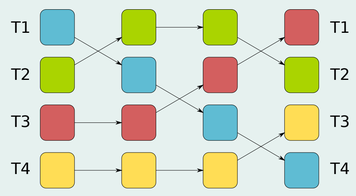
$$E(\{l_i\}) = \sum_{\{ij\}\in\mathcal{N}} V_{ij}(l_i, l_j) + \sum_{\{i\}\in\mathcal{S}} D_i(l_i), \quad l_i \in \mathcal{L} = (\alpha, \beta, ..., \gamma).$$

- The spin lattice is seen as a graph with nodes corresponding to spins and edges corresponding to interactions.
- A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of vertices $\mathcal{V}$ and edges $\mathcal{E}$ connecting them, where edges carry a weight V_{ij}.
- A cut C is a partition of the vertices $\mathcal{V}$ into two sets $\mathcal{R}$ and $\mathcal{Q}$.
- For two labels (spin species), finding the minimum cut can be mapped onto a maximum flow problem that can be solved exactly in polynomial time.
- For more than two labels, the algorithm freezes all but one or two colors and solves the corresponding restricted problem using the flow approach. In the next move, another set of colors is frozen etc.
- The algorithm in general is not guaranteed to find ground states, but very low-lying non-local minima.

Parallel Tempering (PT) Method

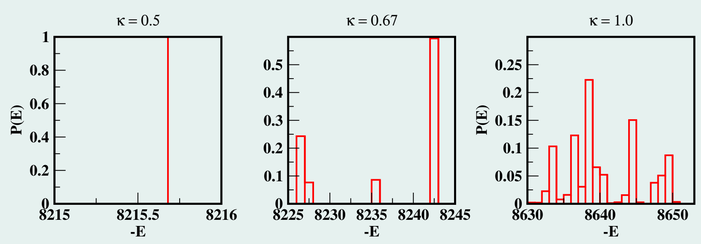
- For RFPM, Metropolis or other local moves do not allow the system to explore all of rugged configuration space efficiently.
- The efficiency can be increased by using PT global moves [4], where we simulate N replicas of the spin system at different temperatures, perform single spin-flip Metropolis sweeps for each replica and exchange (swap) configurations of nearest neighbors with the probability:

$$P_{\rm ex} = e^{-\Delta}, \quad \Delta = (\beta_{m+1} - \beta_m)(E_m - E_{m+1})$$

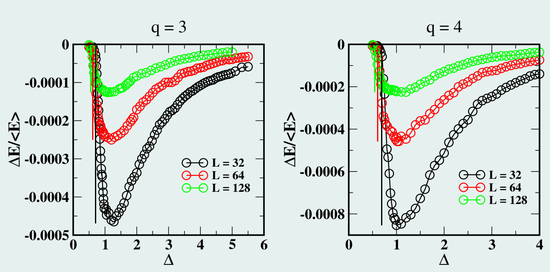


Results GC method

- Energy histogram obtained for a fixed $\{h_i\}$-configuration using GC method for 64^2 lattice and $q = 4$ RFPM. $\kappa = \Delta$ corresponds to the disorder strength.

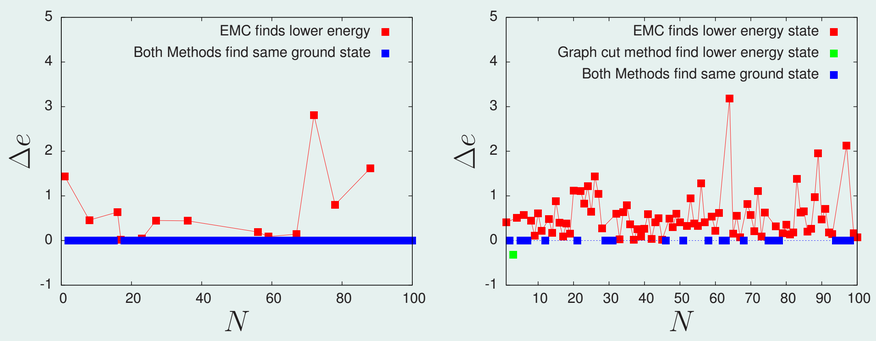


- Coefficient of variance of energy for $q = 3$ and $q = 4$ RFPM obtained for a fixed disorder configuration and 100 different initial spin configurations. The results are then averaged over 1000 disorder samples.

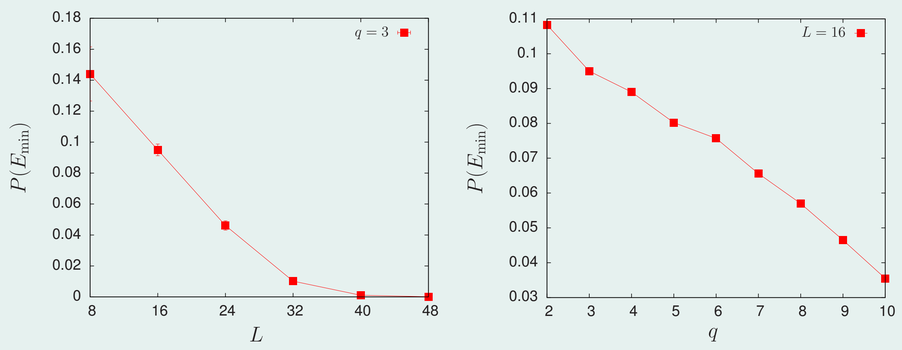


Comparative Results: GC & PT

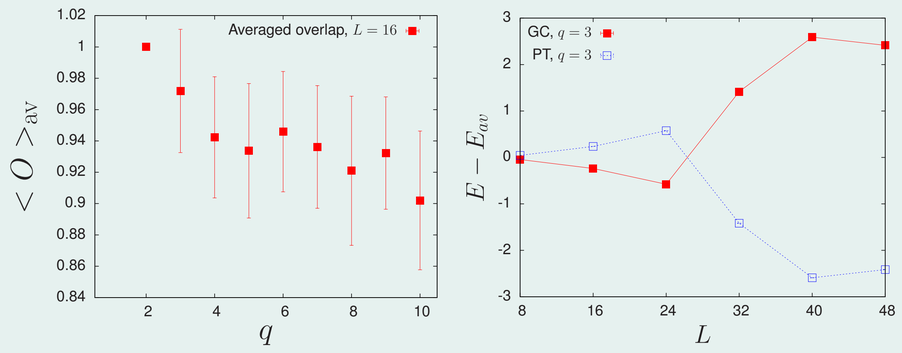
- The differences between lowest energies found in GC and PT runs for small systems are small for $q = 2 - 9$. Below we show plots for $q = 3$ (left) and $q = 10$ (right).



- The success probabilities appear to decay approximately linearly with the number of states q (right).
- They are compatible with an exponential decay with increasing system size L (left).



- For the smaller system sizes, even for the cases where exact ground states are not found, the overlap of lowest-energy state in GC and true ground state is large (left), configurations only differ locally.
- As the system size increases
  - probability of finding ground state decreases for PT
  - the difference, $E - E_{av}$ ($E_{av} = (E_{GC} + E_{PT})/2$) changes sign (right).



Conclusions

- We have studied RFPM using graph-cut and parallel tempering methods.
- PT is asymptotically exact, but the required run time increases exponentially with system size.
- The GC method is found to give an excellent approximation in a linear running time.
- The overlap between the states found by GC and PT is very large for small L where PT finds ground states.
- This allows to study the critical behavior of the random-field Potts model in two and three dimensions (work in progress).

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