The critical Ising model on a torus with a defect line

NIKOLAY IZMAILYAN

Yerevan Physics Institute, Armenia

Poghosyan, Kenna and Izmailian, EPL111 (2015) 60010

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Outline

- **Ising model**
- **Exact solution of Ising model with specific defect line**
- **The finite size corrections to free energy and correlation lengths**
- **Universal amplitude ratios**
- **Summary**

The Ising Model

Definition

- Lattice model with nearest neighbor interactions

$$Z = \sum_{\{s\}} \exp\left(\beta \sum_{<ij>} s_i s_j \right)$$

with $s_i = \pm 1$

- The 2D Ising model has a second order phase transition.

- 2D Ising model solved exactly by Onsager (*Phys. Rev. 65, 117 (1944)*).

- Describe wide range of physical phenomena, including
  - Binary alloys
  - Liquid-vapour transition
  - . . .

Ising model on torus with defect line

The anisotropic two-dimensional Ising model on a lattice with periodic boundary conditions and with specific defect line ("seam") was formulated in
O'Brien, Pearce and Warnaar, Physica A 228 (1996) 63;
Chui, Mercat, Orrick and Pearce, Phys. Lett. B 517 (2001) 429.

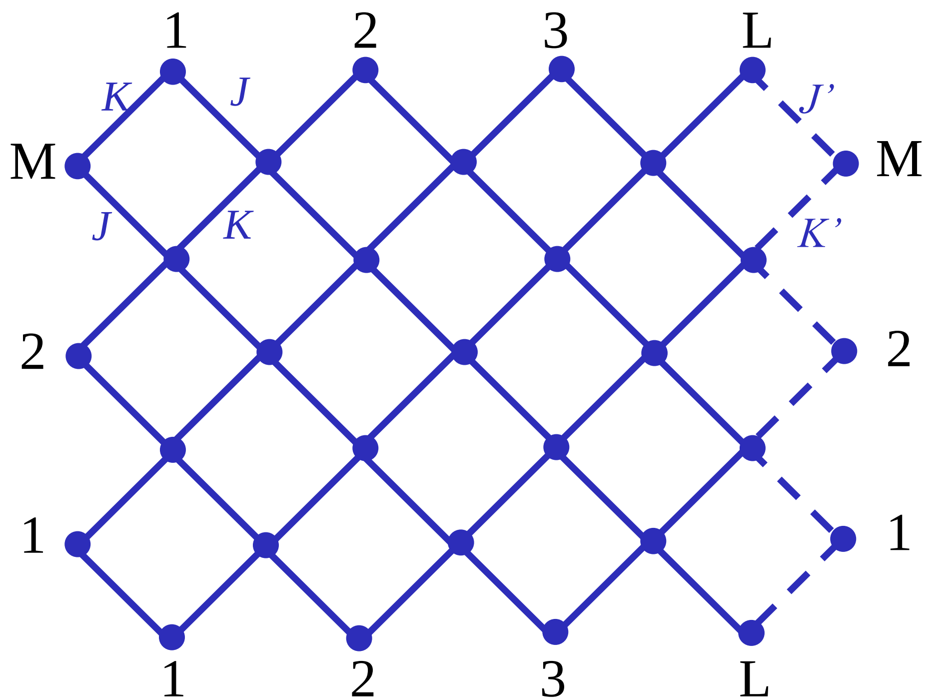
For the Ising model the defect line ("seams") are labelled by the Kac labels (r; s).
r=1,2 and s=1,2,3.
There are six possible seams label (r; s) = (1; 1); (1; 2); (1; 3); (2; 1); (2; 2); (2; 3)

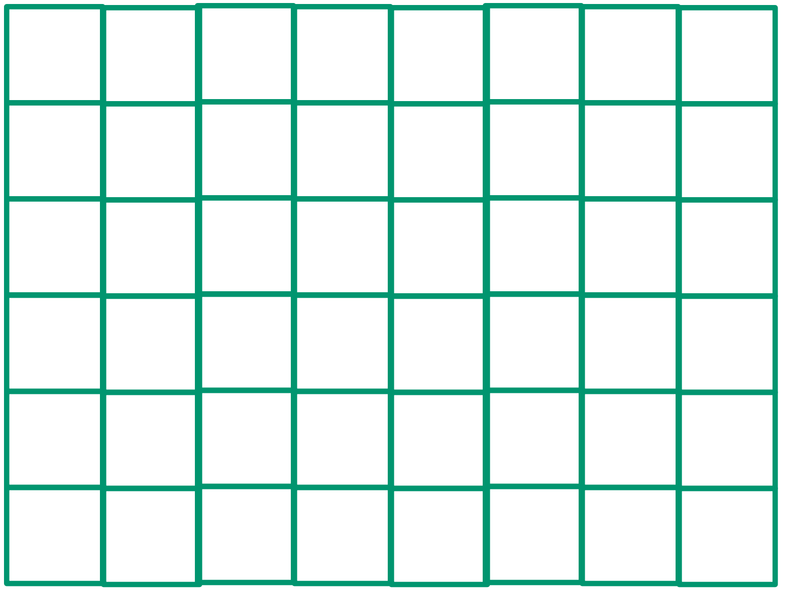
The (r; s) = (1; 1) and (1; 3) seams reproduce the well known partition function of the Ising model with the periodic and antiperiodic boundary conditions respectively.
Little attention has been paid to the boundary condition with the seams
(r; s) = (1; 2); (2; 1); (2; 2) and (2; 3)
.

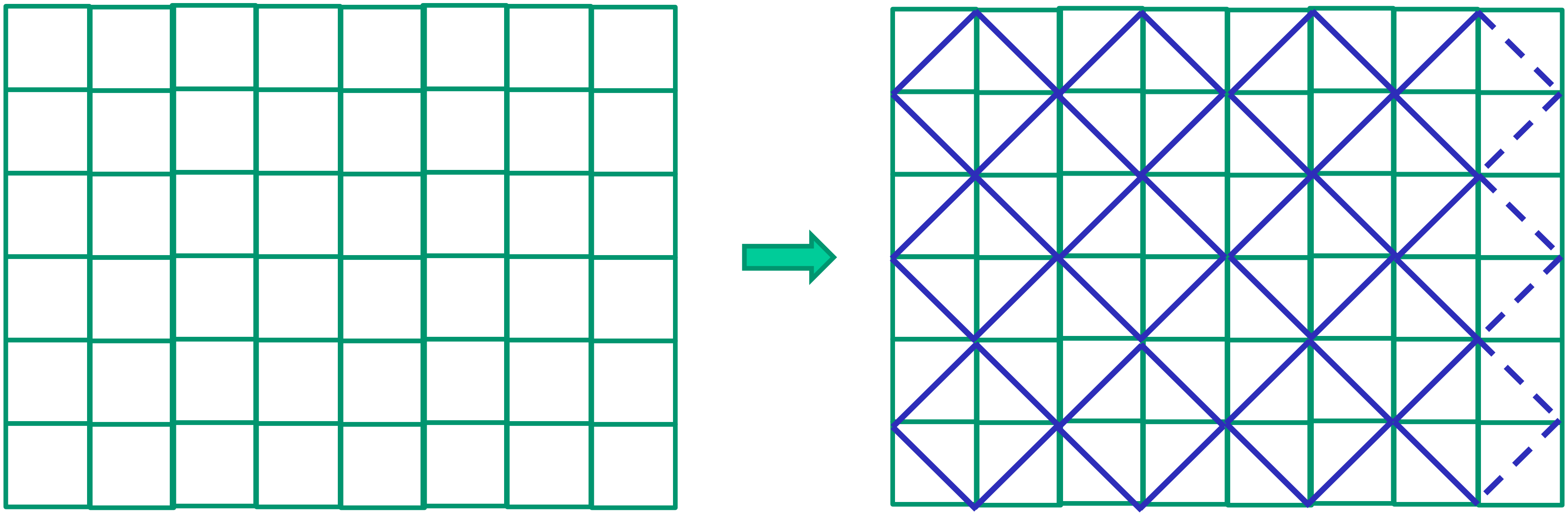
We will show that seam (r; s) = (1; 2) correspond to conformal boundary condition

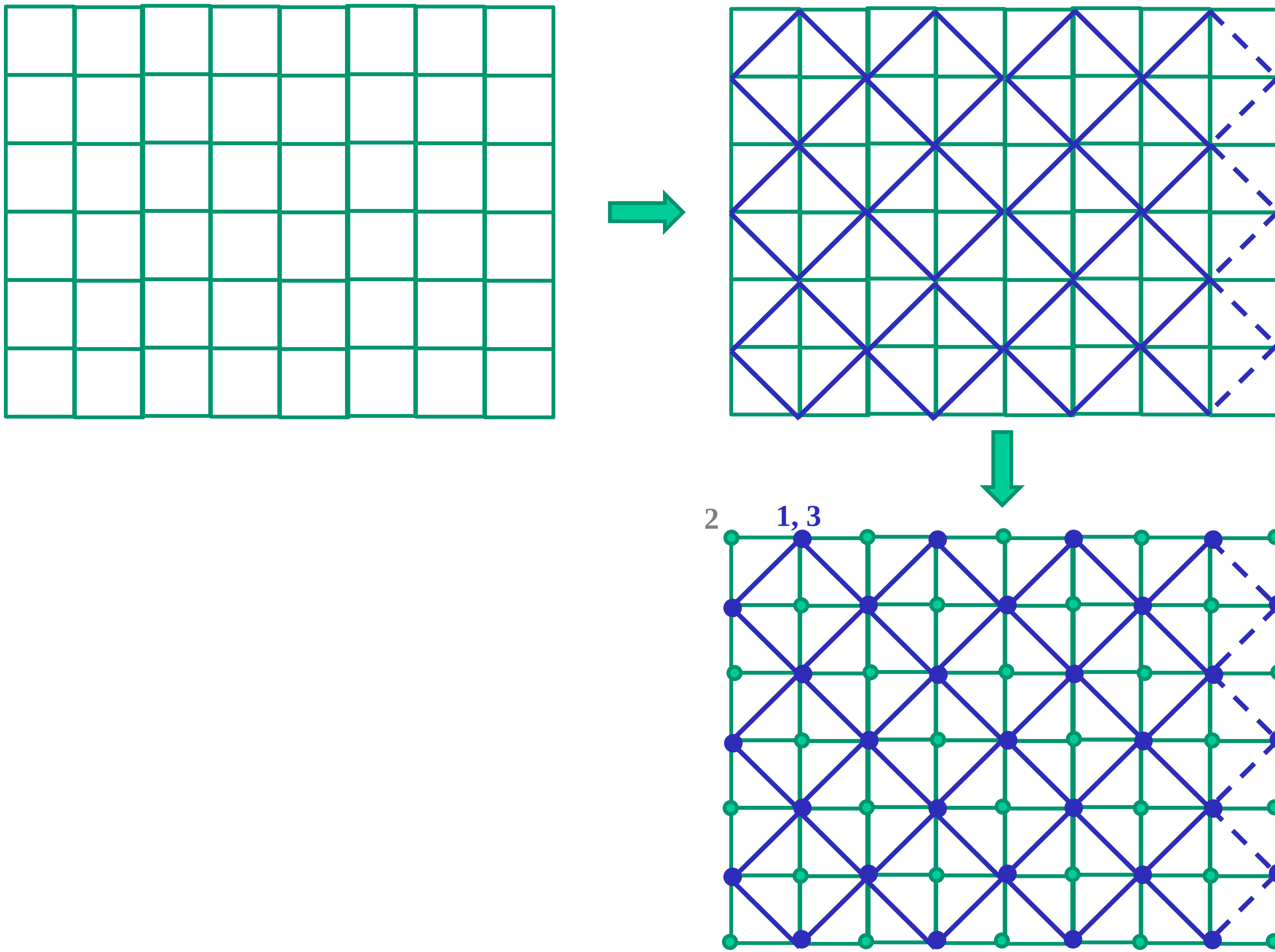
$$(\Delta, \bar{\Delta}) = \left(\frac{1}{16}, 0 \right)$$

Ising model on torus with defect line





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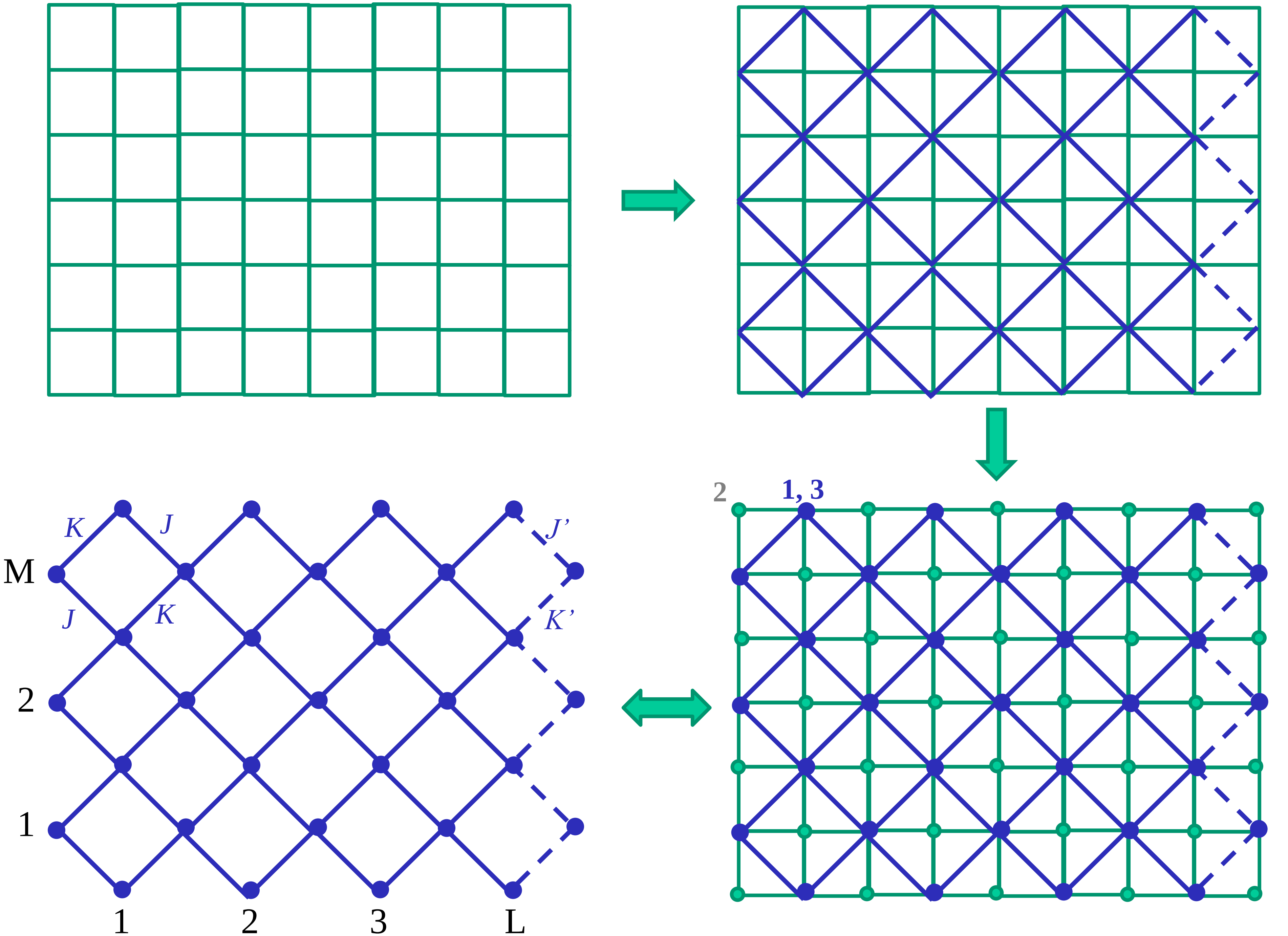

2
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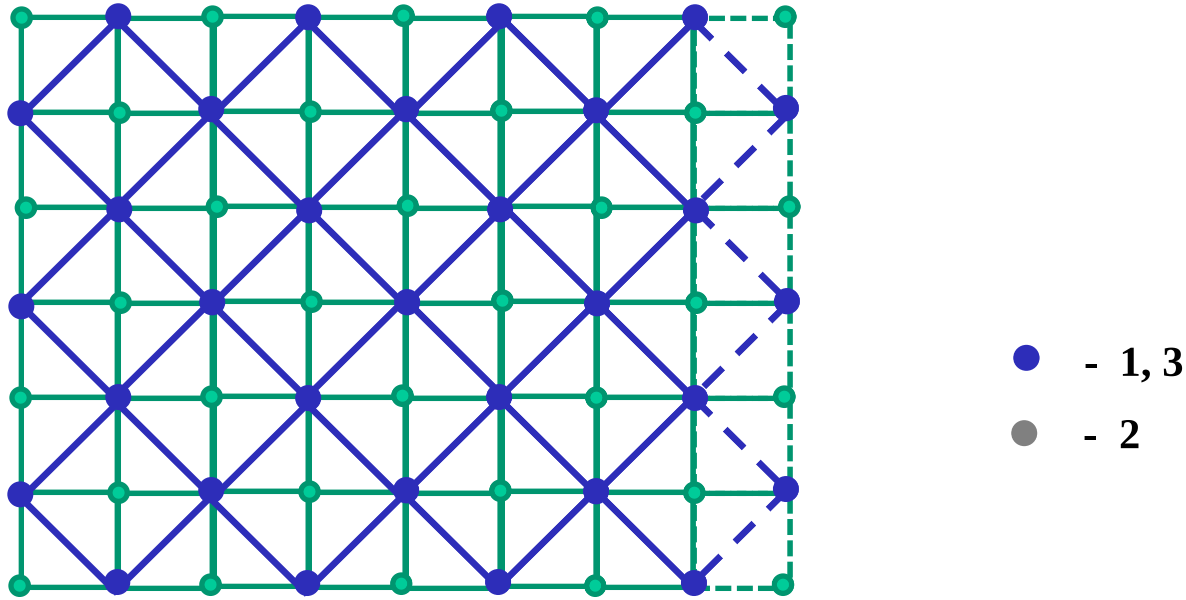
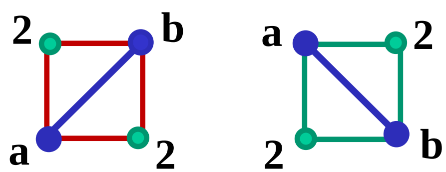


Allowed face configurations:



(a,b)∈(1,1),(3,3),(1,3),(3,1)

Boltzmann weights:

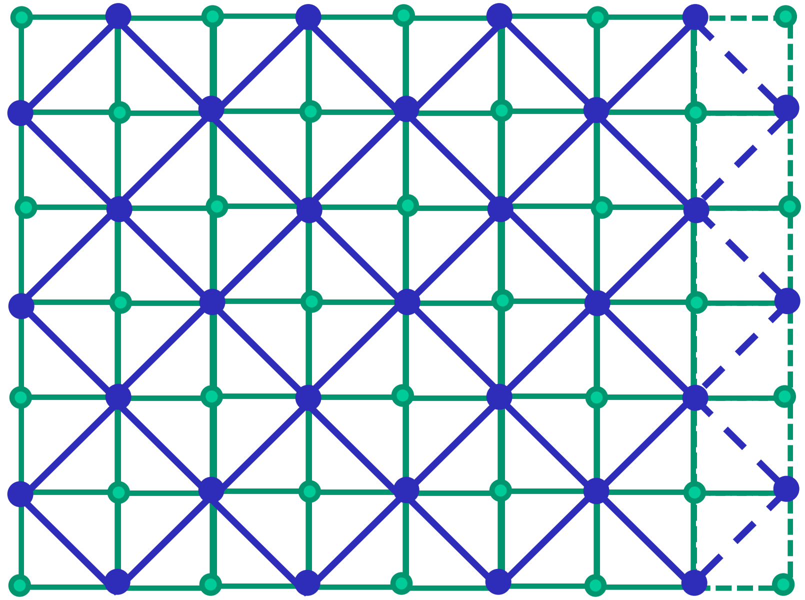
$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\bigg| u = s_1(-u)\delta_{ac} + s_0(u)\frac{\sqrt{\psi_a \psi_c}}{\psi_b}\delta_{\mathrm{bd}}$$

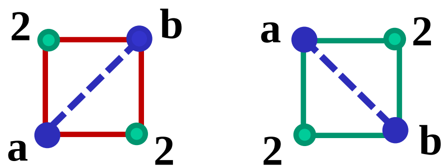
$$s_k(u) = \sqrt{2}\sin\left(u + \frac{\pi k}{4}\right)$$

u is spectral parameter

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix}$$

$\sinh(2J) = \cot(2u)$; $\sinh(2K) = \tan(2u)$





(a,b)∈(1,1),(3,3),(1,3),(3,1)

for seam (r; s) = (1; 2) we have following weights

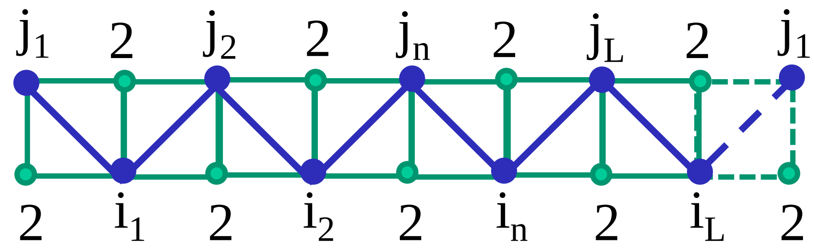
$$W^{(1,2)}\begin{pmatrix} d & c \\ a & b \end{pmatrix} = ie^{i\frac{\pi}{8}}\delta_{ac} - ie^{-i\frac{\pi}{8}}\frac{\sqrt{\psi_a\psi_c}}{\psi_b}\delta_{\mathrm{bd}}$$

For example

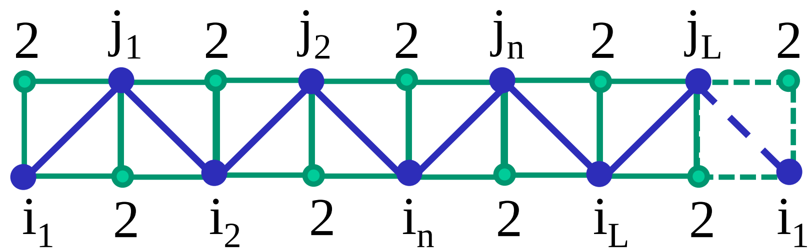
$$W^{(1,2)}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = W^{(1,2)}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = ie^{i\frac{\pi}{8}}$$

The transfer matrix of the model

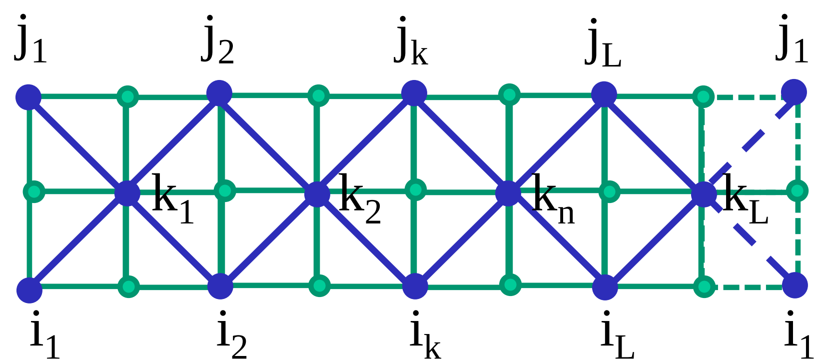
U



V



T(u)=UV



The finite size partition function can be written as

$$Z_{L_x L_y}(u) = Tr\left[T(u)^M\right] = \sum_n e^{-ME_n(u)} = \sum_n e^{-L_y \mathcal{E}_n(u)}$$

Her sum run over all eigenvalues $\lambda_n(u)$ of a transfer matrix $T(u)$ written as

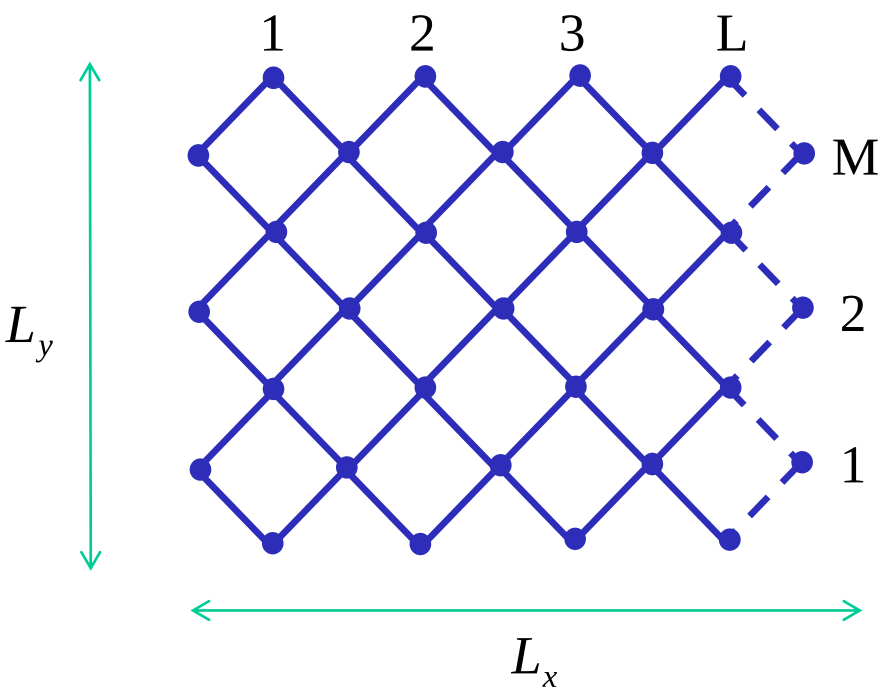
$$\lambda_n(u) = e^{-E_n(u)} = e^{-\sqrt{2}\mathcal{E}_n(u)} \qquad \mathcal{E}_n(u) = \frac{\sqrt{2}}{2} E_n(u)$$

$$\mathcal{E}_n(u) = -\frac{\sqrt{2}}{2} \ln \lambda_n$$

$\mathcal{E}_0(u)$ is the ground state energy

$\mathcal{E}_n(u)$ are the exited state energy

L_x, L_y are physical dimension of the lattice



$$L_x = \sqrt{2}L, \qquad L_y = \sqrt{2}M$$

The double transfer matrix U(u)V(v) satisfy the following functional equation

$$U(u)V\left(u+\frac{\pi}{4}\right)=\cos(2u)^{2L-1}\mathrm{I}+i(-1)^{L}\sin(2u)^{2L-1}\mathrm{R}$$

I is the Identity 2^L x 2^L matrix and **R** is a square 2^L x 2^L matrix with anti-diagonal elements equal to 1 with remaining elements equal to 0.

L=2

$$\mathrm{I}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix};\qquad \mathrm{R}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

Based on functional equation we have obtain all 2^L eigenvalues (λ) of the double transfer matrix U(u)V(v)

$$\lambda=e^{2i\left(u-\frac{\pi}{8}\right)}\prod_{k=1}^{L-1}\left(e^{2i\left(u-\frac{\pi}{8}\right)}\sin\frac{\pi(2k-1)}{2(2L-1)}+\mu_k e^{-2i\left(u-\frac{\pi}{8}\right)}\cos\frac{\pi(2k-1)}{2(2L-1)}\right)^2$$

$\mu_k=\pm1$ can be chosen arbitrary, so we have 2^{L-1} eigenvalues. The remaining 2^{L-1} eigenvalues can be found by taking the complex conjugate of λ

Since λ is complex quantity, we can represent λ as $\lambda = |\lambda| \exp(i\theta)$

$$|\lambda| = \prod_{k=1}^{L-1} \left(1 + \mu_k \sin(4u) \sin \frac{\pi(2k-1)}{2L-1} \right)$$

$$\theta = 2u - \frac{\pi}{4} + \sum_{k=1}^{L-1} \arctan \frac{\cos(4u)\cos \frac{\pi(2k-1)}{2L-1}}{\sin(4u) + \mu_k \sin \frac{\pi(2k-1)}{2L-1}}$$

The largest eigenvalue λ_0 correspond to the case when all $\mu_k=1$

$$|\lambda_0| = \prod_{k=1}^{L-1} \left(1 + \sin(4u) \sin \frac{\pi(2k-1)}{2L-1} \right) = \sqrt{\prod_{k=1}^{2L-2} \left(1 + \sin(4u) \sin \frac{\pi k}{2L-1} \right)}$$

$$\theta_0 = \frac{1}{2} \sum_{k=1}^{2L-2} (-1)^{k+1} \arctan \frac{\cos(4u)\cos \frac{\pi k}{2L-1}}{\sin(4u) + \sin \frac{\pi k}{2L-1}}$$

Asymptotic expansion of ground state

$$\mathcal{E}_0 = -\frac{\sqrt{2}}{2}\ln\lambda_0 = -\frac{\sqrt{2}}{2}\left(\ln|\lambda_0| + i\theta_0\right)$$

$$\ln|\lambda_0| = \frac{1}{2}\sum_{k=1}^{2L-2}\ln\left(1+\sin 4u \sin\frac{\pi k}{2\mathrm{L}-1}\right)$$

Using Euler-Maclaurin summation formula

$$\ln|\lambda_0| = -f_{\text{bulk}}(2L-1) - 2\sum_{k=0}^{\infty}\frac{B_{2k+2}f^{(2k+1)}(0)}{(2k+2)!}\left(\frac{\pi}{2L-1}\right)^{2k+1}$$

where function *f*(x) is given by $f(x) = \frac{1}{2}\log\left[1+\sin(4u)\sin x\right]$

$B_k = B_k(0)$ are Bernoulli numbers $B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ \ldots$

$$\ln|\lambda_0| = -f_{\text{bulk}}(2L-1) - \frac{\pi\sin 4u}{12}\frac{1}{2L-1} - \frac{\pi^3\sin 4u\cos 8u}{720}\frac{1}{(2L-1)^3} + \ldots$$

$$\theta_0 = \frac{1}{2}\sum_{k=1}^{2L-2}(-1)^{k+1}\arctan\frac{\cos(4u)\cos\dfrac{\pi k}{2L-1}}{\sin(4u)+\sin\dfrac{\pi k}{2L-1}}$$

For alternating finite sums one can use the Boole summation formula

$$\theta_0 = -\sum_{k=0}^{\infty}\frac{E_{2k+1}g^{(2k+1)}(0)}{(2k+1)!}\left(\frac{\pi}{2L-1}\right)^{2k+1}$$

where function $g(x)$ is given by $g(x) = \frac{1}{2}\arctan\frac{\cos(4u)\cos x}{\sin(4u)+\sin x}$

and E_n are the Euler numbers $E_0 = 1$, $E_1 = -1/2$, $E_3 = 1/4$, …

$$\theta_0 = -\frac{\pi\cos 4u}{4}\frac{1}{2L-1} + \frac{\pi^3\sin 4u\sin 8u}{48}\frac{1}{(2L-1)^3} + \ldots$$

$$\mathcal{E}_0 = -\frac{\sqrt{2}}{2}\ln\lambda_0 = -\frac{\sqrt{2}}{2}\ln|\lambda_0| - \frac{\sqrt{2}}{2}\, i\,\theta_0$$

$$\mathcal{E}_0 = L_x f_{\text{bulk}} + \frac{2\pi}{L_x}\left[\frac{1}{48}\sin(4u) + \frac{i}{16}\cos(4u)\right] + O\left(\frac{1}{L_x}\right)$$

$$L_x = \frac{\sqrt{2}}{2}(2L-1)$$

$$\mathcal{E}_0 = L_x f_{\text{bulk}} + \frac{2\pi}{L_x}\left[\left(-\frac{1}{24} + \Delta + \bar{\Delta}\right)\sin(4u) + i\left(\Delta - \bar{\Delta}\right)\cos(4u)\right] + O\left(\frac{1}{L_x}\right)$$

$$\begin{cases} -\frac{1}{24} + \Delta + \bar{\Delta} = \frac{1}{48} \\ \Delta - \bar{\Delta} = \frac{1}{16} \end{cases} \Rightarrow \quad \Delta = \frac{1}{16}, \qquad \bar{\Delta} = 0$$

The eigenvalue λ_p correspond to the case when $\mu_p = -1$, while all other $\mu_k = 1$

$$\lambda_p = |\lambda_p| \exp(i\theta_p)$$

$$|\lambda_p| = \prod_{\substack{k=1 \\ k\neq p}}^{L-1} \left(1+\sin(4u)\sin\frac{\pi(2k-1)}{2L-1}\right)\left(1-\sin(4u)\sin\frac{\pi(2p-1)}{2L-1}\right)$$

$$= |\lambda_0| \frac{1-\sin(4u)\sin\frac{\pi(2p-1)}{2L-1}}{1+\sin(4u)\sin\frac{\pi(2p-1)}{2L-1}}$$

$$\theta_p = \theta_0 + 2\arctan\left[\cos(4u)\tan\frac{\pi(2p-1)}{2L-1}\right]$$

For p close to L we replace p by $L-p$

$$|\lambda_{L-p}| = |\lambda_0| \frac{1-\sin(4u)\sin\frac{2\pi p}{2L-1}}{1+\sin(4u)\sin\frac{2\pi p}{2L-1}}$$

$$\theta_{L-p} = \theta_0 + 2\arctan\left[\cos(4u)\tan\frac{2\pi p}{2L-1}\right]$$

Asymptotic expansion of exited energies

$$\mathcal{E}_p = -\frac{\sqrt{2}}{2}\ln\lambda_p \qquad \mathcal{E}_{L-p} = -\frac{\sqrt{2}}{2}\ln\lambda_{L-p}$$

leading finite size correction for $\mathcal{E}_p$ and $\mathcal{E}_{L-p}$

$$\mathcal{E}_p = L_x f_{\text{bulk}} + \frac{2\pi}{L_x}\left[\left(-\frac{23}{48}+p\right)\sin 4u + i\left(\frac{9}{16}-p\right)\cos 4u\right] + O\left(\frac{1}{L_x}\right)^2$$

$$\mathcal{E}_{L-p} = L_x f_{\text{bulk}} + \frac{2\pi}{L_x}\left[\left(\frac{1}{48}+p\right)\sin 4u + i\left(\frac{1}{16}+p\right)\cos 4u\right] + O\left(\frac{1}{L_x}\right)^2$$

Conformal invariance predict that leading finite size correction should be

$$\mathcal{E}_n = L_x f_{\text{bulk}} + \frac{2\pi}{L_x}\left[\left(-\frac{1}{24}+\Delta+\overline{\Delta}+k_n+\overline{k_n}\right)\sin 4u + i\left(\Delta-\overline{\Delta}+k_n-\overline{k_n}\right)\cos 4u\right] + O\left(\frac{1}{L_x}\right)^2$$

For exited state $\mathcal{E}_p$ we have $\left(\Delta = \frac{1}{16}, k_p = 0; \overline{\Delta} = \frac{1}{2}, \overline{k}_p = p-1\right)$

For exited state $\mathcal{E}_{L-p}$ we have $\left(\Delta = \frac{1}{16}, k_{L-p} = p; \overline{\Delta} = 0, \overline{k}_{L-p} = 0\right)$

Universal amplitude ratios

Let us denote by f free energy and by ξ_p, ξ_{L-p} correlation lengths of critical Ising model

$$L_x f = -\frac{\sqrt{2}}{2}\ln|\lambda_0| = \sum_{k=1}^{\infty}\frac{a_k}{L_x^{2k-1}}$$

$$\xi_p^{-1} = \frac{\sqrt{2}}{2}\ln\left|\frac{\lambda_0}{\lambda_p}\right| = \sum_{k=1}^{\infty}\frac{b_k(p)}{L_x^{2k-1}}$$

$$\xi_{L-p}^{-1} = \frac{\sqrt{2}}{2}\ln\left|\frac{\lambda_0}{\lambda_{L-p}}\right| = \sum_{k=1}^{\infty}\frac{c_k(p)}{L_x^{2k-1}}$$

$$a_k = \frac{\pi^{2k-1}B_{2k}}{2^{k-1}(2k)!}f^{(2k-1)}(0)$$

$$b_k(p) = \frac{\pi^{2k-1}(2p-1)^{2k-1}}{2^{k-2}(2k-1)!}f^{(2k-1)}(0)$$

$$c_k(p) = \frac{\pi^{2k-1}p^{2k-1}}{2^{-k-1}(2k-1)!}f^{(2k-1)}(0)$$

$$f(x) = \frac{1}{2}\log[1+\sin(4u)\sin x]$$

$$r_p(k) = \frac{b_k(p)}{a_k} = \frac{4k(2p-1)^{2k-1}}{B_{2k}}$$

$$r_{L-p}(k) = \frac{c_k(p)}{a_k} = \frac{4k(2p)^{2k-1}}{B_{2k}}$$

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The coefficients $a_1, b_1(p), c_1(p)$ are universal, since they related to the central charge (c = 1/2), conformal weights of ground state $\left(\Delta = 1/16, \bar{\Delta} = 0\right)$ and the scaling dimensions x_p and y_p

$$a_1 = 2\pi\left(-\frac{c}{12} + \Delta + \bar{\Delta}\right)\zeta = \frac{\pi}{24}\sin 4u$$

$$b_1(p) = 2\pi x_p \zeta = (2p-1)\pi \sin 4u$$

$$c_1(p) = 2\pi y_p \zeta = 2p\pi \sin 4u$$

$\Rightarrow$

$$\frac{b_1(p)}{a_1} = 24(2p-1)$$

$$\frac{c_1(p)}{a_1} = 48p$$

$$r_p(k) = \frac{b_k(p)}{a_k} = \frac{4k(2p-1)^{2k-1}}{B_{2k}}$$

$$r_{L-p}(k) = \frac{c_k(p)}{a_k} = \frac{4k(2p)^{2k-1}}{B_{2k}}$$

k=1 $\Rightarrow$

$$r_p(1) = \frac{b_1(p)}{a_1} = 24(2p-1)$$

$$r_{L-p}(1) = \frac{c_1(p)}{a_1} = 48p$$

$$r_p(k) = \frac{b_k(p)}{a_k} = \frac{4k(2p-1)^{2k-1}}{B_{2k}}$$

$$r_{L-p}(k) = \frac{c_k(p)}{a_k} = \frac{4k(2p)^{2k-1}}{B_{2k}}$$

k=2 ⟹

$$r_p(2) = \frac{b_2(p)}{a_2} = -240(2p-1)^3$$

$$r_{L-p}(2) = \frac{c_2(p)}{a_2} = -1920p^3$$

The leading finite-size corrections $1/N^3$ can be described by the Hamiltonian with a single perturbative conformal field ϕ_1 with scaling dimension $x_1 = 4$.

$$\phi_1 = L_{-2}^2 + \overline{L}_{-2}^2$$

The energy gap ($\Delta_n = E_n - E_0$) and the ground-state energy (E_0) can be written as

$$\Delta_n = \frac{2\pi}{N} x_n + 2\pi g_1 (C_{n1n} - C_{010}) \left(\frac{2\pi}{N}\right)^3 + \ldots$$

$$E_0 = E_{0,c} + 2\pi g_1 C_{010} \left(\frac{2\pi}{N}\right)^3 + \ldots$$

⟹

$$r_n(2) = \frac{C_{n1n} - C_{010}}{C_{010}}$$

$$\langle 0 | L_{-2}^2 + \overline{L}_{-2}^2 | 0 \rangle = \left(\frac{2\pi}{N}\right)^4 C_{010}, \quad \langle n | L_{-2}^2 + \overline{L}_{-2}^2 | n \rangle = \left(\frac{2\pi}{N}\right)^4 C_{n1n}$$

The excited states $\mathcal{E}_p : \left(\Delta = \frac{1}{16}, r = 0; \bar{\Delta} = \frac{1}{2}, \bar{r} = p - 1 \right)$ can be identified as descendent states of the primary field $\psi(z)$ with conformal weight $\Delta=1/2$.

The excited states $\mathcal{E}_{L-p} : \left(\Delta = \frac{1}{16}, r = p; \bar{\Delta} = 0, \bar{r} = 0 \right)$ can be identified as descendent states generated by the OPE of the primary fields $\psi(z)\sigma(0)$ where $\sigma(0)$ is the spin operator with conformal dimension $\Delta = 1/16$

The universal structure constants C_{n1n} for descendent states $\left(L_{-1}^{n}\middle|\Delta\right\rangle)$ generated by primary field $\psi(z)$ with conformal dimension Δ have already computed by Reinicke (*J. Phys. A 20 (1987) 5325*)

$$C_{n1n} = \left(\frac{c}{24}\right)^2 + \frac{11c}{1440} + (\Delta + r)\left(\Delta - \frac{2+c}{12} + \frac{r(2\Delta + r)(5\Delta + 1)}{(\Delta+1)(2\Delta+1)}\right) + \frac{r}{30}\left[r^2(5c-8) - 5c - 28\right]\delta_{\Delta,0} + \left(\Delta \to \bar{\Delta}, r \to \bar{r}\right)$$

ground state $\mathcal{E}_0 : \Delta = \frac{1}{16}, r = 0; \bar{\Delta} = 0, \bar{r} = 0$

$$C_{010} = -\frac{7}{11520}$$

exited state $\mathcal{E}_p : \Delta = \frac{1}{16}, r = 0; \bar{\Delta} = \frac{1}{2}, \bar{r} = p - 1$

$$C_{p1p} = -\frac{7}{11520} + \frac{7(2p-1)^3}{48}$$

$$r_p = \frac{C_{p1p} - C_{010}}{C_{010}} = -240(2p-1)^3 = r_p(2)$$

Naive application for the exited state $\mathcal{E}_{L-p} : \left(\Delta = \frac{1}{16}, r = p; \bar{\Delta} = 0, \bar{r} = 0\right)$ lead to

$$C_{(L-p)1(L-p)} = -\frac{7}{11520} + \frac{7p(16p^2 + 3p - 2)}{102}$$

$$r_{L-p} = \frac{C_{(L-p)1(L-p)} - C_{010}}{C_{010}} = -\frac{1920p\left(16p^2 + 3p - 1\right)}{17} \neq r_{L-p}(2) = -1920p^3$$

We have calculate the universal structure constants C_{n1n} for descendent states generated by the OPE of the primary fields $\psi(z)\sigma(0)$

$$C_{n1n} = -\frac{7}{1440} + \frac{7p^3}{6} + \left(p \to \bar{p}\right),$$

Let us now consider the exited state $\mathcal{E}_{L-p} : \left(\Delta = \frac{1}{16}, r = p; \bar{\Delta} = 0, \bar{r} = 0 \right)$

The universal structure constant $C_{(L\text{-}p)1(L\text{-}p)}$ consist from holomorphic $C_{L\text{-}p}$ part with $\Delta_0 = 1/16$ and $r = p$ and anti-holomorphic $\bar{C}_{L-p}$ part with $\bar{\Delta}_0 = 0$ and $\bar{r} = 0$

$$C_{(L-p)1(L-p)} = C_{L-p} + \bar{C}_{L-p}$$

$$C_{L-p} = -\frac{7}{1440} + \frac{7p^3}{6}$$

$$\bar{C}_{L-p} = \frac{49}{11520}$$

$$\Longrightarrow \quad C_{(L-p)1(L-p)} = -\frac{7}{11520} + \frac{7p^3}{6}$$

For the ground state $C_{010} = C_0 + \bar{C}_0 = -\frac{7}{1440} + \frac{49}{11520} = -\frac{7}{11520}$

$$r_{L-p} = \frac{C_{(L-p)1(L-p)} - C_{010}}{C_{010}} = -1920p^3 = r_{L-p}(2), \quad \text{for all } p$$

Summary

- We have solve exactly the critical Ising model on torus with specific defect line.
- We have obtain the new set of universal amplitude ratios
- We have show that the value of the universal amplitude ratios can be obtained from conformal field theory.

Thank you for your attention!