Random quantum systems with long-range interactions

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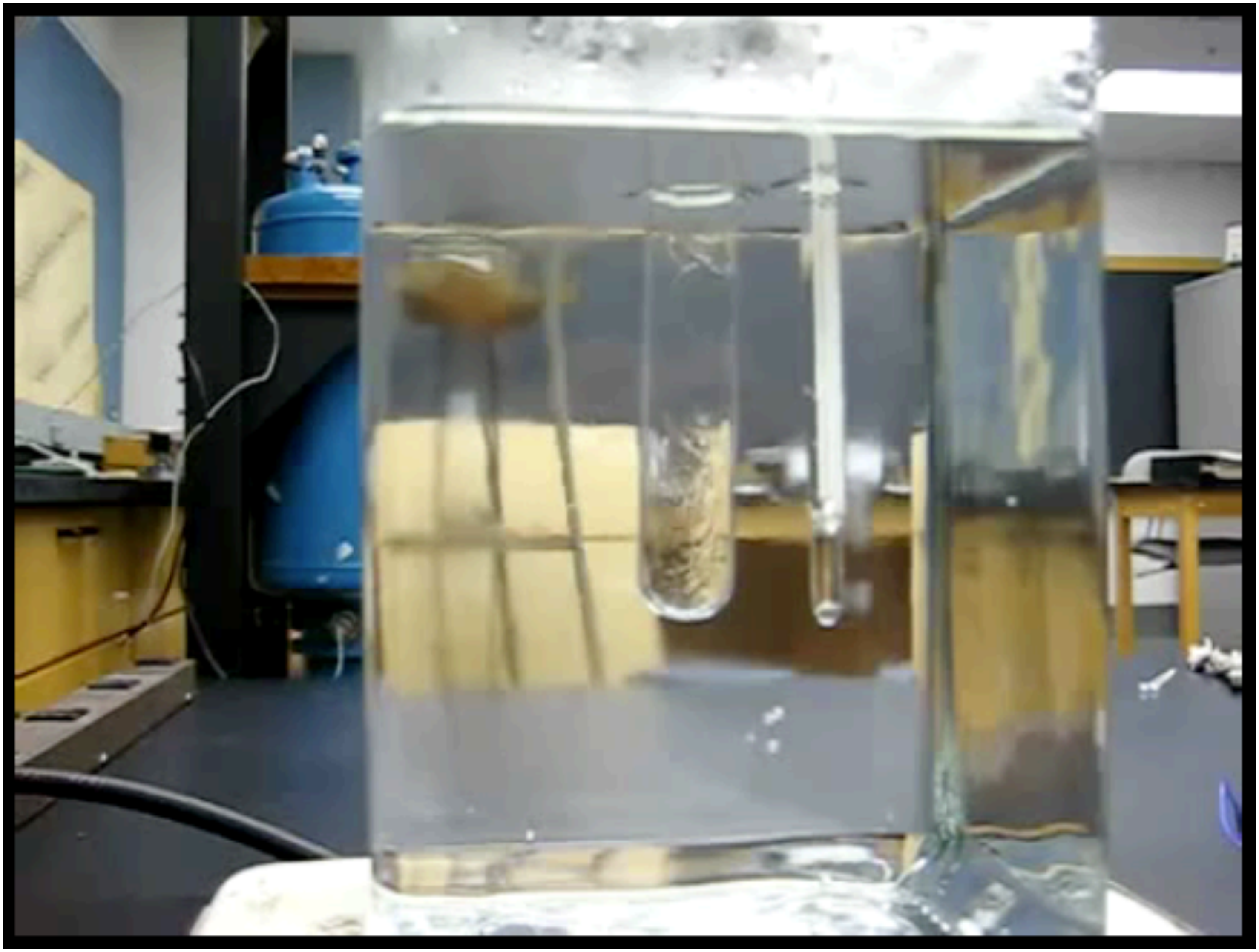
Thermal fluctuations

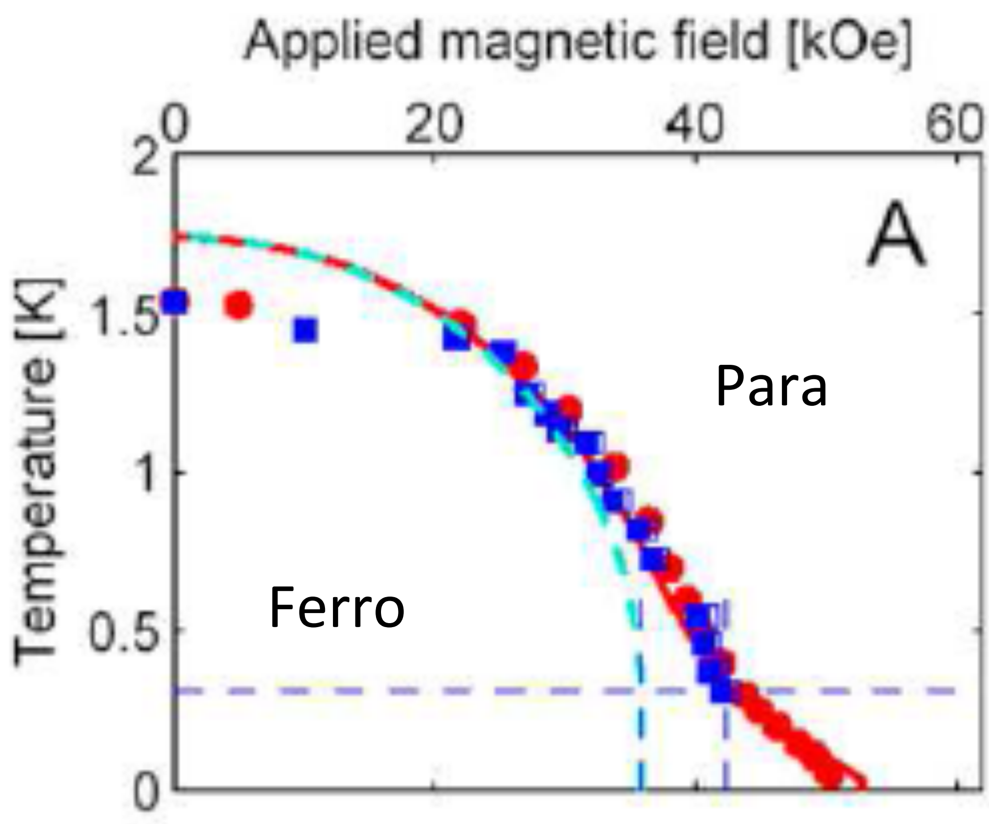


Critical opalescence

Thermal & quantum fluctuations

$LiHoF_4$ dipol-bonded quantum Ising ferromagnet

$$\mathcal{H} = -\sum_{i,j} J_{ij}\sigma_i^x\sigma_j^x - H_t^2\sum_i \sigma_i^z$$



(After Ronnow et al., 2005.)

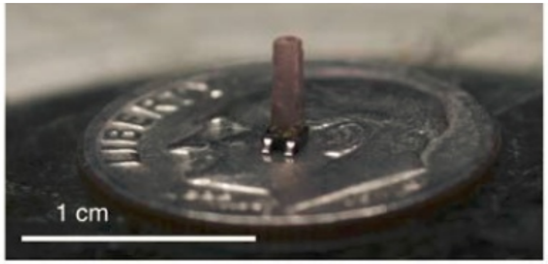
Thermal & quantum & disorder fluctuations

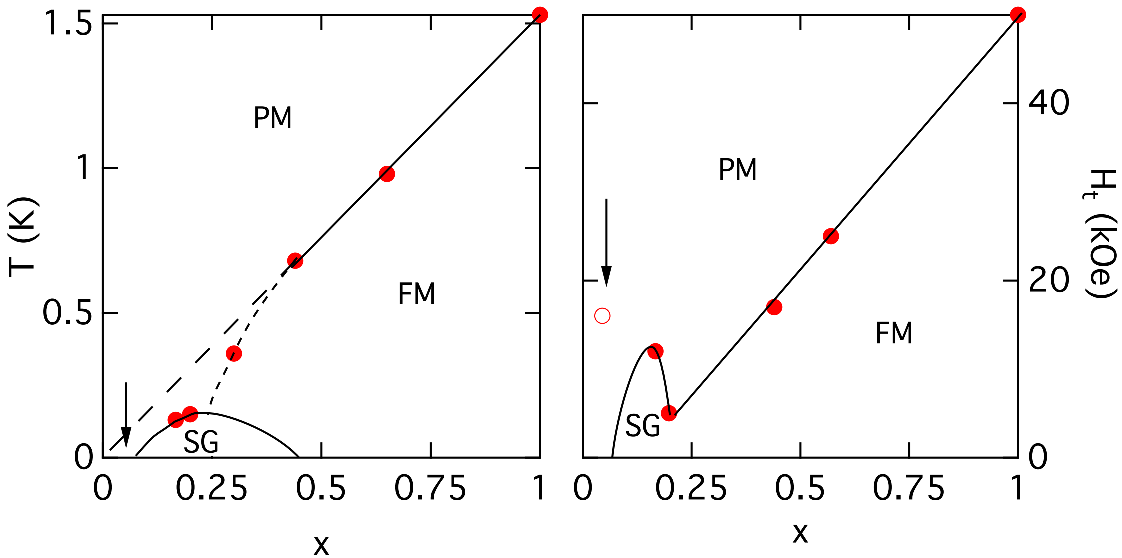
$LiHo_xY_{1-x}F_4$ dipol-bonded diluted quantum Ising ferromagnet

$$\mathcal{H} = -\sum_{i,j} J_{ij}\epsilon_i\epsilon_j\sigma_i^x\sigma_j^x - H_t^2\sum_i \sigma_i^z$$

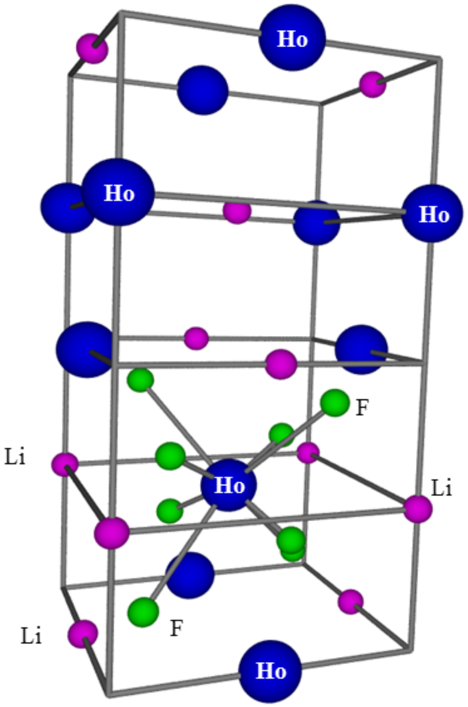
$\epsilon_i = 1, \quad x \quad \text{probability}$

$\epsilon_i = 0, \quad 1-x \quad \text{probability}$





(After Ancona-Torres et al, 2008)



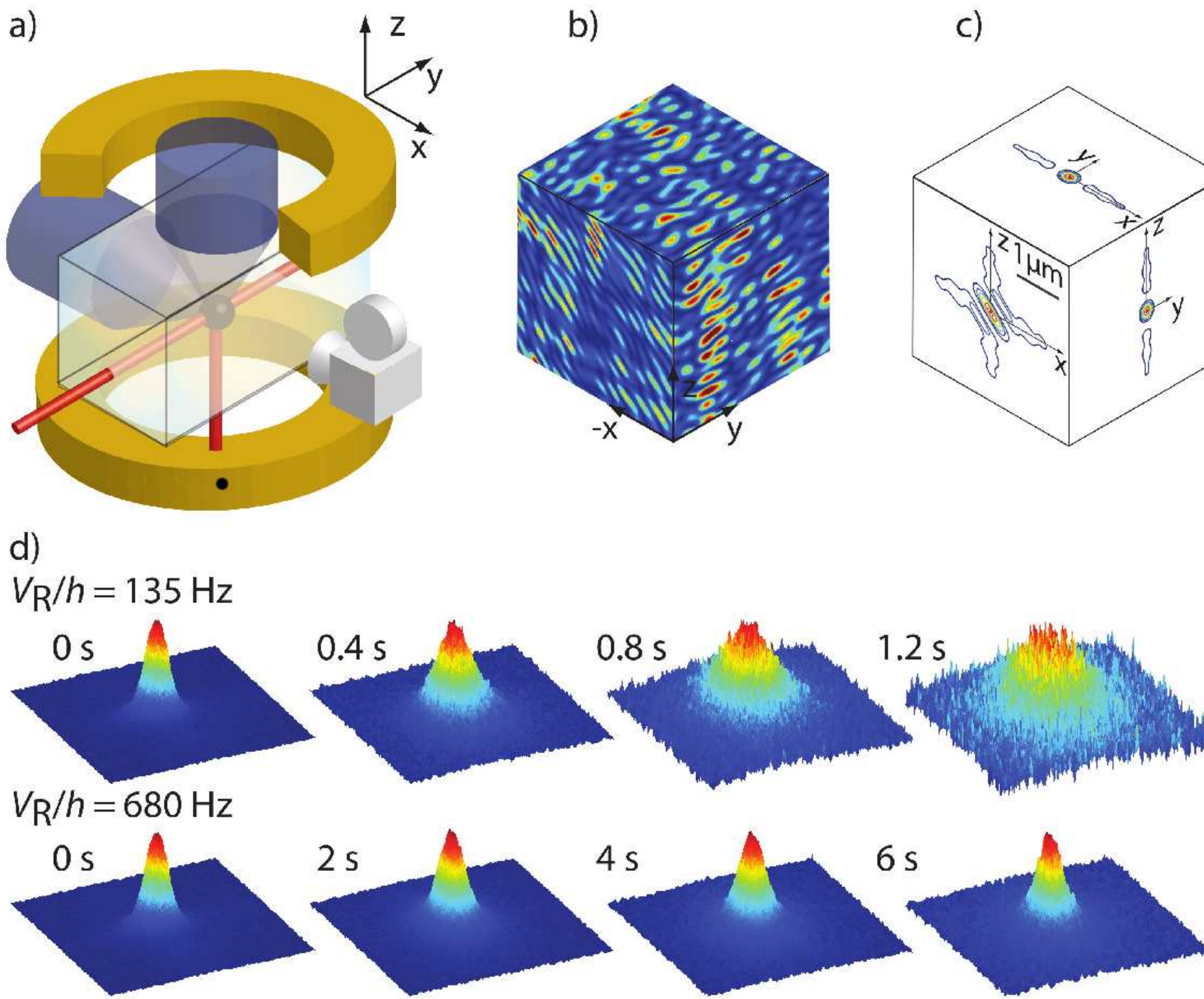
$a = b = 5.176$ Å

$c = 10.75$ Å

Quantum & disorder fluctuations

3D Anderson localisation with cold atoms

Palaiseau (Aspect group)



disorder

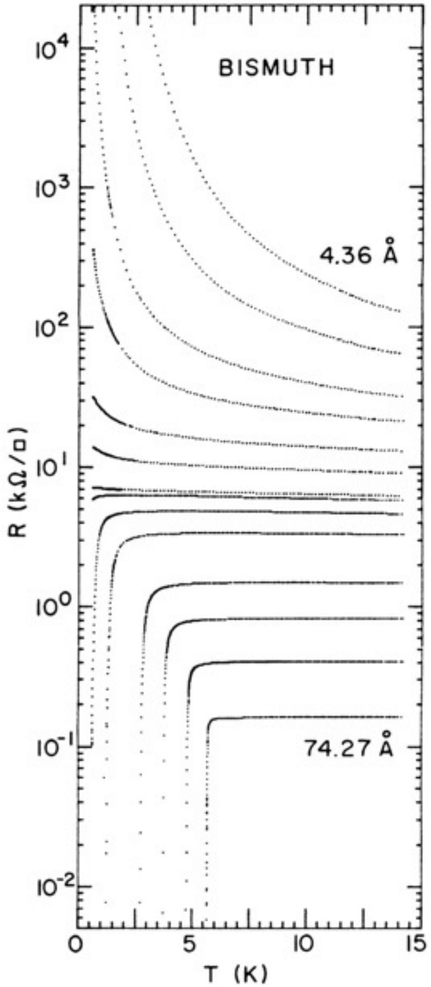
weak

strong

Quantum & disorder fluctuations

2D superconductor-insulator transition

width

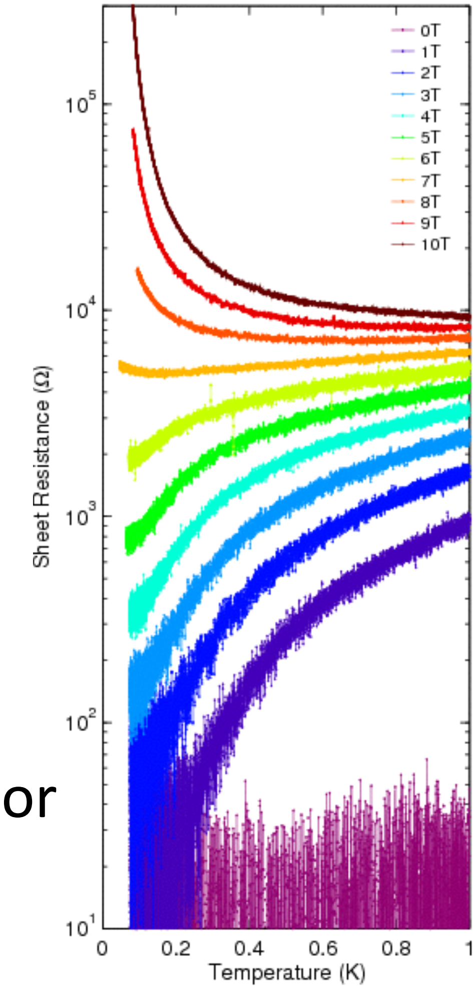


(Haviland, Liu and Goldman)

magnetic field

insulator

superconductor



(Shahar)

Random quantum Ising model –
short-range interactions

$$\mathcal{H} = -\sum_{(ij)} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i\sigma_i^z$$

J_{ij}, h_i random variables

(ij) nearest neighbours

In a finite system of linear size L :

Statics: magnetization $[m]_{av}$

O(1) | L^{-x} | $\exp(-L^d)$

ordered phase critical point disordered phase

Dynamics : (energy gap) $\exp(-L^d)$ $\exp(-L^{\psi})$ $L^{-z(\delta)}$ O(1) $[\ln(h)]_{\mathrm{av}} - [\ln(J)]_{\mathrm{av}}$

Griffiths phase

Theoretical studies

$\epsilon-$expansion does not work

quantum MC - possible

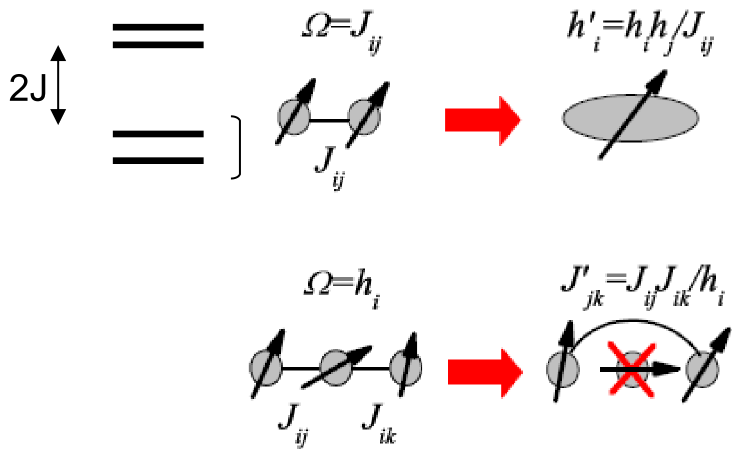
Stong disorder RG recommended

Strong Disorder RG

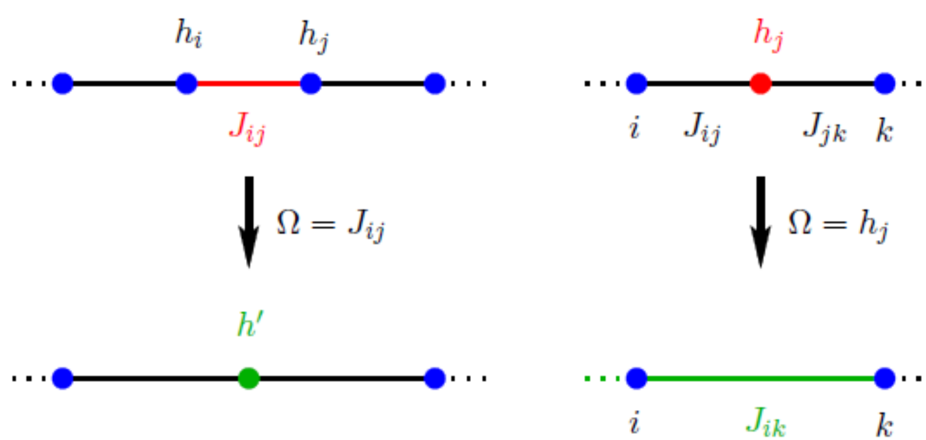
$$\mathcal{H} = -\sum_{(i,j)} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i \sigma_i^z$$

Local renormalisation: quantum and disorder fluctuations are treated at the same time

Rules:



chain → chain



D. S. Fisher (1994): analytical solution in 1D at the critical point

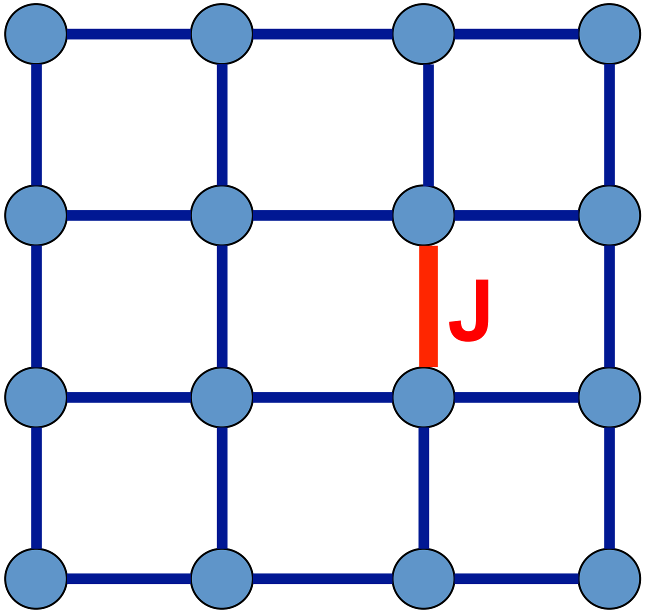
Infinite disorder fixed point:

- the ratio of two effective couplings tends to infinity
- the renormalization steps are asymptotically exact

What does happen in higher dimension?

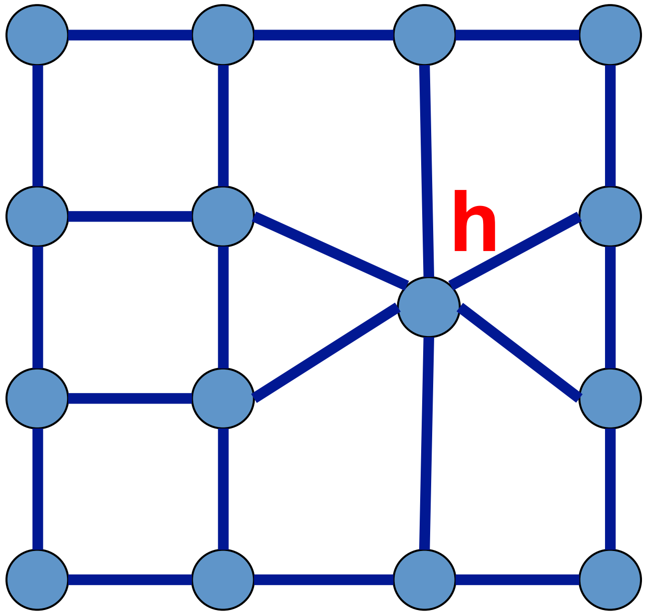
Renormalisation in higher dimensions

J is the strongest



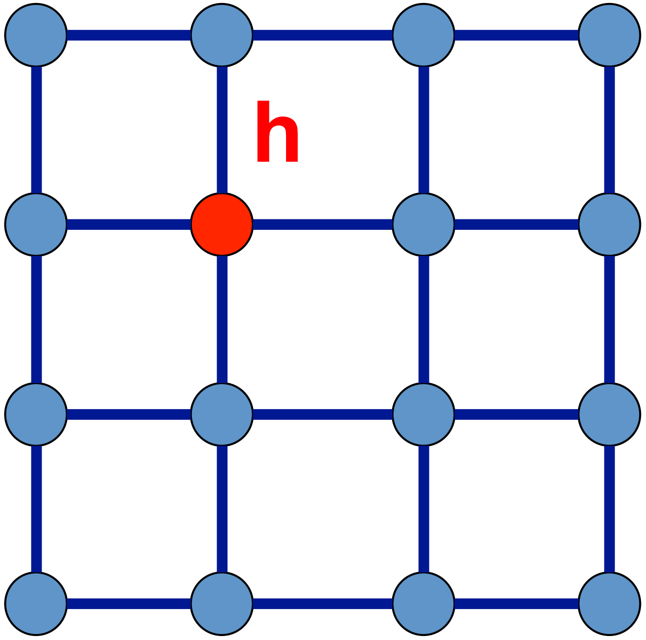
Renormalisation in higher dimensions

J is the strongest



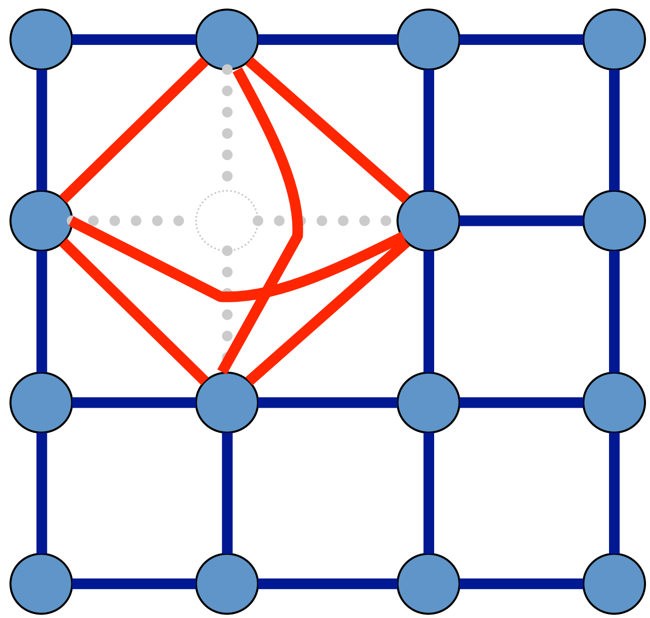
Renormalisation in higher dimensions

h is the strongest

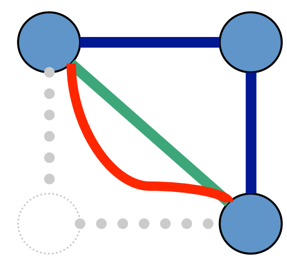


Renormalisation in higher dimensions

h is the strongest



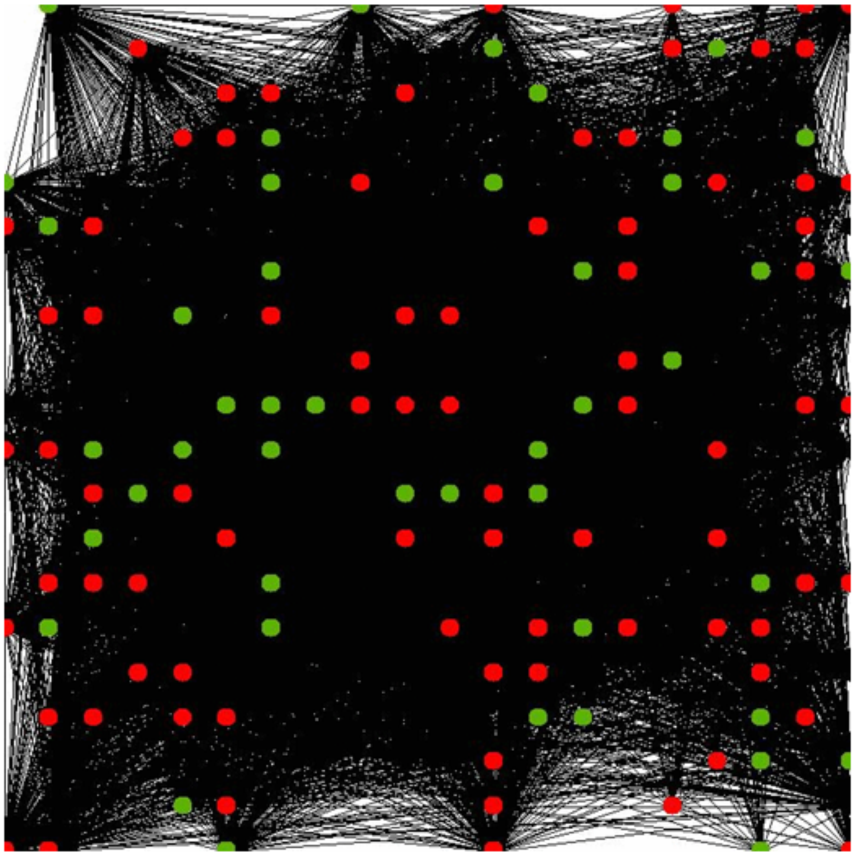
If there was a coupling
maximum rule



In each step the number of spins is reduced,
but the number of couplings could strongly increase!

Efficient RG algorithm

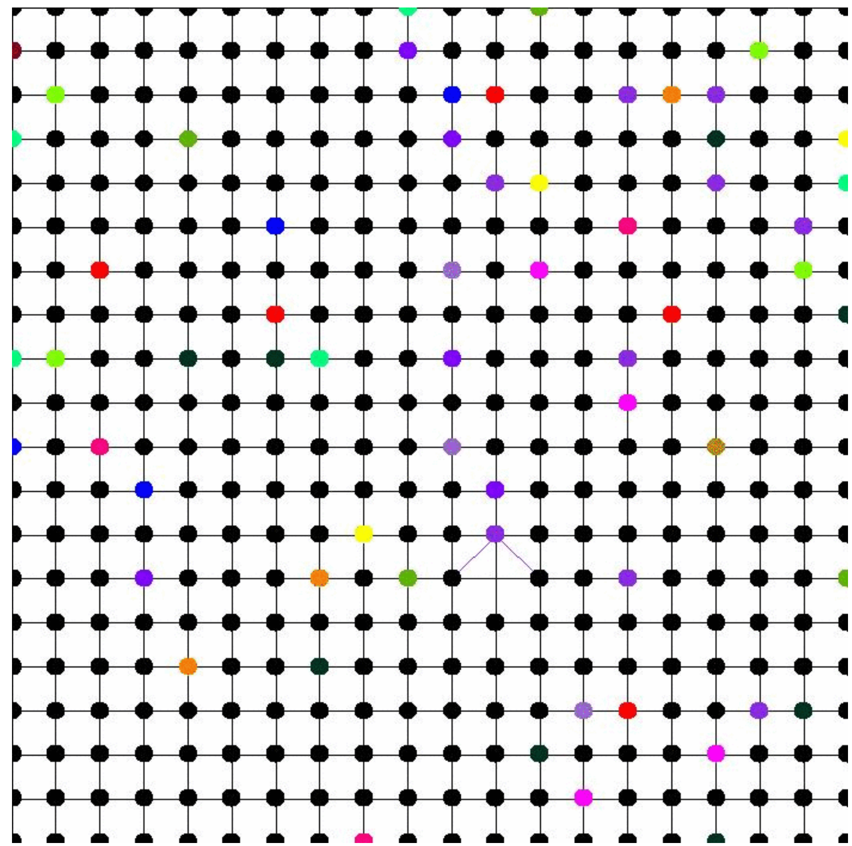
„Traditional"



$$t \sim N^3$$

(Kovács & Iglói, 2011a,b)

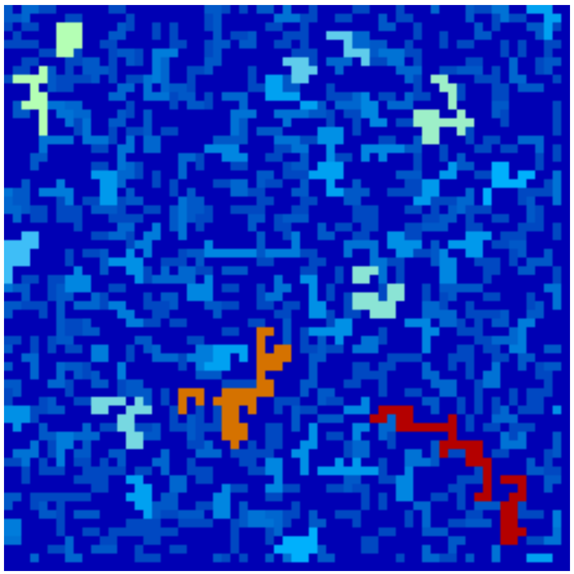
„Novel"



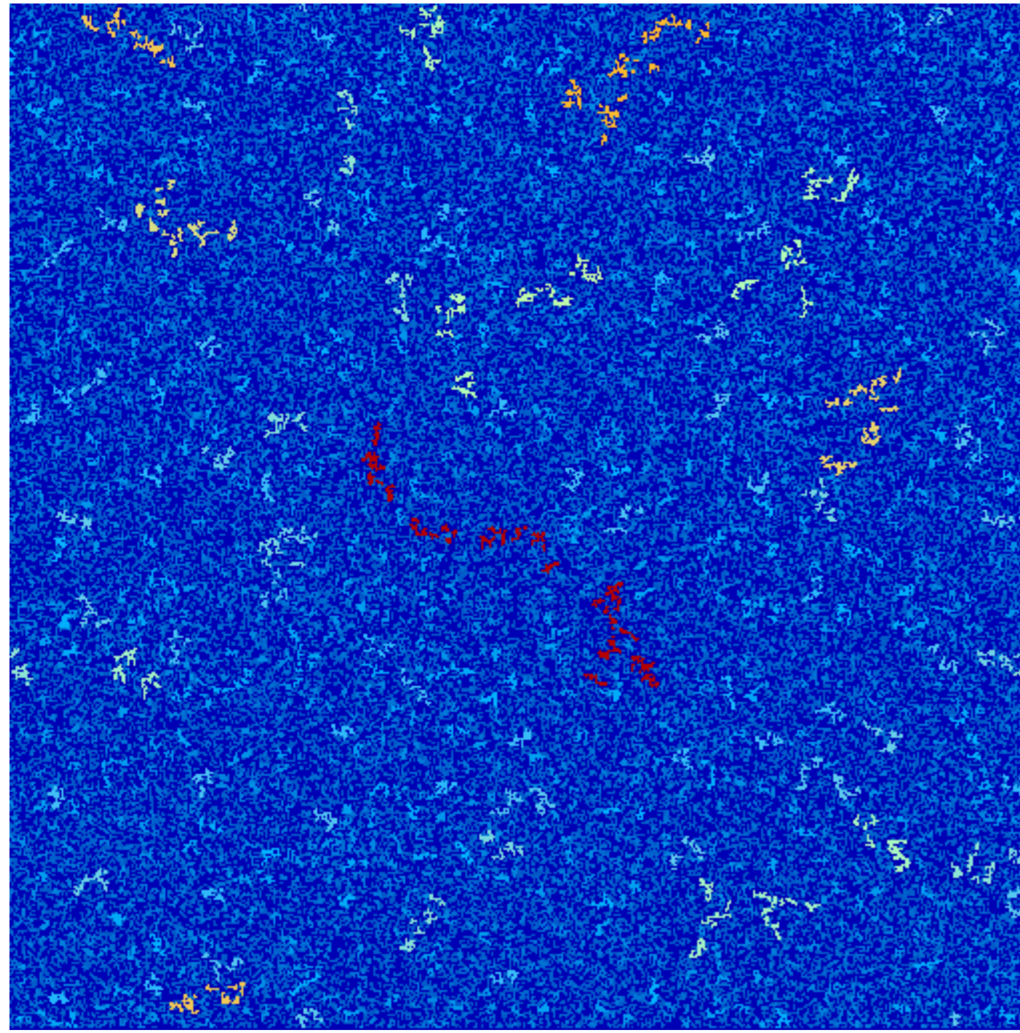
$$t \sim N \log N$$

The number of couplings does not increase

Structure of clusters



L=64



L=512

Physical quantities

- Fractal dimension, d_f
- Magnetization

$$m \sim L^{d_f - d}$$

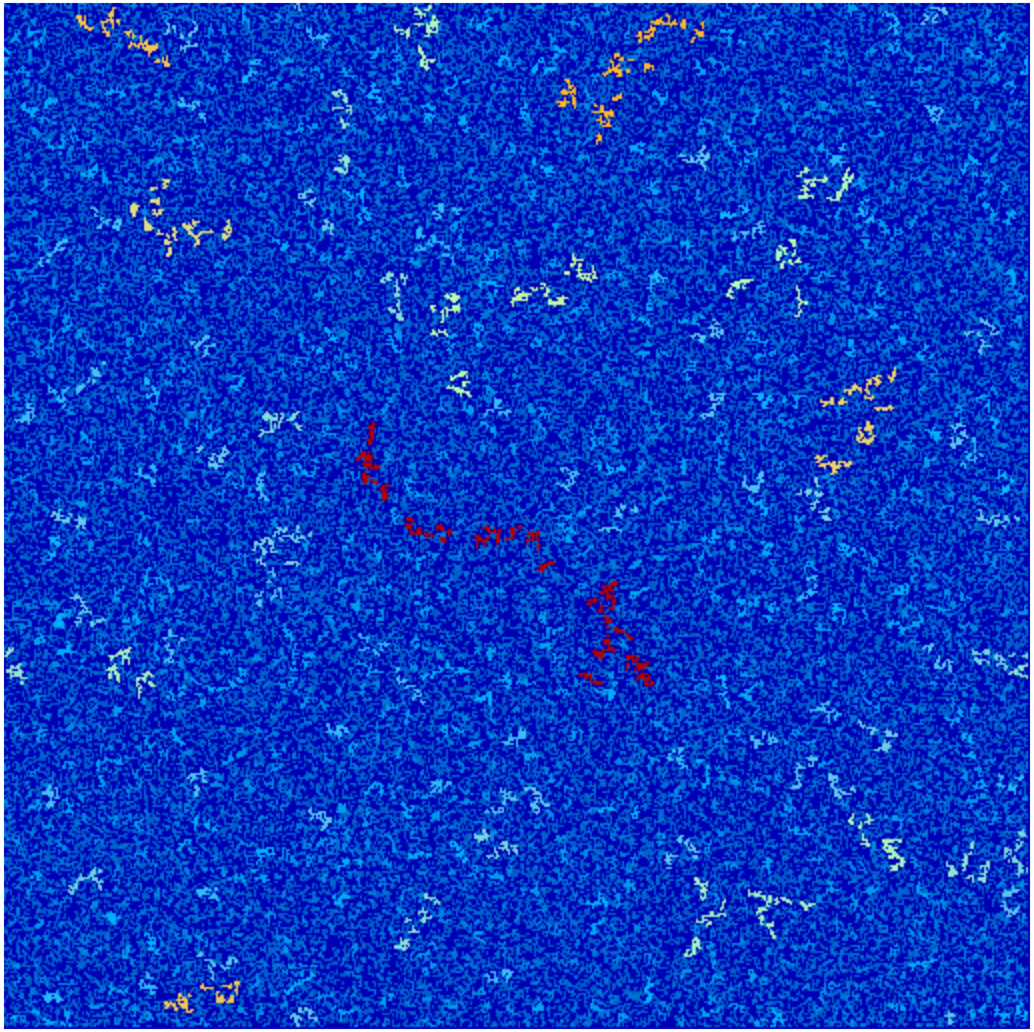
- Extension of clusters
- Correlation length

$$\xi \sim |\delta|^{-\nu}$$

- energy of clusters
- Energy gap

$$\ln \epsilon \sim L^{\psi}$$

(Kovács & Iglói, 2010)



L=512

Long-range forces

$$\mathcal{H} = -\sum_{i \neq j} \frac{b_{ij}}{r_{ij}^{\alpha}} \sigma_i^x \sigma_j^x - \sum_i h_i \sigma_i^z$$

with $\alpha > d$ (extensive energy)

b_{ij} and h_i i.i.d. random numbers

In d=3 possible relation with $\mathrm{LiHo_xY_{1-x}F_4}$

Numerical strong disorder RG study with the maximum rule

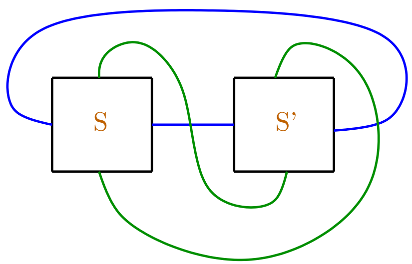
Juhász, Kovács & Iglói:

EPL **107**, 47008 (2014)
Phys. Rev. E **91**, 032815 (2015)

Phys. Rev. B**93**, 184203 (2016)

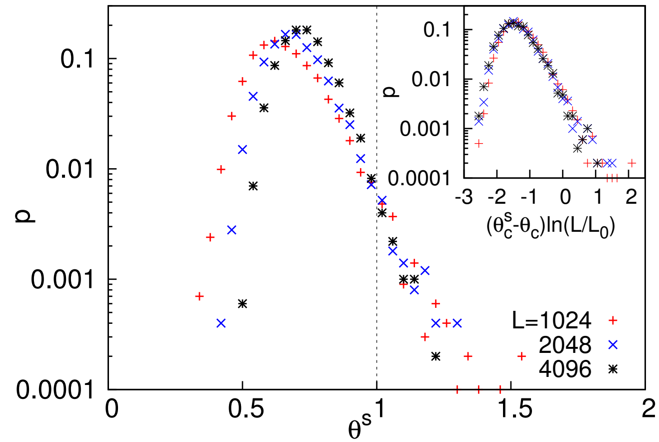
Critical point
Scaling of lengths

- Finite-size critical points - $\theta_c(S,L)$
  - two-copies of the same sample (S and S') are coupled together



  - continuously increase θ and monitor the clusters, which are built of identical sites in the copies
  - at $\theta_c(S,L)$ the last correlated cluster disappears, thus for $\theta > \theta_c(S,L)$ we are in the paramagnetic phase

- Distribution of pseudocritical points



- Finite-size scaling
  - shift of the mean: (conv. scaling)

$$\left|\theta_c - \overline{\theta_c}(L)\right| \sim 1/\ln L \qquad (L^{-1/\nu_s})$$

  - width of the distribution:

$$\Delta\theta_c(L) \sim 1/\ln L \qquad (L^{-1/\nu_w})$$

  - KT-like scaling of the correlation length

$$\boxed{\xi \sim \exp\left(\text{const}/|\theta - \theta_\text{c}|\right)} \qquad \xi \sim |\theta - \theta_c|^{-\nu}$$

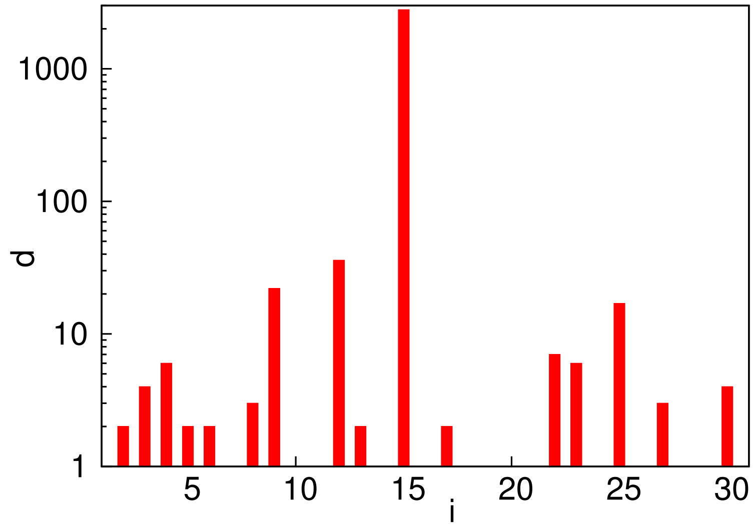
Critical point: numerical SDRG results

ratio of the frequency of the bond and field decimations at the critical point

$$r_{\theta_c}(L) = \frac{N^{\#}_{bond}(L)}{N^{\#}_{field}(L)} \sim 1/\ln^2(L)$$

SR system $(\sim \mathcal{O}(1))$

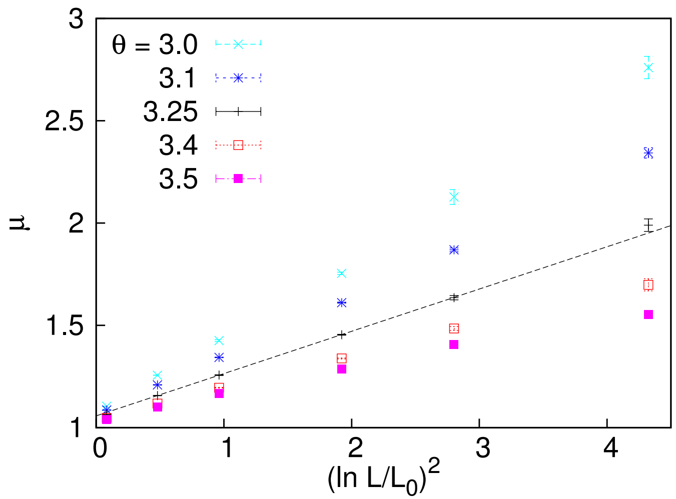
structure of the largest non-decimated cluster: (μ=32 and L=8192)



average magnetic moment of the last remaining cluster

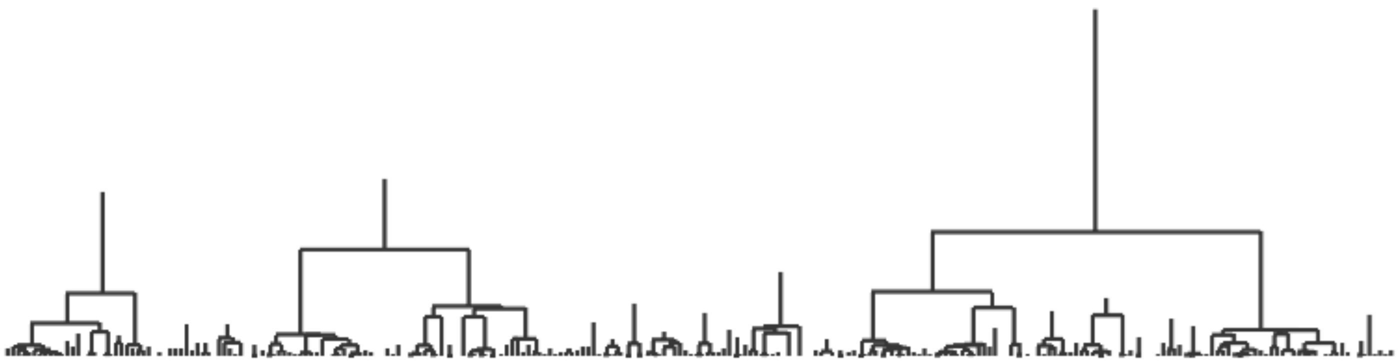
$$\mu(L) \sim \ln^2 L$$

$(\sim L^{d_f})$ SR system

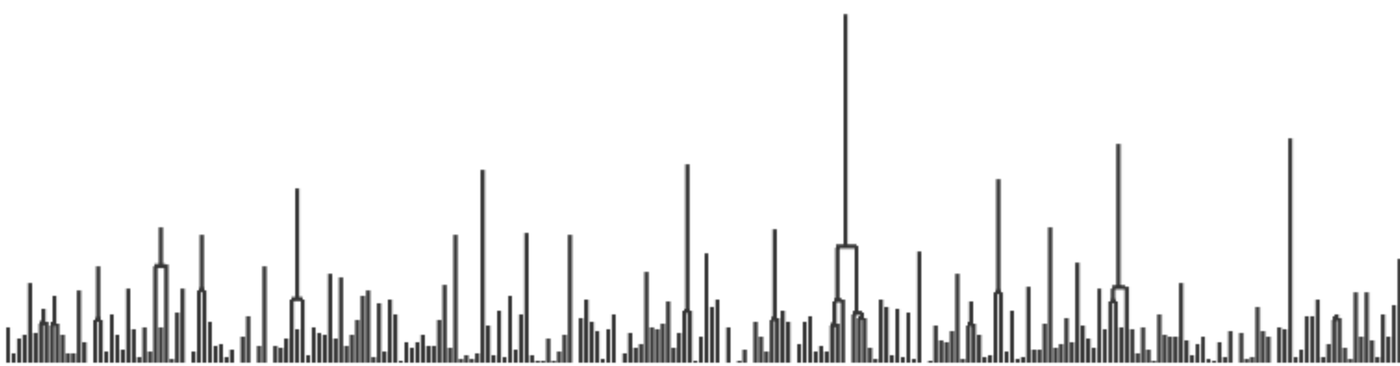


3D

Critical point: numerical SDRG results



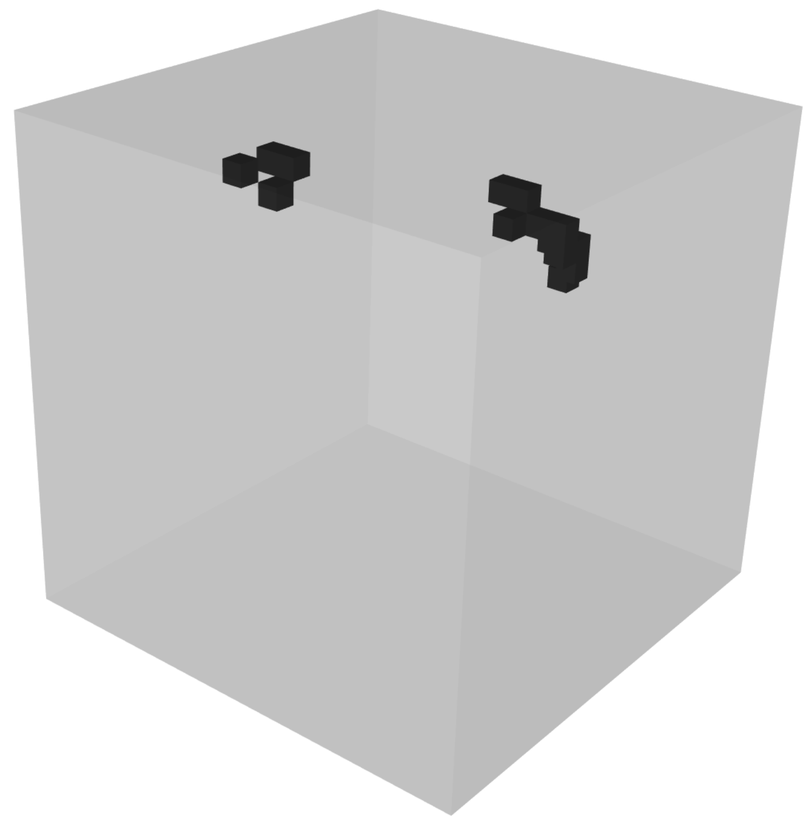
1D SR model: decimated spin-vertical spike
fused spins - horizontal line



1D LR model: dominantly field decimations

Largest spin clusters

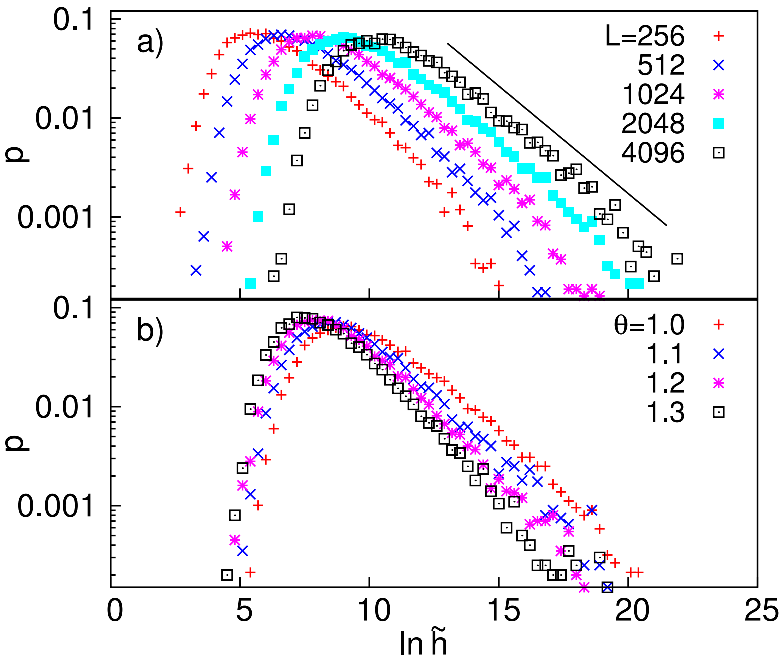
2D



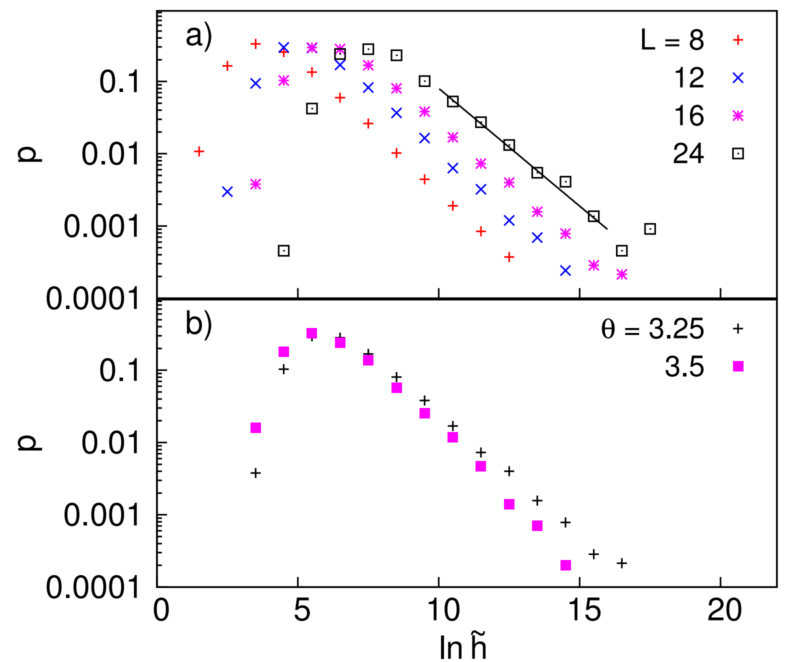
3D

Sparse and quasi-1D

Energy scaling



1D



3D

$$\epsilon \sim L^{-z(\theta)}$$ strong disorder fixed point $$z_c \equiv z(\theta_c) \simeq \alpha$$

SR: $$\ln \epsilon \sim L^{\psi}$$ Infinite disorder fixed point $$z_c = \infty$$

Analytical derivation of the results - Primary model

- observations in the RG procedure

  1. almost always transverse fields are decimated

  2. after a field decimation, the maximum rule leads almost always to $\tilde{J}_{jk} = J_{jk}$

  3. the extension w_i of (non-decimated) clusters are typically much smaller than the distances between them

- construction of the primary model

  - take $b_{ij} = b = 1$, but let h_i random

  - according to $2)$ we have from the additivity of the bond lengths:

    $$\tilde{J}_{i-1,i+1}^{-1/\alpha} = J_{i-1,i}^{-1/\alpha} + J_{i,i+1}^{-1/\alpha} + w_i$$

    and w_i is neglected due to $3)$.

  - Using reduced variables

    $$\zeta = \left(\frac{\Omega}{J}\right)^{1/\alpha} - 1$$
    $$\beta = \frac{1}{\alpha} \ln \frac{\Omega}{h}$$

  - the RG equations reads:

    $$\tilde{\zeta} = \zeta_{i-1,i} + \zeta_{i,i+1} + 1$$
    $$\tilde{\beta} = \beta_i + \beta_{i+1}$$

- equivalent to a $1d$ disordered $O(2)$ quantum rotor model ($1d$ disordered bosons) (E. Altman, Y. Kafri, A. Polkovnikov, and G. Refael (2004)) with

  - grain charging energy $U_i \leftrightarrow J_{i,i+1}^{1/\alpha}$

  - Josephson coupling $\mathscr{J}_{i,i+1} \leftrightarrow h_i^{1/\alpha}$

- Note, that site and bond variables are interchanged in the two problems.

Solution of the primary model $\mathbf{d = 1}$

- Let us change the log-energy-scale: $\Gamma \equiv \frac{1}{\alpha} \ln \frac{\Omega_0}{\Omega} \to \Gamma + \mathrm{d}\Gamma$

- the distributions $g_\Gamma(\beta)$ and $f_\Gamma(\zeta)$ will follow the equations:

$$\frac{\partial g_\Gamma(\beta)}{\partial \Gamma} = \frac{\partial g_\Gamma(\beta)}{\partial \beta} + f_0(\Gamma) \int d\beta' g_\Gamma(\beta') g_\Gamma(\beta - \beta') + g_\Gamma(\beta)[g_0(\Gamma) - f_0(\Gamma)]$$

$$\frac{\partial f_\Gamma(\zeta)}{\partial \Gamma} = (\zeta + 1) \frac{\partial f_\Gamma(\zeta)}{\partial \zeta} + g_0(\Gamma) \int d\zeta' f_\Gamma(\zeta') g_\Gamma(\zeta - \zeta' - 1) + f_\Gamma(\zeta)[f_0(\Gamma) + 1 - g_0(\Gamma)],$$

- the fixed-point solutions ($\Gamma \to \infty$) are:

$$g_\Gamma(\beta) = g_0(\Gamma) e^{-g_0(\Gamma)\beta},$$
$$f_\Gamma(\zeta) = f_0(\Gamma) e^{-f_0(\Gamma)\zeta}$$

- which satisfy the ordinary differential equations

$$\frac{dg_0(\Gamma)}{d\Gamma} = -f_0(\Gamma) g_0(\Gamma),$$
$$\frac{df_0(\Gamma)}{d\Gamma} = f_0(\Gamma)(1 - g_0(\Gamma)),$$

- thus $f_0(\Gamma) = g_0(\Gamma) - \ln g_0(\Gamma) - 1 + \varepsilon$
  $\varepsilon = -a + \ln(1 + a)$

- The boundary conditions in the $\Gamma \to \infty$ limit:

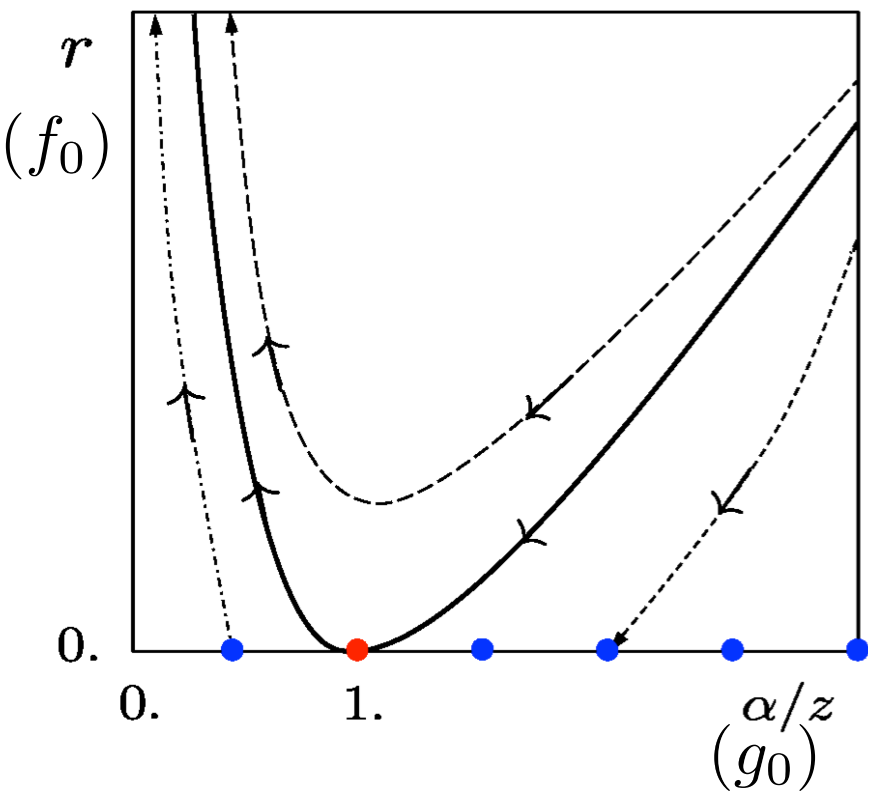
$$f_0(\Gamma) \to 0,$$
$$g_0(\Gamma) \to 1 + a$$

  - paramagnetic phase $a > 0$
  - critical point $a = 0$

- The solutions in leading order in Γ:

$$\begin{aligned} g_0(\Gamma) &\simeq 1 + a\coth[(\Gamma + C)a/2], \\ f_0(\Gamma) &\simeq \frac{a^2}{2\sinh^2[(\Gamma + C)a/2]}, \end{aligned}$$

RG flow-diagram



$$r(L) = \frac{N^{\#}_{bond}(L)}{N^{\#}_{field}(L)}$$

L increases along the flow

$$p(h) = \frac{d}{z} h^{-1+d/z}$$

r=0: line of fixed points

α/z > 1 : paramagnetic phase - stable

α/z < 1 : ferromagnetic phase - unstable

α/z = 1 : critical point

Diverging correlation length

Jump in the magnetization

Mixed-order transition

Interpretation through extreme value statistics (EVS)

- chain of length L, renormalized to a cluster of μ spins
- its effective field is given by:

$$\tilde{h} \sim \prod_{i=1}^{\mu} h_i / \prod_{i=1}^{\mu-1} J_i, \qquad J_i = b_i r_i^{-\alpha}$$

- limit distribution of the fields

$$g(h) \sim h^{-1+(1+a)/\alpha}$$

- h_i is the *smallest* out of r_i variables
- according to EVS $\rightarrow$ $h_i \simeq \kappa_i r_i^{-\alpha/(1+a)}$
- κ_i follow Fréchet statistics

$$P(\kappa) = \alpha^{-1} \kappa^{1/\alpha - 1} \exp(-\kappa^{1/\alpha})$$

- the asymptotic behavior of $\tilde{h}$ is different
  - $\overline{\ln h} > \overline{\ln J}$ (paramagnet)
  - $\overline{\ln h} < \overline{\ln J}$ (ferromagnet)

- criticality: $\boxed{a = 0}$ and $\boxed{\overline{\ln b} = \overline{\ln \kappa}}$
- at the critical point:
  - $\tilde{h} \sim \prod_{i=1}^{\mu} \kappa_i / \prod_{i=1}^{\mu-1} b_i \sim \exp(-c\mu^{1/2})$ from the central limit theorem
  - from energy scaling: $\tilde{h} \sim L^{-\alpha}$ which implies $\boxed{\mu \sim \ln^2 L}$

- in the paramagnetic phase $0 < a \ll 1$
  - $\xi(a)$ is the length of the longest decimated bond, r_l
  - $J_l / h_l \sim (b_l r_l^{-\alpha}) / (r_l^{-\alpha/(1+a)} \kappa_l) \sim r_l^{-\alpha a} / \kappa_l > 1$
  - thus the smallest value: $\kappa_l < \xi^{-\alpha a}$

$$\mathrm{Prob}(\kappa_l < \xi^{-\alpha a}) = \int_0^{\xi^{-\alpha a}} P(\kappa) \mathrm{d}\kappa$$
$$= 1 - e^{-\xi^{-a}} \sim \xi^{-a} = e^{-C'} = \mathscr{O}(1)$$

  - $\boxed{\xi \sim \exp(C'/a) \sim \exp\left(\mathrm{const}/|\theta - \theta_{\mathrm{c}}|\right)}$

- in the ferromagnetic phase $0 < -a \ll 1$
  - there is a giant connected cluster
  - $\xi(a)$ is the length of the longest hole in it, r_l
  - all the transverse fields are decimated out in this region

- $J_l/h_l \sim (b_l r_l^{-\alpha})/(r_l^{-\alpha/(1-|a|)}\kappa_l) \sim r_l^{\alpha|a|}/\kappa_l < 1$
- thus the smallest value: $\kappa_l > \xi^{\alpha|a|}$

$$\mathrm{Prob}(\kappa_l > \xi^{\alpha|a|}) = \int_{\xi^{\alpha|a|}}^{\infty} P(\kappa)\mathrm{d}\kappa$$
$$= \exp\left(-\xi^{|a|}\right) = \exp\left(-e^{-C'}\right) = \mathscr{P}$$

- thus $\xi \sim \exp(-C'/|a|)$.
- ξ has different scaling behaviours for
  * $C' < 0$ $(0 < \mathscr{P} < 1/e)$, thus $\xi \to \infty$
  * $C' > 0$ $(1/e < \mathscr{P} < 1)$, thus $\xi \to 0$ and the magnetization is of $\mathscr{O}(1)$

- two possible scenarios:
  * $\boxed{\text{second-order transition}}$

    if $\mathscr{P} < 1/e$ in every samples

    $$\boxed{\xi \sim \exp(-C'/|a|) \sim \exp\left(\mathrm{const}/|\theta - \theta_{\mathrm{c}}|\right)}$$

  * $\boxed{\text{mixed-order transition}}$

    if $\mathscr{P} > 1/e$
    the average magnetization is of $\mathscr{O}(1)$
    but the correlation length is divergent.

Conclusions

- critical behaviour is controlled by a strong disorder fixed point
  - dynamical exponent $z_c = \alpha$
  - KT-like scaling: $\xi \sim \exp(\mathrm{const}/|\theta - \theta_\mathrm{c}|)$
  - critical cluster is a logarithmic fractal: $\mu_L \sim (\ln L)^2$
  - Mixed-order transition

- Griffiths region in the paramagnetic phase with $z < \alpha$

- identical behaviour in models with a discrete order parameter (contact process, Potts model, etc.)
  - numerical verification for the LR random contact process

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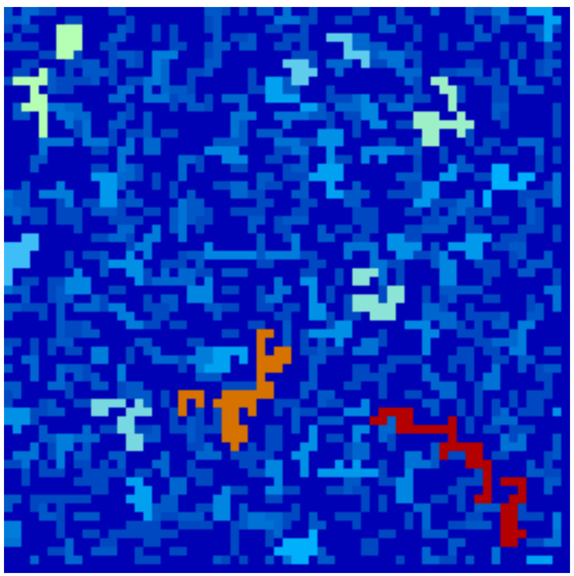
Thank you for your attention!

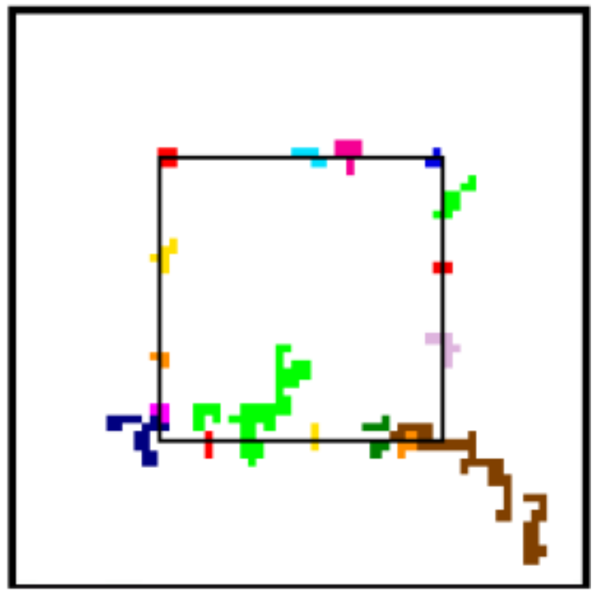
Entanglement: number of clusters

Quantification: von Neumann entropy

$$\mathcal{S} = -\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}} \log_2 \rho_{\mathcal{A}}) \qquad \rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}} |\Psi\rangle\langle\Psi|$$

Area law: $\mathcal{S}(\ell) \sim \ell^{d-1}$





At the critical point universal logarithmic corrections

$$\mathcal{S}_{1\mathrm{D}}(\ell) = \frac{c}{3} \log_2 \ell$$

$$\mathcal{S}_{2\mathrm{D}}(\ell) = a\ell + b \ln \ell$$

$$\mathcal{S}_{3\mathrm{D}}(\ell) = a\ell^2 + b\ell + c \ln \ell$$

$$\mathcal{S}_{4\mathrm{D}}(\ell) = a\ell^3 + b\ell^2 + c\ell + d \ln \ell$$

The area law is satisfied, but due to corners there is a universal logarithmic correction

(Kovács & Iglói, 2012)