Griffiths phase in a Potts model with correlated disorder

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Different kinds of disorder

Pure Potts model

Classical “spins” lying on the nodes of a square lattice.

\[ H = -J \sum_{(i,j)} \delta_{\sigma_i, \sigma_j} \quad (\sigma_i = 0, \ldots, q - 1) \]

\(\mathbb{Z}_q\)-symmetry is spontaneously broken in the low temperature phase.

Second-order phase transition for \(q \leq 4\), first-order above. 

\(q\)-dependent universality class.
Homogeneous uncorrelated disorder

Quenched disorder coupled to the energy density of the Potts model

\[ \mathcal{H} = - \sum_{(i,j)} J_{ij} \delta_{\sigma_i,\sigma_j} \]

where \( J_{ij} > 0 \) (no frustration) are random variables, distributed for e.g. as

\[ \varphi(J_{ij}) = \frac{1}{2} [\delta(J_{ij} - J_1) + \delta(J_{ij} - J_2)] \]

Critical line in the plane \( J_1 - J_2 \) is given by self-duality.

Two averages (thermal fluctuations and disorder):

\[ \langle X \rangle = \int \frac{1}{Z[J]} \sum_{\{\sigma\}} X(\{\sigma\}) e^{-\beta \mathcal{H}[\sigma,J]} \prod_{(i,j)} \varphi(J_{ij}) dJ_{ij} \]
Homogeneous disorder (2)

- **Regime** $q \leq 4$: New universality class for $q < 2 \leq 4$ in agreement with the **Harris criterion** (disorder is relevant if energy-energy correlation functions decay faster than disorder correlations, equiv. $1/\nu > d/2$ or $\alpha > 0$).
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New $q$-dependent universality classes.
McCoy-Wu model

Homogeneous distribution of couplings $J_{ij}$ in one direction and random in the second direction:

$q$-independent universality class (Senthil-Majumdar)!

$$\beta = (3 - \sqrt{5})/2, \quad \nu = 2$$

Griffiths phase:
Singularity of free energy in a finite range of temperatures, due to the existence of macroscopic regions with a high concentration of strong couplings and acting as super-paramagnets.
Weinrib-Halperin RG calculations

Renormalisation-Group study of the $\phi^4$ model with (Gaussian) correlated disorder:

$$\frac{(J_{ij} - \bar{J})(J_{kl} - \bar{J})}{||\vec{r}_{ij} - \vec{r}_{kl}||^{-a}}$$

Disorder is relevant when $a < d$ and

$$\frac{1}{\nu} > \frac{a}{2}$$

New universality class:

$$\nu = \frac{2}{a} \quad \text{(exact)}, \quad \eta = O(\varepsilon^2).$$

Monte Carlo simulations for the 3D Ising model.
No results for the Potts model.
How to construct a configuration of correlated random couplings?
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Simulate another spin model at a critical point!
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Simulate another spin model at a critical point!

Ashkin-Teller model (two coupled Ising models):

$$-\beta H^{\text{AT}} = \sum_{(i,j)} \left[ J^{\text{AT}} \sigma_i \sigma_j + J^{\text{AT}} \tau_i \tau_j + K^{\text{AT}} \sigma_i \sigma_j \tau_i \tau_j \right]$$

Two broken $\mathbb{Z}_2$-symetries so two order parameters: $m = \sum_i \sigma_i$ and $p = \sum_i \sigma_i \tau_i$. Self-dual critical line with varying critical exponents:

$$\beta^{\text{AT}}_\sigma = \frac{2 - y}{24 - 16y}, \quad \beta^{\text{AT}}_{\sigma \tau} = \frac{1}{12 - 8y}, \quad \nu^{\text{AT}} = \frac{2 - y}{3 - 2y}$$

where $y \in [0; 4/3]$ and $\cos \frac{\pi y}{2} = \frac{1}{2} \left[ e^{4K^{\text{AT}}} - 1 \right]$
Polarisation-polarisation correlation function

\[ \langle \sigma_i \tau_i \sigma_j \tau_j \rangle \sim |\vec{r}_i - \vec{r}_j|^{-2\beta_{\sigma\tau}^{AT} / \nu^{AT}} \]

Generate Ashkin-Teller spin configurations and associate a coupling configuration to each of them by

\[ J_{ij} = \frac{J_1 + J_2}{2} + \frac{J_1 - J_2}{2} \sigma_i \tau_i, \]

so that

\[ (J_{ij} - \bar{J})(J_{kl} - \bar{J}) \sim |\vec{r}_i - \vec{r}_k|^{-a} \]

Self-duality of the random Potts model is preserved.
Temperature behaviour of the 8-state Potts model
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Griffiths phase!
Algebraic Finite-Size Scaling in the Griffiths region.

Critical exponent $\beta/\nu$ depends on disorder correlations but not on $q$.

The hyperscaling relation is not satisfied!

$$\frac{\gamma}{\nu} = d - 2\frac{\beta}{\nu}$$
Self-averaging ratio (sample-to-sample relative fluctuations)

\[ R_m = \frac{\langle m \rangle^2 - \langle m \rangle^2}{\langle m \rangle^2} \]
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Magnetization is non-self averaging in the Griffiths region. A constant ratio \( R_m \) implies

\[ \langle m \rangle^2 - \langle m \rangle^2 = R_m \langle m \rangle^2 \sim L^{2\beta/\nu} \]
Decompose the susceptibility as

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\[
\bar{\chi} = \beta L^d \left[ \langle m^2 \rangle - \langle m \rangle^2 \right] = \beta L^d \left[ \langle m^2 \rangle - \langle m \rangle^2 \right] - \beta L^d \left[ \langle m \rangle^2 - \langle m \rangle^2 \right] = \chi_1 - \chi_2
\]

Not only \( \chi_1 \) and \( \chi_2 \) have the same scaling behavior \( L^{d-2\beta/\nu} \) (hyperscaling holds) but they also have the same amplitude \( A \), i.e.

\[
\chi_i = A L^{d-2\beta/\nu} (1 + B_i L^{-\omega_i} + \ldots), \quad (i = 1, 2).
\]

As a consequence, their difference behaves at large lattice sizes as

\[
\overline{\chi} = \chi_1 - \chi_2 \sim A B_1 L^{d-2\beta/\nu - \omega_1} - A B_2 L^{d-2\beta/\nu - \omega_2}
\]
Same mechanism as in the 3D Random-Field Ising model.

Violation of the hyperscaling relation is also observed in the energy sector.