Critical behavior of bond-diluted negative-weight percolation

O. Melchert[1], L. Apolo[2], A.K. Hartmann[1]

[1]Institut für Physik, Universität Oldenburg
[2]City College of the City University of New York, New York

Outline

- Introduction
- Percolation problem
- Results
  - – Diluted negative-weight percolation (DNWP)
  - – Densely packed loops (DPLs)
  - – NWP in higher dimensions
- Summary

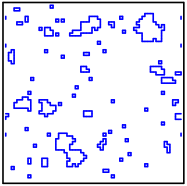
Model

- $L \times L$ lattice, fully periodic boundary conditions
- Undirected edges, weight (cost) distribution:

$$P(\omega) = \rho\, (2\pi)^{-1/2} \exp(-\omega^2/2) \;+\; (1-\rho)\, \delta(\omega - 1)$$

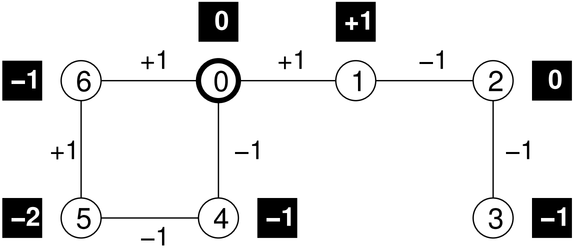
- Allows for loops $\mathcal{L}$ with negative weight $\omega_{\mathcal{L}}$
- Agent on lattice edges: pay/receive resources
- Configuration $\mathcal{C}$ of loops, with

$$E \equiv \sum_{\mathcal{L}\in\mathcal{C}} \omega_{\mathcal{L}} \stackrel{!}{=} \min$$



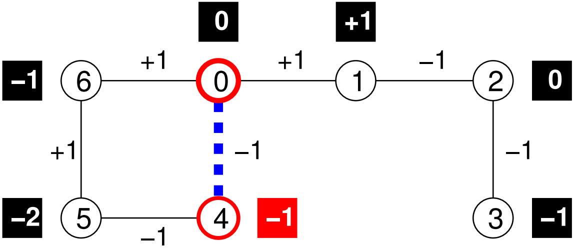
- Obtain $\mathcal{C}$ through mapping to minimum weight perfect matching problem

Minimal distances



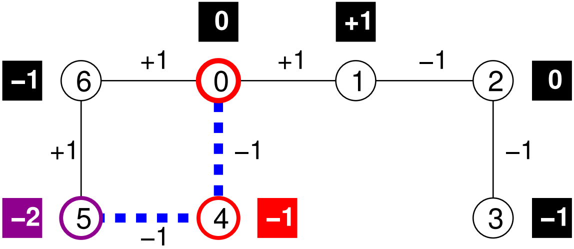
- $d(i) = \min_{j \in N(i)} \left[d(j) + \omega(i,j) \right]$ not fulfilled

Minimal distances



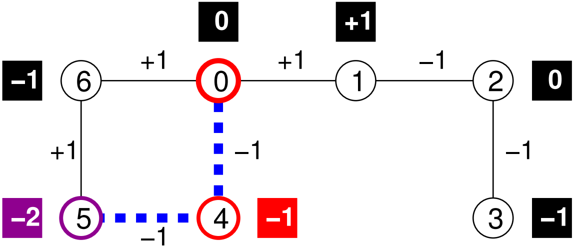
- $d(i) = \min_{j \in N(i)} \left[d(j) + \omega(i,j) \right]$ not fulfilled

Minimal distances



- $d(i) = \min_{j \in N(i)} \left[d(j) + \omega(i,j) \right]$ not fulfilled

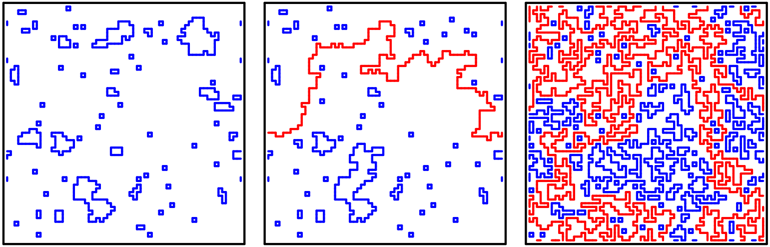
Minimal distances



- $d(i) = \min_{j \in N(i)} \left[d(j) + \omega(i,j) \right]$ not fulfilled
- Standard minimum-weight path algorithms, e.g. Dijkstra, Bellman-Ford, Floyd-Warshall, don't work
- Minimum-weight path problem requires matching techniques

  [R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network flows*]

Loop percolation



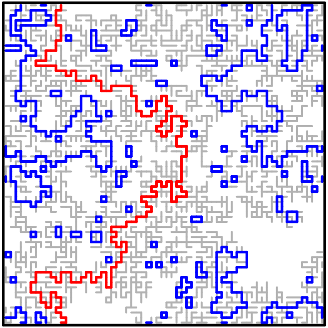
($L = 64$ at $\rho = 0.335,\ 0.340,\ 0.750$)

- Observe system spanning loops above critical ρ
- Disorder induced, geometric transition
- Characterize loops using observables from percolation theory (finite-size scaling (FSS) analysis)

[D. Stauffer, A. Aharony, *Introduction to Percolation Theory*]

Bond diluted NWP

- Consider diluted lattice: fraction p of *absent* bonds
- Modified lattice topology: $p = 0 \rightarrow$ regular lattice $p > 0 \rightarrow$ fractal lattice structure
- Random percolation has impact on NWP of loops
- $p > 1/2$: unlikely to find spanning (red) loops

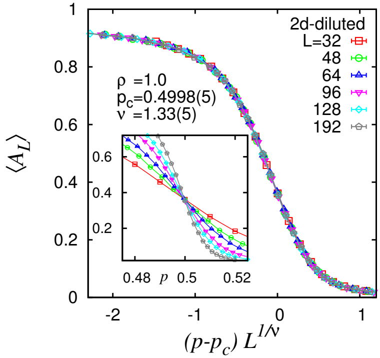


($L = 64$, $p = 0.5$, $\rho = 1.0$)

- Figure: minimum-weight loop configuration on largest cluster of bonds (gray edges)

Effect of bond dilution on NWP

- Scaling analysis for relative size of smallest box that fits largest loop $\langle A_L \rangle \equiv \langle R_x \times R_y \rangle / L^2$



- Finite-size scaling:

  $\langle A_L \rangle \sim f_1[(p - p_c)L^{1/\nu}]$

  p_c : critical point

  ν : correlation length exponent

- Data collapse for $\rho = 1.0$:

  $S = 1.25$

  $p_c = 0.4998(5)$

  $\nu = 1.33(5)$

  (no dilution: $\nu = 1.49(7)$)

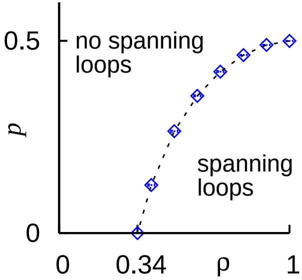
- S = "quality" of the scaling assumption (good if $S \leq 2$)
- FSS analysis via `autoScale.py` [OM, arXiv:0910.5403]

Effect of bond dilution on NWP

- Scaling analysis: probe critical points p_c and correlation length exponents ν at fixed disorder parameter ρ

ρ	p_c	ν	S
0.340(1)	0.0	1.49(7)	0.91
0.4	0.125(2)	1.49(9)	0.97
0.5	0.265(2)	1.49(15)	1.02
0.6	0.357(2)	1.49(11)	1.03
0.7	0.420(2)	1.47(11)	1.02
0.8	0.463(1)	1.47(11)	0.93
0.9	0.4893(9)	1.41(8)	1.20
1.0	0.4998(5)	1.33(5)	0.91

[Apolo, Melchert, Hartmann, PRE (2009)]



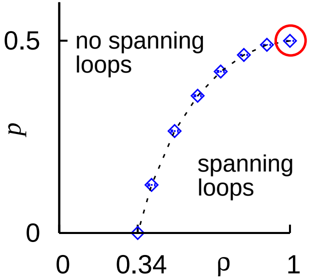
- ν at $\rho = 1.0$: characteristic for $2d$ random percolation
  Here: NWP → different compared to usual percolation
- Dilution changes universality class of NWP

Effect of bond dilution on NWP

- Scaling analysis: probe critical points p_c and correlation length exponents ν at fixed disorder parameter ρ

ρ	p_c	ν	S
0.340(1)	0.0	1.49(7)	0.91
0.4	0.125(2)	1.49(9)	0.97
0.5	0.265(2)	1.49(15)	1.02
0.6	0.357(2)	1.49(11)	1.03
0.7	0.420(2)	1.47(11)	1.02
0.8	0.463(1)	1.47(11)	0.93
0.9	0.4893(9)	1.41(8)	1.20
1.0	0.4998(5)	1.33(5)	0.91

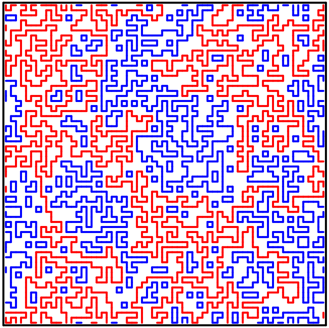
[Apolo, Melchert, Hartmann, PRE (2009)]



- Detailed analysis at $\rho = 1.0$, $p = 0.4998(5)$ up to $L = 512$:
  Scaling of loop-length: $\langle \ell \rangle \sim L^{d_f} \rightarrow d_f = 1.333(2)$
  Similar to self-avoiding walks: $d_f^{\mathrm{SAW}} = 4/3$

Densely-packed loops (DPLs)

- Consider NWP at large values of "Gaussian" disorder parameter ρ
- Similarity to fully-packed loop (FPL) model [Zeng *et. al.*, PRL (1998)]
- Comparison to FPL model: DPL model ...
  $\rightarrow$ exhibits local "impurities"
  $\rightarrow$ has more general disorder
  $\rightarrow$ possible on various lattice geometries

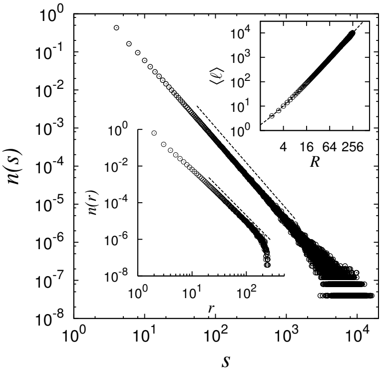


($L = 64$, $\rho = 1.0$)

- Figure: configuration of densely packed loops (red: single spanning loop, blue: nonspanning "small" loops)

Results for densely packed loops

- Scaling analysis for square system with $L = 256$ at $\rho = 1.0$



- Finite-size scaling:

length distrib.: $n_s \sim s^{-\tau}$,
rough. distrib.: $n_r \sim r^{\zeta - 3}$,
loop length: $\langle \ell \rangle \sim R^{d_f}(1 + cR^{-\omega})$

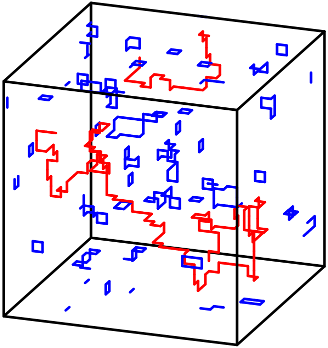
scaling relation: $\tau - 1 = (2 - \zeta)/d_f$

		DPL	FPL
τ	=	2.14(2)	2.15(1)
d_f	=	1.751(5)	1.75(1)
ω	=	0.92(5)	–
ζ	=	0.00(3)	0.00(1)

- DPL: square lattice, FPL: hexagonal lattice
- General disorder distribution + local "impurities": DPLs exhibit same scaling as FPLs

NWP in higher dimensions

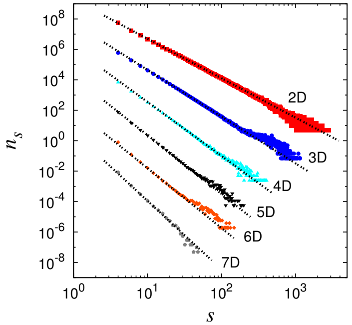
- Consider NWP on regular lattice graphs
- Here: hypercubic lattices with dimension $d = 2 \ldots 7$
- Scaling analysis: estimate geometric properties of loops



($d = 3$, $L = 16$, $\rho = 0.13$)

- Figure: minimum-weight configuration of loops (red: spanning loop, blue: nonspanning “small” loops)

Results for NWP in higher dimensions



- **Finite-size scaling analysis:**

  box size: $\langle A_L \rangle \sim f_1[(\rho - \rho_c)L^{1/\nu}]$,
  loop numbers: $\langle N \rangle \sim L^{(2-\alpha)/\nu}$,
  loop length: $\langle \ell \rangle \sim L^{d_f}$ $(\sim L^{d-\beta/\nu})$,
  suszept.: $\chi_L \sim L^{\gamma/\nu}$,
  length distrib.: $n_s \sim s^{-\tau}$

- **Scaling relations:**

  $\gamma + 2\beta = d\nu$, $\tau = 1 + d/d_f$
  and $2 - \alpha = d\nu$

- Resulting critical points and exponents in $d = 2 \ldots 7$:

d	ρ_c	ν	α	β	γ	d_f	τ
2	0.340(1)	1.49(7)	-0.99(14)	1.07(6)	0.77(7)	1.266(2)	2.59(3)
3	0.1273(3)	1.00(2)	-1.00(3)	1.54(5)	-0.09(3)	1.459(3)	3.07(1)
4	0.0640(2)	0.80(3)	-1.20(12)	1.91(11)	-0.66(5)	1.60(1)	3.55(2)
5	0.0385(2)	0.66(2)	-1.30(10)	2.10(12)	-1.06(7)	1.75(3)	3.86(3)
6	0.0265(2)	0.50(1)	-1.00(6)	1.92(6)	-0.99(3)	2.02(1)	4.00(2)
7	0.01977(6)	3/7 (?)				2.01(2)	4.50(1)

Summary

- Negative-weight percolation of loops
- Distinct from random bond/site percolation
- Results:
  – dilution changes universality class of $2d$ NWP
  – densely packed loops: scaling as for FPL model
  – scaling properties for various dimensions
- More details:
  OM & A.K. Hartmann, NJP 10 (2008) 043039
  L. Apolo, OM & A.K. Hartmann, PRE 79 (2009) 031103

Summary

- Negative-weight percolation of loops
- Distinct from random bond/site percolation
- Results:
  – dilution changes universality class of 2*d* NWP
  – densely packed loops: scaling as for FPL model
  – scaling properties for various dimensions

- More details:
  OM & A.K. Hartmann, NJP 10 (2008) 043039
  L. Apolo, OM & A.K. Hartmann, PRE 79 (2009) 031103

- Thank you!