Disordered Potts model on the diamond hierarchical lattice

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Phys. Rev. B **80**, 134201 (2009), arXiv:0908.3622

Leipzig, 27th November 2009

AGENDA

- Disordered Potts model
  - random ferromagnet (RF)
  - spin glass (SG)
- Diamond hierarchical lattice
  - renormalization of the Potts model
  - large-q limit
- Phase diagram
  - Pure system
  - Random system
    - numerical study by the pool method
    - phases
  - fixed points - numerically exact treatment
- Conclusions

Disordered Potts model

Hamiltonian:

$$\mathscr{H} = -\sum_{\langle i,j \rangle} J_{ij}\delta(\sigma_i, \sigma_j)$$

$\sigma_i = 1, 2, \ldots, q$, at site i
J_{ij} independent and identically distributed random numbers.

Phases and phase transitions in regular lattices

Pure system: $J_{ij} = J > 0$

- $q \leq q_c$: 2^{nd}-order transition
- $q > q_c$: 1^{st}-order transition

$q_c(d=2) = 4$, $q_c(d=3) < 3$

Random ferromagnet (RF): $J_{ij} > 0$

- $d = 2$: 2^{nd}-order transition for all q
- $d = 3$ and $q \geq 3$
  - weak disorder: 1^{st}-order transition
  - strong disorder: 2^{nd}-order transition

Spin glass (SG): $J_{ij} < 0$ and $J_{ij} > 0$

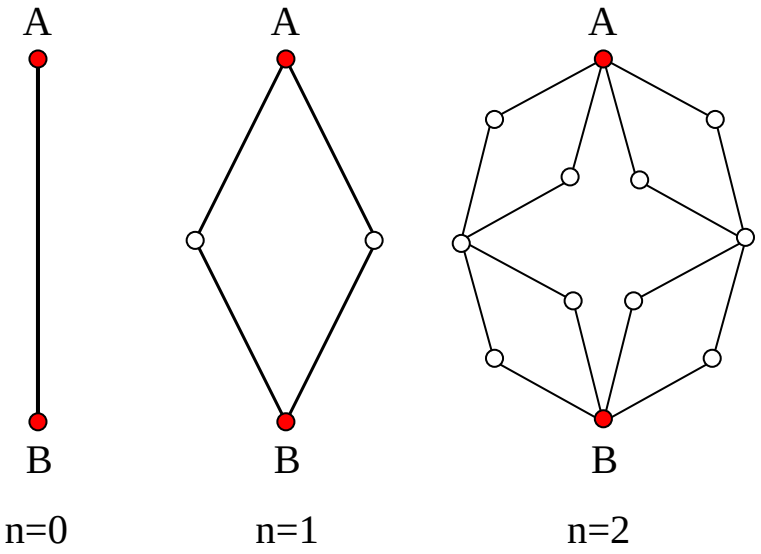
- $d = 3$
  - $q = 3$: SG phase
  - $q = 10$: no SG phase
- $d = 2 < d_c$ no SG phase

Potts model in the large-q limit

Motivations of its study

- experimental side
  - suggested as plausible model for supercooled liquids
    - valid in the mean-field approach
    - not for the nearest-neighbour model
- theoretical side
  - simpler to study
    - $1/q$-expansion
    - high-temperature expansion is dominated by one diagram
  - $q \to \infty$ is non-singular limit

Diamond hierarchical lattice



- branching number: $b = 2$

- generation: n
- length (between A and B): $L_n = 2^n$
- total number of bonds: $B_n = (2b)^n = L_n^{d_{eff}(b)}$
- effective dimension: $d_{eff}(b) = \ln(2b)/\ln(2)$.

Renormalization of the Potts model

Fix one boundary $\sigma_A = +1$, then the partition function reads

$$\begin{aligned} Z_n^{(1,.)} &= Z_n^+ + Z_n^- \\ Z_n^+ &= Z_n^{(1,1)} \\ Z_n^- &= \sum_{\sigma_B \neq 1} Z_n^{(1,\sigma_B)} = (q-1) Z_n^{(1,2)} \end{aligned}$$

The exact RG equations for the Z

$$\begin{aligned} Z_{n+1}^{(1,1)} &= \prod_{i=1}^{b} \left[Z_n^{(1,1)}(i_1) Z_n^{(1,1)}(i_2) + (q-1) Z_n^{(1,2)}(i_1) Z_n^{(1,2)}(i_2) \right] \\ Z_{n+1}^{1,2} &= \prod_{i=1}^{b} \left[Z_n^{(1,1)}(i_1) Z_n^{(1,2)}(i_2) + Z_n^{(1,2)}(i_1) Z_n^{(1,1)}(i_2) + (q-2) Z_n^{(1,2)}(i_1) Z_n^{(1,2)}(i_2) \right] \end{aligned}$$

It is easier to work in terms of:

$$x_n = \frac{\mathscr{Z}_n^{1,1}}{\mathscr{Z}_n^{1,2}} = e^{\beta F_n^{\text{inter}}} ,$$

where $F_n^{\text{inter}} = F_n^{1,2} - F_n^{1,1}$ is the interface free energy. The ratio x_n obeys the recursion equation

$$x_{n+1} = \prod_{i=1}^{b} \left[\frac{x_n^{(i_1)} x_n^{(i_2)} + (q-1)}{x_n^{(i_1)} + x_n^{(i_2)} + (q-2)} \right]$$

Renormalization for large-q

rescale the temperature:
$T' = T \ln q$, so that $e^{\beta} = q^{\beta'}$ where $\beta' = 1/T'$ $(k_B = 1)$
high-temperature series expansion of $\mathscr{Z}$ is dominated by one diagram:

$$\mathscr{Z} \underset{q\to\infty}{\simeq} q^{\phi} + \text{subleading terms}$$

which is related to the free energy through $\phi = -\beta' F$.

Similarly, the interface free energy:

$$x_n = q^{\beta' F_n^{\text{inter}}}, \qquad I_n = \beta' F_n^{\text{inter}}$$

In terms of I_n the recursion equations are:

$$q^{I_{n+1}} = \prod_{i=1}^{b} \left[\frac{q^{I_n^{(i_1)}} q^{I_n^{(i_2)}} + (q-1)}{q^{I_n^{(i_1)}} + q^{I_n^{(i_2)}} + (q-2)} \right]$$

which have a simpler form for large-q:

$$I_{n+1} = \sum_{i=1}^{b} \Phi \left[I_n^{(i_1)}, I_n^{(i_2)} \right]$$

where the auxiliary function is:

$$\Phi \left[I^{(1)}, I^{(2)} \right] = \begin{cases} 0 & \text{if } I_{max} + I_{min} < 1 \text{ and } I_{max} < 1, \\ 1 - I_{max} & \text{if } I_{max} + I_{min} < 1 \text{ and } I_{max} > 1, \\ I_{max} + I_{min} - 1 & \text{if } I_{max} + I_{min} > 1 \text{ and } I_{max} < 1, \\ I_{min} & \text{if } I_{max} + I_{min} > 1 \text{ and } I_{max} > 1. \end{cases}$$

with $I_{max} = max(I^{(1)}, I^{(2)})$ and $I_{min} = min(I^{(1)}, I^{(2)})$.

The initial condition is given by

$$I_0^{(i)} = \beta' J_i$$

where J_i is the value of the ith coupling.

Phase diagram: pure system

$$I_{n+1} = \begin{cases} 0 & \text{if } 0 < I_n \leq 1/2, \\ b(2I_n - 1) & \text{if } 1/2 < I_n \leq 1, \\ bI_n & \text{if } I_n > 1. \end{cases}$$

- fixed-point at $I_c = b/(2b-1)$
- first-order transition

Random system: Numerical study

Distribution of the couplings

$$\mathscr{P}(J) = \begin{cases} 1 & \text{if } \frac{p}{1-p} < J < \frac{1}{1-p}, \\ 0 & \text{otherwise.} \end{cases}$$

with $p \leq 1$

mean value is given by:

$$\overline{J} = \frac{1}{2}\left(\frac{1+p}{1-p}\right).$$

- $p > 0$ all couplings are random ferromagnetic
- $p \to 1$ we have the pure system
- $p < 0$ there are also negative bonds
- $p = -1$ the distribution is symmetric

Numerical pool method

- start with $N = 5 \times 10^6$ elements
- first itereation: generate N new elements
- second iteration: input is taken from the first iteration pool
- iterate up to $n \sim 70 - 80$ steps

Phase diagram for $b = 2$ and $d_{eff} = 2$

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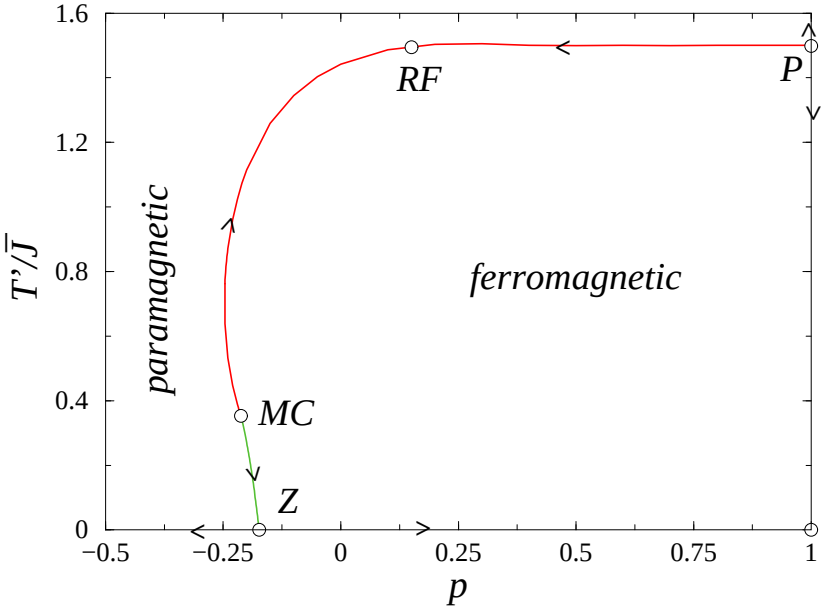


	$\bar{I}$	ΔI	ν
Ferro	L^{d_s}	L^{θ}	
Para	0	0	
P	$b/(2b-1)$	0	$1/d$
RF	$O(1)$	$O(1)$	1.31
MC	$O(1)$	$O(1)$	3.61
Z	L^{θ_Z}	L^{θ_Z}	1.73

Fixed points

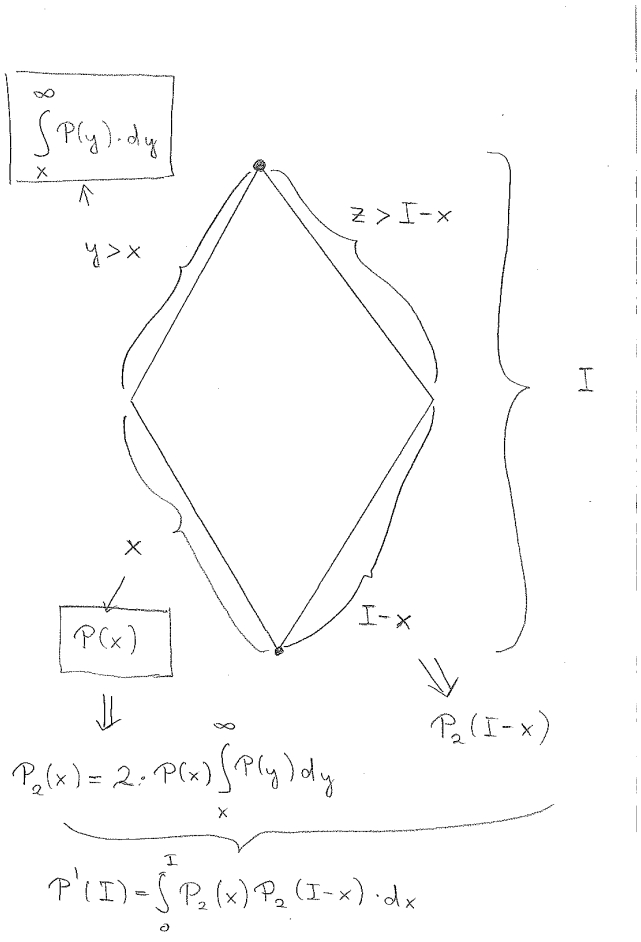
- P: pure system
- RF: random ferromagnet
  driving force: temperature
- MC: multicritical (Nishimori)
- Z: zero-temperature (spin-glass)
  driving force: frustration

Ferromagnetic phase

- I_n grows without limit
- for each bond $I_{max} \geq I_{min} > 1$
- $I_{n+1} = I_n^{(1)}(min) + I_n^{(2)}(min)$
- distribution function:

$$P'(I) = \int_0^I dx P_2(x) P_2(I-x)$$

$$P_2(I) = 2P(I) \int_I^\infty dx P(x), \quad I > 1$$



- solution

$$P'(2\bar{I}+I_1) = \frac{1}{\lambda} P(\bar{I}+I_1/\lambda)$$

- numerical solution: $\lambda = 1.230091(1)$

- scaling behaviour:

$$\bar{I} \sim L^{d_s}, \quad I_1 = \Delta I \sim L^{\theta}$$

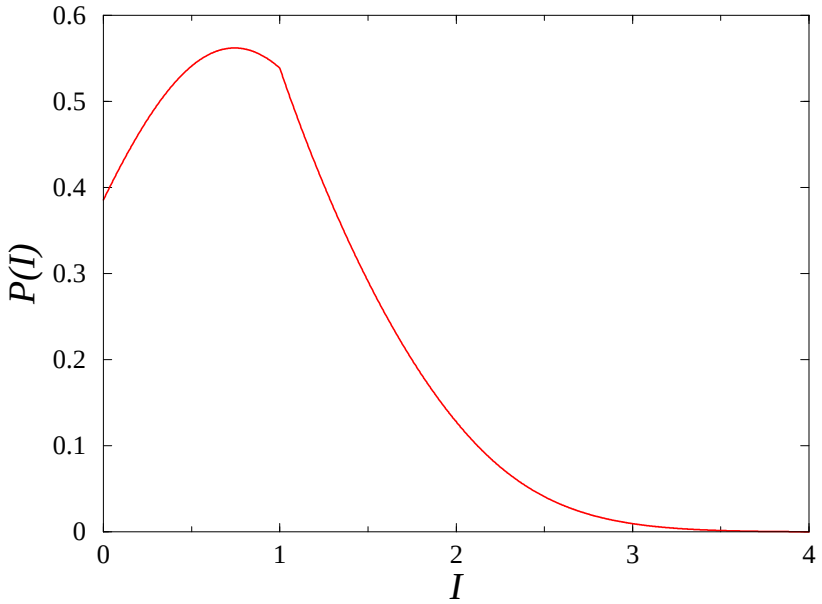
- exponents:
  - $d_s = d_{eff} - 1 = 1$
  - droplet exponent:
    - $\theta = \log(\lambda)/\log(2) = 0.298765(1)$
    - θ is the same as for the directed polymer in the same lattice.

Random ferromagnet (RF) fixed point

- scaling to $I_n \geq 0$
- different recursion relations at $I = 0$ and in the regions $0 < I < 1$, $1 < I < 2$, and $I > 2$.
- solution:

$$P'(I) = P(I)$$

- $p_0 = 0.1280795$



Thermal critical exponent

- Jacobian: $J(x,y) = \delta P'(x)/\delta P(y)$
- eigenvalue problem:

$$\int dy J(x,y) f_i(y) = \lambda_i^{\mathrm{RF}} f_i(x)$$

- numerical solution:
  $\lambda_1^{\mathrm{RF}} = 1.6994583(1)$
- thermal eigenvalue:

$$y_t^{\mathrm{RF}} = \frac{\log \lambda_1^{\mathrm{RF}}}{\log 2} = 0.7650750(1)$$

- correlation length exponent:
  $\nu_{\mathrm{RF}} = 1/y_t^{\mathrm{RF}} = 1.307061(1)$.
- scaling of $\bar{I}(T',L)$ for $T' < T_c'$:

$$\bar{I}(T',L) = \left[\frac{L}{\xi_{av}(T')}\right]^{d_s}$$

  with: $\xi_{av}(T') \sim (T' - T_c')^{-\nu_{\mathrm{RF}}}$.
- scaling of $\Delta I(T',L)$ for $T' < T_c'$:

$$\Delta I(T',L) = \left[\frac{L}{\xi_{var}(T')}\right]^{\theta}$$

  and $\xi_{var}(T') \sim \xi_{av}(T')$.

Zero-temperature (Z) fixed point

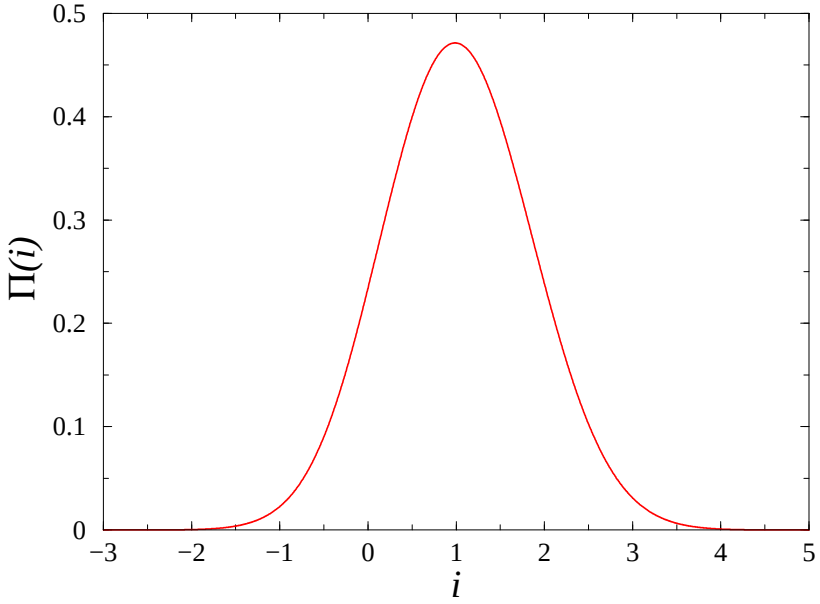
- scaling to $|I_n| \to \infty$
- new scaling variable: $i_n \equiv I_n/\bar{I}_n$ and parameter: $\alpha_{n+1} = \bar{I}_{n+1}/\bar{I}_n$.
- scaling behaviour:
  - $p > p_Z$: $\alpha_n \to b$
  - $p < p_Z$: $\alpha_n \to 0$
  - $p = p_Z$: $\alpha_n \to \alpha_Z$
- probability distribution transforms as:

$$\Pi'(i) = \alpha_Z \Pi(i)$$

with $\alpha_Z = 1.10661(1)$

- the droplet exponent:

$$\theta_Z = \frac{\log \alpha_Z}{\log 2} = 0.14615(1)$$



- correlation length exponent:

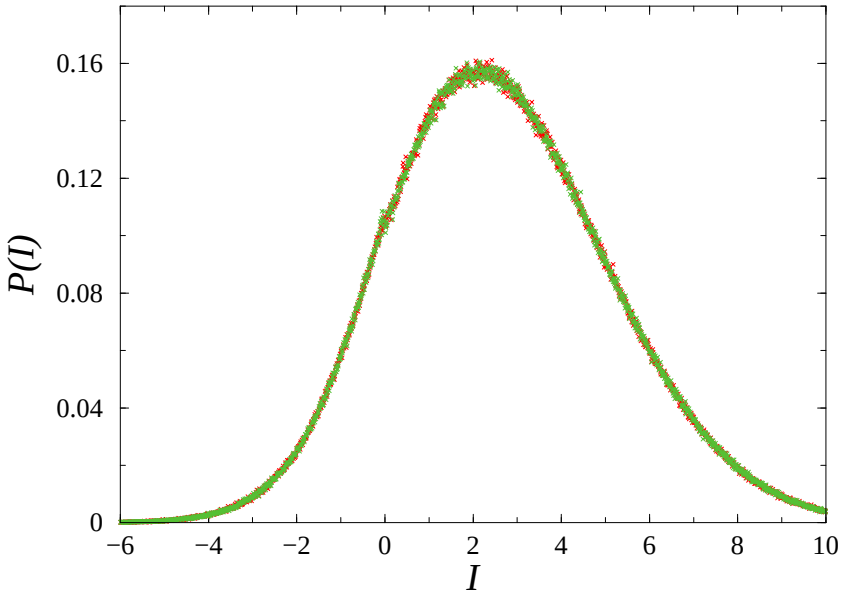
$$\nu_Z = 1/y_t^Z = 1.72906(1)$$

-

$$\bar{I}(T', L) = \frac{L^{d_s}}{[\xi_{av}(T')]^{d_s - \theta_Z}}$$

Multicritical (MC) fixed point

- scaling of $I_n = O(1)$
- both $I_n > 0$ and $I_n < 0$
- numerical solution by the pool method



- correlation length:

$$\xi(t) \sim (t \ln^{\kappa} t)^{-\nu_{\mathrm{MC}}}$$

- exponents: $\kappa \approx 1.5$ and $\nu_{\mathrm{MC}} = 3.61$.

Conclusions

- q-state Potts-model on the diamond hierarchical lattice
- numerically exact solution of the random model for large-q
- ferro & paramagnetic phases - similarity with directed polymers
- four non-trivial fixed points - similarity with the $\pm J$, $q = 3$ model
- RF fixed point - driving force: temperature
- Z fixed point - driving force: disorder
- MC fixed point - similar to the Nishimori point
- possible extensions
  - larger branching number, $b > 2$.
  - other quantities: magnetization, correlations, etc.
  - for smaller values of q, $1/q$ expansion.