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Quantum Field Theory — Problem Sheet 8

2 pages — Problems 8.1 to 8.3

Problem 8.1

Let $P : C^\infty(M) \to C^\infty(M)$ be a normally hyperbolic operator on a globally hyperbolic spacetime (M, g) and consider the dual pairing

$$C^\infty(M) \times C^\infty(M) \ni (f, g) \mapsto \langle f, g \rangle \doteq \int d\mu_g(x) f(x) g(x)\,,$$

which is well-defined for f and g with compact intersecting support; $d\mu_g = \sqrt{|\det g|}dx$ is the invariant volume measure. Assume that P is formally selfadjoint with respect to this pairing, i.e. for all $f, g \in C^\infty$ with compact intersecting support,

$$\langle f, Pg \rangle = \langle Pf, g \rangle\,.$$

Verify the following statements.

(1) The retarded/advanced Green's operators $\Delta_{R/A}$ are mutual adjoints of one another, i.e., for all $f, g \in C_0^\infty(M)$

$$\langle f, \Delta_{R/A} g \rangle = \langle \Delta_{A/R} f, g \rangle\,.$$

Their distributional kernels thus satisfy

$$\Delta_R(x, y) = \Delta_A(y, x)\,.$$

(2) $\Delta_{R/A}$ are not only right-inverses of P, but also left-inverses, i.e. for all $f \in C_0^\infty(M)$

$$\Delta_{R/A} P f = f\,.$$

(3) The causal propagator $\Delta \doteq \Delta_R - \Delta_A$ is formally skewadjoint

$$\langle f, \Delta g \rangle = -\langle \Delta f, g \rangle\,, \qquad \Delta(x, y) = -\Delta(y, x)$$

(4) The kernel of Δ on C_0^∞ is the image of P on C_0^∞, and the image of Δ on C_0^∞ are all solutions of the equation $Pu = 0$ with compactly supported Cauchy data.

Hint: Decompose $u = u_+ + u_-$ where $u_\pm$ has past/future compact support, whereby a set K is called future/past compact if $J^\pm(x) \cap K$ is compact for all $x \in M$. Use that Δ_R is an inverse of P not only on $C_0^\infty(M)$ but in fact on all functions with past compact support, and that Δ_A is an inverse of P on functions with future compact support.

Remark: In fact, the properties (2) and (4) do not depend on the formal selfadjointness of P.

Problem 8.2

Let ω be a quasifree state on the Borchers-Uhlmann algebra $\mathscr{A}(M)$ and let $f \in C_0^\infty(M, \mathbb{R})$ be an arbitrary test function. Define a new state on $\mathscr{A}(M)$ by the "insertion"

$$\omega_f(A) \doteq \frac{\omega(\phi(f)A\phi(f))}{\omega(\phi(f)\phi(f))}$$

for all $A \in \mathscr{A}(M)$. Investigate whether ω_f is quasifree.

Problem 8.3

Let α_t be a one-parameter group of $*$-automorphisms of the Borchers-Uhlmann algebra $\mathscr{A}(M)$ and let ω be a quasifree, α_t-invariant state on $\mathscr{A}(M)$. Prove that ω is a β-KMS state for α_t if an only if, for all $f_1, f_2 \in C_0^\infty(M)$ the functions $F_{f_1,f_2}(t)$, $G_{f_1,f_2}(t)$ defined by

$$F_{f_1,f_2}(t) \doteq \omega\big(\alpha_t(\phi(f_1))\phi(f_2)\big) \qquad G_{f_1,f_2}(t) \doteq \omega\big(\phi(f_2)\alpha_t(\phi(f_1))\big)$$

extend to functions $F_{f_1,f_2}(z)$, $G_{f_1,f_2}(z)$ on the complex plane which are analytic on the strips $-\beta > \Im z > 0$ and $0 < \Im z < \beta$ respectively, continuous on the boundaries of these strips and satisfy

$$F_{f_1,f_2}(t - i\beta) = G_{f_1,f_2}(t)\,.$$