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Inst. f. Theoretische Physik
Summer Term 2016

Quantum Field Theory — Problem Sheet 7

2 pages — Problems 7.1 to 7.3

Problem 7.1
The Wightman two-point function of the real scalar field on Minkowski spacetime in the vacuum state is given by

$$\Delta_+(x-y) \doteq \langle \phi(x)\phi(y)\rangle_\Omega$$

$$\Delta_+(z) \doteq \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \exp(-i\omega_{\vec{p}} z_0)\exp(i\vec{p}\cdot\vec{x})\exp(-\omega_{\vec{p}}\epsilon)$$

$$\omega_{\vec{p}} \doteq \sqrt{|\vec{p}|^2+m^2}$$

Compute $\Delta_+(z)$ explicitly for the case $m=0$. Use the result to compute the causal propagator $\Delta(z) = -i(\Delta_+(z) - \Delta_+(-z))$ and the retarded and advanced Green's functions $\Delta_R(z) = \Theta(z_0)\Delta(z)$, $\Delta_A(z) = \Delta_R(-z)$ for the massless case.

Hint: Use the distributional identity

$$\lim_{\epsilon\to 0^+}\left(\frac{1}{f(x)+i\epsilon} - \frac{1}{f(x)-i\epsilon}\right) = -2\pi i\delta(f(x))\,,$$

valid for any smooth and real-valued function f on $\mathbb{R}^n$.

Problem 7.2
We consider the scalar vacuum two-point function in the massive case. Let

$$\sigma(z) \doteq \frac{z_0^2 - |\vec{z}|^2}{2}\,, \qquad \sigma_\epsilon(z) \doteq \frac{(z_0 - i\epsilon)^2 - |\vec{z}|^2}{2}\,.$$

One may show that $\Delta_+(z)$ can be expanded as

$$\Delta_+(z) = \lim_{\epsilon\to 0^+}\frac{1}{8\pi^2}\left(\frac{1}{\sigma_\epsilon(z)} + V_0\log(m^2\sigma_\epsilon(z)) + \sigma(z)f_1(\sigma(z)) + \sigma(z)f_2(\sigma(z))\log(m^2\sigma_\epsilon(z))\right)$$

where f_1 and f_2 are smooth functions and V_0 is a suitable constant. Determine V_0.

Hint: Compute $\lim_{z\to 0}\sigma(z)\Box\Delta_+(z)$.

Problem 7.3
We define the Dirac operator and its adjoint as

$$D \doteq i\partial\!\!\!/ - m\,, \qquad D^* \doteq -i\partial\!\!\!/ - m\,.$$

The retarded and advanced fundamental solutions of the Dirac equations are the unique solutions of

$$\begin{aligned} DS_R(z) &= \delta(z)1_4\,, \qquad & DS_A(z) &= \delta(z)1_4 \\ \mathrm{supp}(S_R(z)) &\subset J^+(0)\,, \qquad & \mathrm{supp}(S_A(z)) &\subset J^-(0) \end{aligned}$$

where 1_4 is the unit matrix in $\mathbb{C}^4$.

(1) Construct S_R and S_A by means of D^*, 1_4 and the scalar fundamental solutions Δ_R, Δ_A.

(2) The causal propagator S of the Dirac equation is defined as $S \doteq S_R - S_A$ and defines the covariant canonical anticommutation (CAR) relations of the quantized Dirac field (here given in unsmeared form for simplicity) by

$$\{\psi^a(x), \bar{\psi}_b(y)\} = S^a{}_b(x-y)1\,,$$

where upper/lower indices refer to $\mathbb{C}^4$ respectively its dual space, 1 is the identity operator on the Hilbert space and $\bar{\psi} \doteq \psi^+\gamma^0$ is the Dirac conjugation with ψ^+ being the adjoint with respect to the inner product in $\mathbb{C}^4$. The covariant CAR may be shown to be equivalent to the equal-time CAR

$$\{\psi^a(x), \psi^+_b(y)\} \restriction_{x_0=y_0} = i\delta(\vec{x}-\vec{y})\delta^a_b 1\,.$$

(In both versions of the CAR the anticommutators of like fields are required to vanish.) Verify the following properties of S which must hold for consistency with the CAR.

$$\begin{aligned} \gamma^0 S^+(z)\gamma^0 &= S(-z) \\ S(z_0 = 0, \vec{z})\gamma^0 &= i\delta(\vec{z})1_4\,. \end{aligned}$$