A sum rule for perfect electromagnetic conductor (PEMC) Casimir forces

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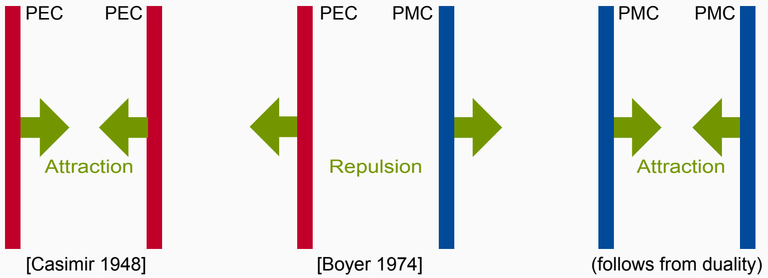
Workshop: *Dispersion forces and dissipation*

Alexander von Humboldt Stiftung/Foundation

Motivation



Generalised concept of a perfect electromagnetic conductor (PEMC) interpolates between these cases

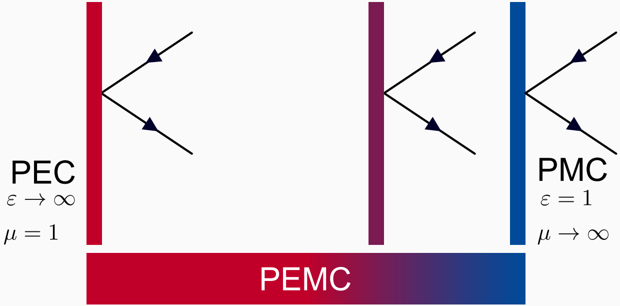
Outline

- What exactly are PEMCs?
- How do we calculate Casimir forces between them?
- What do we mean by a 'sum rule'?

What are PEMCs?

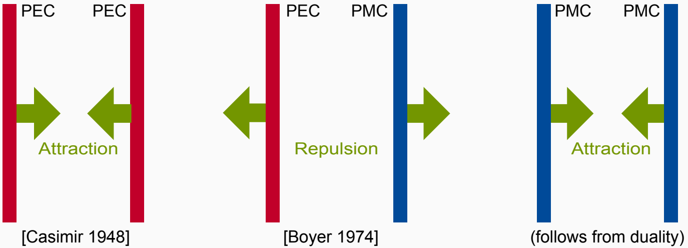
Perfect electromagnetic conductor (PEMC)

- Introduced: *I. V. Lindell, A. H. Sihvola, J. Electromagn. Waves Appl. 19, 861869 (2005)*
- Main idea: Interpolation between the perfect electric conductor and the perfect magnetic conductor



Perfect electromagnetic conductor (PEMC)

Known Casimir force characteristics of PECs and PMCs



We study;

Perfect electromagnetic conductor (PEMC)

PEMC is a polarization-mixing material

$$\hat{\mathbf{D}} = \varepsilon_0 \varepsilon \hat{\mathbf{E}} + \frac{1}{c}\xi \hat{\mathbf{H}},$$
$$\hat{\mathbf{B}} = \mu_0 \mu \hat{\mathbf{H}} + \frac{1}{c}\zeta \hat{\mathbf{E}}.$$

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and is defined by constraints in the limit $\varepsilon \to \infty, \mu \to \infty$;

$$\xi = \zeta = \pm \sqrt{\mu\varepsilon},$$
$$M = \frac{\xi}{\mu} = \pm \sqrt{\frac{\varepsilon}{\mu}} \quad \longleftarrow \text{kept finite!}$$

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Leads to boundary conditions;

$$\mathbf{n} \cdot (Z_0 \mathbf{D} - M\mathbf{B}) = 0,$$
$$\mathbf{n} \times (Z_0 \mathbf{H} + M\mathbf{E}) = 0.$$

($Z_0 = \mu_0/\varepsilon_0$, normal vector $\mathbf{n}$).

Compare with

$$\text{PEC:}\ \mathbf{n} \times \mathbf{E} = 0\,, \mathbf{n} \cdot \mathbf{B} = 0$$
$$\text{PMC:}\ \mathbf{n} \times \mathbf{H} = 0\,, \mathbf{n} \cdot \mathbf{D} = 0$$

PEMCs and duality

PEMCs and duality

Maxwell's equations can be written as;

$$\nabla \cdot \begin{pmatrix} Z_0\mathbf{D} \\ \mathbf{B} \end{pmatrix} = 0, \quad \nabla \times \begin{pmatrix} \mathbf{E} \\ Z_0\mathbf{H} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} Z_0\mathbf{D} \\ \mathbf{B} \end{pmatrix} = 0$$

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Invariant under an SO(2) transformation;

$$\mathbb{U} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

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PEMC boundary conditions are reproduced by beginning from a PEC ($\varepsilon \to \infty, \mu = 1$, implying $M \to \infty$) with;

$$\boxed{M = \cot\theta}$$

The PMC ($\varepsilon = 1, \mu \to \infty$, implying $M \to 0$) has $\theta = \pi/2$

Casimir forces between PEMCs

Modes?

PEC: $\mathbf{n} \times \mathbf{E} = 0\,, \mathbf{n} \cdot \mathbf{B} = 0$
PMC: $\mathbf{n} \times \mathbf{H} = 0\,, \mathbf{n} \cdot \mathbf{D} = 0$:

Allowed values of k;

- PEC-PEC :

$$k = \frac{n\pi}{L}$$

- PEC-PMC:

$$k = \frac{(n + \frac{1}{2})\pi}{L}$$

Try to combine PEC and PMC boundary conditions using adjustable coefficient α;

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$$\frac{k}{\alpha} + \tan(kL) = 0$$

$\alpha \to \infty$: PEC, $\alpha \to 0$: PMC
Transcendental equation!

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Transcendental equation!

- Abandon mode picture, move to Green's function and stress tensor

General expressions

Electric field is

$$\hat{\mathbf{E}}(\mathbf{r},\omega) = \mathrm{i}\mu_0\omega \int d^3\mathbf{r}' \mathbb{G}(\mathbf{r},\mathbf{r}',\omega) \cdot \hat{\mathbf{j}}_\mathrm{N}(\mathbf{r}',\omega)$$

where $\hat{\mathbf{j}}_\mathrm{N}$ is a noise-current source.

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where $\hat{\mathbf{j}}_\mathrm{N}$ is a noise-current source. We evaluate

$$\mathbf{F} = \int_{\partial V} \mathbf{dA}\cdot\langle\hat{\mathbb{T}}\rangle$$

where

$$\hat{\mathbb{T}} = \varepsilon_0\hat{\mathbf{E}}\otimes\hat{\mathbf{E}} + \frac{1}{\mu_0}\hat{\mathbf{B}}\otimes\hat{\mathbf{B}} - \frac{1}{2}(\varepsilon_0\hat{\mathbf{E}}^2 + \frac{1}{\mu_0}\hat{\mathbf{B}}^2)\mathbb{I}.$$

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giving...

$$\langle\hat{\mathbb{T}}\rangle = -\frac{\hbar}{2\pi}\int_0^\infty d\xi \int_{\partial V} dA \Bigg\{ \frac{\xi^2}{c^2}[\mathbb{G}^{(1)}(\mathbf{r},\mathbf{r},\mathrm{i}\xi) + \mathbb{G}^{(1)\mathsf{T}}(\mathbf{r},\mathbf{r},\mathrm{i}\xi)] + \vec{\nabla}\times[\mathbb{G}^{(1)}(\mathbf{r},\mathbf{r}',\mathrm{i}\xi) + \mathbb{G}^{(1)\mathsf{T}}(\mathbf{r}',\mathbf{r},\mathrm{i}\xi)]\times\nabla'\Big|_{\mathbf{r}'\to\mathbf{r}}$$
$$-\frac{1}{2}\mathrm{tr}\left[\frac{\xi^2}{c^2}[\mathbb{G}^{(1)}(\mathbf{r},\mathbf{r},\mathrm{i}\xi) + \mathbb{G}^{(1)\mathsf{T}}(\mathbf{r},\mathbf{r},\mathrm{i}\xi)] + \vec{\nabla}\times[\mathbb{G}^{(1)}(\mathbf{r},\mathbf{r}',\mathrm{i}\xi) + \mathbb{G}^{(1)\mathsf{T}}(\mathbf{r}',\mathbf{r},\mathrm{i}\xi)]\times\nabla'\Big|_{\mathbf{r}'\to\mathbf{r}}\right]\mathbb{I}\Bigg\}.$$

Green's tensor

Polarisation-mixing medium: reflection coefficients → reflection matrices

$$\mathbf{v}_{\text{refl}} = R \cdot \mathbf{v}_{\text{inc}} = \begin{pmatrix} r_{ss} & r_{sp} \\ r_{ps} & r_{pp} \end{pmatrix} \cdot \begin{pmatrix} v_s \\ v_p \end{pmatrix}$$

Green's tensor

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Using the PEMC boundary conditions

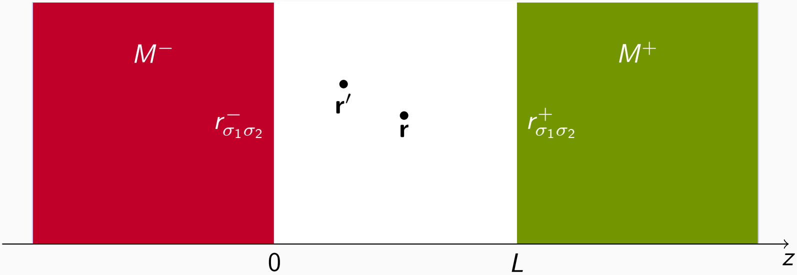
$$\mathbf{n} \cdot (Z_0 \mathbf{D} - M\mathbf{B}) = 0,$$
$$\mathbf{n} \times (Z_0 \mathbf{H} + M\mathbf{E}) = 0.$$

one eventually finds;

$$R = \frac{1}{1 + M^2} \begin{pmatrix} 1 - M^2 & -2M \\ -2M & M^2 - 1 \end{pmatrix}$$

Green's tensor

We need to construct the Green's tensor in the region between two dissimilar PEMC slabs separated by vacuum



Begin with a multi-reflection denominator;

$$D^{\pm}_{\sigma_i\sigma_j} = \left(\mathbb{I} - R^{\pm} \cdot R^{\mp} e^{-2\mathrm{i}k^{\perp}L}\right)^{-1}_{\sigma_i\sigma_j} \qquad R^{\pm} = \begin{pmatrix} r^{\pm}_{ss} & r^{\pm}_{sp} \\ r^{\pm}_{ps} & r^{\pm}_{pp} \end{pmatrix}$$

Green's tensor

Final result for scattering Green's tensor is...

$$
\begin{aligned}
\mathbb{G}^{(1)}(\mathbf{r},\mathbf{r}',\omega) = & \frac{1}{8\pi^2}\int \frac{d^2k^\parallel}{k^\perp} e^{i\mathbf{k}^\parallel\cdot(\mathbf{r}-\mathbf{r}')} \\
\times \quad & \left[\sum_{\sigma_1\sigma_2} \mathbf{e}_{\sigma_1+}\cdot R^+\cdot(D^\mp)^{-1}\cdot R^-\cdot\mathbf{e}_{\sigma_2+} e^{\mathrm{i}k^\perp(2L+z-z')}\right. \\
& + \sum_{\sigma_1\sigma_2} \mathbf{e}_{\sigma_1-}\cdot R^-\cdot(D^\pm)^{-1}\cdot R^+\cdot\mathbf{e}_{\sigma_2-} e^{\mathrm{i}k^\perp(2L-z+z')} \\
& + \sum_{\sigma_1\sigma_2} \mathbf{e}_{\sigma_1-}\cdot R^-\cdot(D^\mp)^{-1}\cdot\mathbf{e}_{\sigma_2+} e^{\mathrm{i}k^\perp(z+z')} \\
& \left. + \sum_{\sigma_1\sigma_2} \mathbf{e}_{\sigma_1+}\cdot R^+\cdot(D^\pm)^{-1}\cdot\mathbf{e}_{\sigma_2-} e^{\mathrm{i}k^\perp(2L-z-z')}\right].
\end{aligned}
$$

Multi-reflection denominators

General form;

$$D^{\pm}_{\sigma_i\sigma_j} = \left(\mathbb{I} - R^{\pm} \cdot R^{\mp} e^{-2\mathrm{i}k^{\perp}L}\right)^{-1}_{\sigma_i\sigma_j}$$

PEMC reflection matrix is simple

$$R^{\pm} = \frac{1}{1+M_{\pm}^2}\begin{pmatrix} 1-M_{\pm}^2 & -2M_{\pm} \\ -2M_{\pm} & M_{\pm}^2-1 \end{pmatrix}$$

gives;

$$(D^{\pm})^{-1} = \frac{\varphi}{1-2\varphi\cos(2\delta)+\varphi^2}\begin{pmatrix} \varphi-\cos(2\delta) & \pm\sin(2\delta) \\ \mp\sin(2\delta) & \varphi-\cos(2\delta) \end{pmatrix}$$

with $\varphi = e^{-2\mathrm{i}k^{\perp}L}$ and $\delta = \theta^{+} - \theta^{-} = \mathrm{arccot}(M^{+}) - \mathrm{arccot}(M^{-})$

Casimir force

The force per unit area reduces to...

$$f = -\frac{\hbar c}{\pi^2 L^4} \int_0^\infty dx\; x^3 \frac{e^{2x}\cos(2\delta) - 1}{1 - 2e^{2x}\cos(2\delta) + e^{4x}}$$

which can be analytically integrated

$$\mathbf{f}(\theta^+, \theta^-) = -\frac{3\hbar c}{8\pi^2 L^4} \mathrm{Re}\left(\mathrm{Li}_4\left[e^{2i(\theta^+ - \theta^-)}\right]\right)\mathbf{e}_z$$

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Using

$$\mathrm{Re}\,\mathrm{Li}_4(e^{i\phi}) = \sum_{k=1}^{\infty}\frac{\cos^k(k\phi)}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2\phi^2}{12} + \frac{\pi\phi^3}{12} - \frac{\phi^4}{48}$$

we get...

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we get...

Main result

$$\mathbf{f}(\delta) = -\frac{\hbar c}{8\pi^2 L^4}\left[\frac{\pi^4}{30} - \delta^2(\pi - \delta)^2\right]\mathbf{e}_z$$

Casimir force

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From here we obtain the special cases of Casimir ($\delta = 0$)

$$\mathbf{f}(0) = -\frac{\hbar c}{240\pi^2 L^4}\mathbf{e}_z$$

and Boyer ($\delta = \pi/2$)

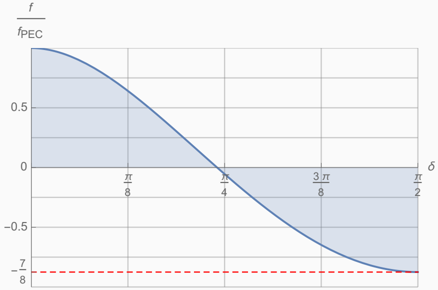
$$\mathbf{f}(\pi/2) = +\frac{7}{8}\cdot\frac{\hbar c}{240\pi^2 L^4}\mathbf{e}_z$$

Casimir force — PEMC parameter dependence

Main result

$$\mathbf{f}(\delta) = -\frac{\hbar c}{8\pi^2 L^4}\left[\frac{\pi^4}{30} - \delta^2(\pi-\delta)^2\right]\mathbf{e}_z$$

$$\delta = \theta^+ - \theta^-$$
$$= \mathrm{arccot}(M^+) - \mathrm{arccot}(M^-)$$



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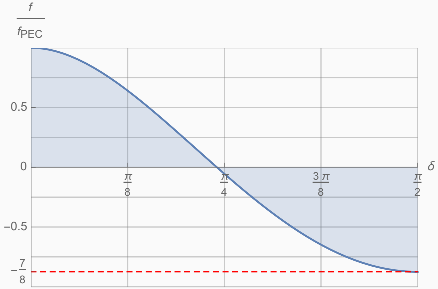

$$\delta(f=0) = \frac{\pi}{2}\left(1 - \sqrt{1 - 2\sqrt{\frac{2}{15}}}\right) \approx 0.96\cdot\frac{\pi}{4}$$

Casimir force — 'Sum Rule'

Main result

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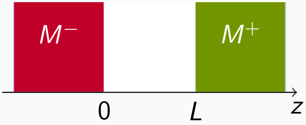

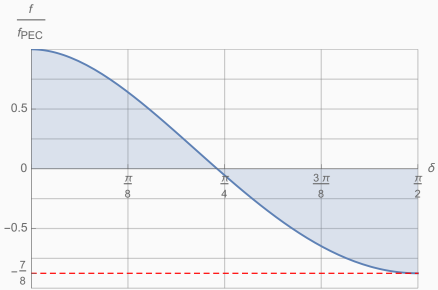

Sum rule

$$\int_0^{\pi/2} \mathbf{f}(\delta)\mathrm{d}\delta = 0$$

Conclusion

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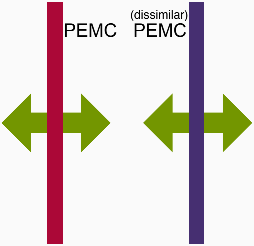

Casmir force between PEMCs

$$\mathbf{f}(\delta) = -\frac{\hbar c}{8\pi^2 L^4}\left[\frac{\pi^4}{30} - \delta^2(\pi-\delta)^2\right]\mathbf{e}_z$$

Zero-force parameter

$$\delta(f=0) = \frac{\pi}{2}\left(1-\sqrt{1-2\sqrt{\frac{2}{15}}}\right) \approx 0.96\cdot\frac{\pi}{4}$$

Sum rule

$$\int_0^{\pi/2}\mathbf{f}(\delta)\mathrm{d}\delta = 0$$