These exercises are intended to be solved emulating a real-time exam situation. Take the time it takes for you to solve each exercise and compare it to the suggested time in parenthesis.

1. **Potentials and phase-space portraits**

   Sketch the effective potential

   $$V_{eff} = -\cos x + \sec x$$

   and the corresponding phase-space portrait for $(-2\pi \leq x \leq 2\pi)$. *Hint*: use trigonometric identities to reduce the potential to sines and cosines for an easier sketching. Pay attention to the asymptotes of the plot. (*15-20 min*)

2. **Two springs in series**

   Consider a mass m attached to two springs connected in series, of constants k_1 and k_2, respectively. The mass is oscillating in one dimension. The distance from the origin to the junction of the springs is $s_1(t)$ and the distance from the origin to the mass is $s_2(t)$, i.e. the distance from the junction to the mass is $s_2 - s_1$

   (a) Find the Lagrangian of the system and show that the EOM for the mass is

   $$\ddot{s_2}(t) = -\frac{k_{eff}\ s_2(t)}{m}$$

   where $k_{eff} = \frac{k_1 k_2}{k_1+k_2}$

   Hint: Use $s_1(t)$ and $s_2(t)$ as generalized coordinates. Place a virtual mass m_0 at the junction of the springs while you are finding the Lagrangian, and make it vanish after you find the Euler-Lagrange equations for both coordinates. Then solve for s_2. (*10-15 min*)

   (b) Sketch the isolines and the gradient of the energy of the system. (*5-10 min*)

   (c) Solve the equation of motion from (a) with $k_1 = 1$, $k_2 = 3$, $m = \frac{3}{4}$, and initial conditions

   $$s_2(t_0) = \dot{s_2}(t_0) = 2$$

   (*5-10 min*)

3. **Binary star** (*35-45 min*)

The EOM for a a binary star system (or a planet and a moon) composed of stars A and B are

$$m_A \ddot{\mathbf{x}}_A = \mathbf{F}$$
$$m_B \ddot{\mathbf{x}}_B = -\mathbf{F}$$

where $\mathbf{x}_{A,B}$ and $m_{A,B}$ are the positions and masses of star A and B, respectively. The only force acting is the gravity force

$$\mathbf{F} = \frac{G m_A m_B}{|\mathbf{r}|^3}\mathbf{r}$$

where $\mathbf{r}(t) = \mathbf{x}_B(t) - \mathbf{x}_A(t)$. A problem with this type of systems is that the parameters (such as the masses and G) are very big, and the rotation can be either very small or very large. Therefore, nondimensionalization is quite useful.

(a) Consider the dimensionless positions $\boldsymbol{\xi}_A = \frac{\boldsymbol{x}_A}{L}$ and $\boldsymbol{\xi}_B = \frac{\boldsymbol{x}_B}{L}$, with $L = \mathbf{r}(0)$, then the gravity force takes the form $\mathbf{F} = \frac{G m_A m_B}{L^2 |\tilde{\mathbf{r}}|^3}\tilde{\mathbf{r}}$, where $\tilde{\mathbf{r}} = \boldsymbol{\xi}_B - \boldsymbol{\xi}_A$.
Show that the nondimensionalized EOM for A and B take the general form

$$\ddot{\boldsymbol{\xi}}_A = \frac{G m_B T^2}{L^3} \frac{\tilde{\mathbf{r}}}{|\tilde{\mathbf{r}}|^3}$$
$$\ddot{\boldsymbol{\xi}}_B = -\frac{G m_A T^2}{L^3} \frac{\tilde{\mathbf{r}}}{|\tilde{\mathbf{r}}|^3}$$

for some time scale T. Here $\ddot{\boldsymbol{\xi}}_{A,B} = \frac{\mathrm{d}^2 \boldsymbol{\xi}_{A,B}}{\mathrm{d}\tau^2}$, for some dimensionless time τ.

(b) Define $\alpha = \frac{m_A}{m_B}$. Find a value for the time scale T such that the EOM become

$$\ddot{\boldsymbol{\xi}}_A = \frac{1}{1+\alpha} \frac{\tilde{\mathbf{r}}}{|\tilde{\mathbf{r}}|^3}$$
$$\ddot{\boldsymbol{\xi}}_B = \frac{1}{1+\alpha^{-1}} \frac{\tilde{\mathbf{r}}}{|\tilde{\mathbf{r}}|^3}$$

(c) Show that in the limit $m_A \ll m_B$ we have that $\boldsymbol{\xi}_B = 0$ (what does it mean?) and the EOM for A becomes

$$\ddot{\boldsymbol{\xi}}_A = -\frac{\boldsymbol{\xi}_A}{|\boldsymbol{\xi}_A|^3}$$

(d) Introduce polar coordinates $\boldsymbol{\xi}_A = r\cos\theta \mathbf{i} + r\sin\theta \mathbf{j}$, and therefore $\boldsymbol{\xi}_A = r\mathbf{e}_r$. Show that the EOM for A become

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{r^2}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

what kind of movement occurs for the case $r = 1$?