Lecture Notes by Jürgen Vollmer

Theoretical Mechanics

— Working Copy — Chapter 5 —

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Contents

5
Impact of Spatial Extension

In Chapter 4 we discussed the motion of point particles. However, in our environment the spatial extension of particles in crucial. Physical objects always keep a minimum distance due to their spatial extension. When they had zero extension, one could neither blow up water droplets by impact with a laser (Figure 5.1), nor work clackers (Figure 5.2) or hit a ball with a tennis racket (Figure 5.3). Even giving spin to a ball only works due to the distance between the surface of the racket and the center of the ball.

At the end of this chapter we will be able to discuss the evolution of balls with spin, and their reflections from flat surfaces. Why is spin of so much importance in table tennis? How can a “Kreisläufer” score a goal in Handball, even when the goal keeper is fully blocking he direct path to the goal?

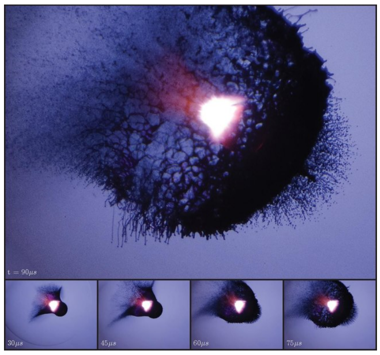


Figure 5.1: Impact of a laser pulse on a microdrop of opaque liquid that is thus blown up; cf. Klein, et al, Phys. Rev. Appl. 3 (2015) 044018

Punt/Anefo, Amsterdam 1971, CC0

Figure 5.2: Girl playing with clackers.

Charlie Cowins from Belmont, NC, USA, CC BY 2.0

Figure 5.3: Man running to return a tennis ball.

What is the magic of Beckham’s banana kicks?

5.1 *Motivation and outline: How do particles collide?*

In order to get a first impression about this idea we consider the case of two particles at the positions $\boldsymbol{q}_i$, $i \in \{1, 2\}$ that interact by a repulsive Coulomb force that derives from a potential $\Phi_C(|\boldsymbol{R}|)$ with $\boldsymbol{R} = \boldsymbol{q}_2 - \boldsymbol{q}_1$,

$$\Phi_C(|\boldsymbol{R}|) = \frac{C}{|\boldsymbol{R}|} \quad \Rightarrow \quad \boldsymbol{F}_c(\boldsymbol{q}_i) = -\nabla_{\boldsymbol{q}_i} \Phi_C(|\boldsymbol{q}_i - \boldsymbol{q}_{2-i}|) = \frac{C\,(\boldsymbol{q}_i - \boldsymbol{q}_{2-i})}{|\boldsymbol{q}_i - \boldsymbol{q}_{2-i}|^3}$$

Here, $2 - i$ is the index of the other particle (1 for $i = 2$ and 2 for $i = 1$), and the constant C is the product of the permittivity of the vacuum and the particle charges. For charges of opposite signs this force has agrees with the gravitational force when one substitutes $C \to -m_1\, m_2\, G$. This results in the same dimensionless equations of motion as obtained for the Kepler problem, with the important difference that the length and time units adopted to defined the dimensionless units take vastly different values.

When the two particles carry charges of equal signs the force is repulsive, giving rise to the EOM

$$0 = w''(\theta) + w(\theta) + \frac{\mu\, C}{L^2}\, w(\theta)$$

such that

$$R(\theta) = \frac{1}{W(\theta)} = \frac{R_0}{-1 + \epsilon\, \cos(\theta - \theta_0)} \quad \text{where} \quad R_0 = \frac{L^2}{\mu\, C}$$

agrees with Equation (4.9.3) up to a change of the sign of the one in the denominator and the length unit R_0.

Remark 5.1. It is illuminating to adopt a different perspective on the origin of the minus sign in front of the one. Let us write the force on particle 1 as $\boldsymbol{F}_1 = F_1\, \hat{e}(\theta)$ where $\hat{e}(\theta)$ is the vector pointing from particle 1 to particle 2. The strength of the scalar force F_1 will be positive for an attractive force and negative for a repulsive force. In the dimensionless force $F t_0^2 / \mu\, R_0$ the change of sign is taken into account by the sign of C in $R_0 = L^2 / \mu\, C$ and the solution takes the form of Equation (4.9.3). In order to obtain a positive length scale $|R_0| = \pm R_0$ we multiply the numerator and denominator of the solution by the ± 1 and absorb the factor in front of ϵ in a rotation of the angle by π such that the polar coordinates are always aligned with the direction of the force. Hence, one finds

$$R(\theta) = \frac{1}{W(\theta)} = \frac{|R_0|}{\pm 1 + \epsilon\, \cos(\theta - \theta_0)} \quad \text{where} \quad \pm 1 = \text{sign}(C)$$

At this point dimensionless units play out their strength. We obtain the solution of the nontrivial EOM by an analysis of the ODE and mapping of parameters to a known problem, rather than going again through the involved analysis.

The phase-space portrait and the shape of the orbits for repulsive interactions are plotted in Figure 5.4. We observe that the trajectory

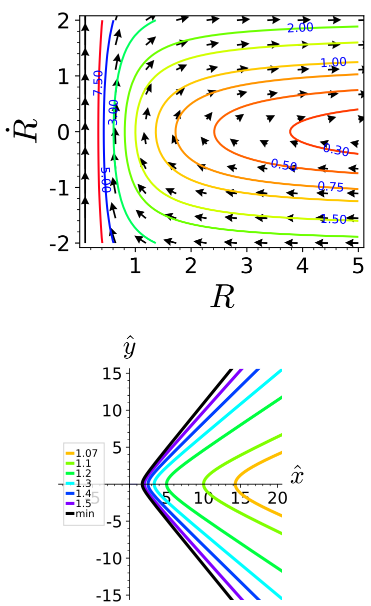


Figure 5.4: Phase-space flow and the shape of trajectories for scattering with a repulsive Coulomb potential.

shape describes the approach of the other particle from a perspective of an observer that sits on a particle located in the origin. When the observer sits on a particle that has a much larger mass than the approaching particle, then an outside observer will see virtually no motion of the mass-rich particle and the lines in Figure 5.4 describe the lines of the trajectories of the light particle in a plane selected by the initial angular momentum of the scattering problem. In general, two particles of masses m_1 and m_2 will be at opposite sides of the center of mass. In a coordinate system with its origin at the center of mass the lines in Figure 5.4 describe the particle trajectories up to factors $m_1/(m_1 + m_2)$ and $-m_2/(m_1 + m_2)$ for the first and second particle, respectively. A pair of trajectories for $m_1 = 0.3\,(m_1 + m_2)$ and $\epsilon = 1.2$ is shown in Figure 5.5. The approximation as point particles is well justified when the sum of the particle radii is much smaller than their closest approach $R_0/(\epsilon - 1)$.

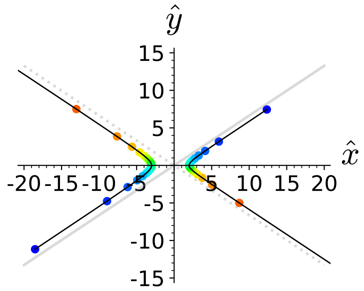


Figure 5.5: The two black lines show the scattering trajectories of two particles with $\epsilon = 1.2$ and relative mass $m_1 = 0.3\,(m_1 + m_2)$. They approach each other along the solid gray line and separate along the dotted line. Particle 1 is initially at the top right. Corresponding positions are marked by dots of matching color.

Outline

In Section 5.2 we study the collision of spherical hard-ball particles that only interact by a force kick vertical to the surfaces at their contact point when they touch. Then we compare the Coulomb case and the force-kick case in order to explore which features of the outgoing trajectories are provided by conservation laws, irrespective of the type of interaction. In Section 5.3 we discuss the forces of an extended object (Earth) on a point particle moving without further interactions in its gravitational field. In Section 5.4 we further explore the impact spatial extension of solid particles: How does their shape matter? How are particles set into spinning motion, and how does the spin evolve? Section 5.5 addresses the motion of particles with internal degrees of freedom. Finally, in Section 5.6 we wrap up the findings of this section by discussing the reflections of balls: How do balls pick up spin in collisions? What happens upon multiple collisions in a channel with parallel walls? How should one return a ping-pong ball arriving with severe spin? How much energy is dissipated into vibrations of the ball?

check and update upon finalizing Chapter

5.1.1 *Self Test*

Problem 5.1. **Scattering angle for the Coulomb potential**

For the choice of coordinates adopted in Figures 5.4 and 5.5 the trajectories have an asymptotic angle θ with the $\hat{x}$-axis when they approach each other and they separate with an asymptotic angle $-\theta$.

a) Show that

$$\tan^2\theta = \frac{2\,E\,L^2}{\mu\,C^2} \tag{5.1.1}$$

b) The parameter dependence of the scattering angle θ is shown in Figure 5.6. What happens to the line for very large values of $2\,E\,L^2/\mu\,C^2$?

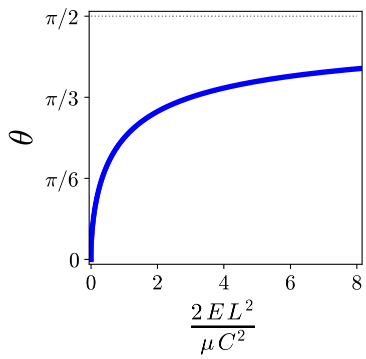


Figure 5.6: Scattering angle θ for a collision of two particles that interact by a repulsive Coulomb potential..

c) How would the scattering trajectories in Figure 5.5 look like for $\theta = \pi/2$? Does this comply with your finding in b)?

5.2 *Collisions of hard-ball particles*

We consider two spherical particles and denote their radii and masses as R_i and m_i with $i \in \{1,2\}$, respectively. At the initial time $t = t_0$ the particles motion is not (yet) subjected to a force such that

$$\boldsymbol{q}_i(t) = \boldsymbol{q}_i(t_0) + v_i\,(t - t_0)\,, \quad \text{for} \;\; i \in \{1,2\}$$

5.2.1 *Center of mass motion*

Analogous to the treatment of the Kepler problem, we decompose the motion of the particles into a center-of-mass motion $\boldsymbol{Q}(t)$ and a relative motion $\boldsymbol{r}(t)$. Introducing the notion $M = m_1 + m_2$ the former amounts to

$$M\,\boldsymbol{Q}(t) = m_1\,\boldsymbol{q}_1(t) + m_2\,\boldsymbol{q}_2(t) = M\,\boldsymbol{Q}(t_0) + \dot{\boldsymbol{Q}}(t_0)\,(t - t_0) \qquad (5.2.1)$$

Since there are not external forces the total momentum $M\,\dot{\boldsymbol{Q}}(t)$ is conserved (cf. Theorem 3.5) such that Equation (5.2.1) applies for all times – even when the particles collide. A collision will therefore only impact the evolution relative to the center of mass. Equation (5.2.1) holds for all times.

5.2.2 *Condition for collisions*

To explore the relative motion we write $\boldsymbol{q}_i = \boldsymbol{Q} + \boldsymbol{x}_i$, and we introduce the momentum $\boldsymbol{p} = m_1\,\dot{\boldsymbol{x}}_1 = -m_2\,\dot{\boldsymbol{x}}_2$ and the distance coordinate $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. With these notations the angular momentum of the relative motion reads $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, and it is conserved when the collision force is acting along the line connecting the centers of the particles (cf. Theorem 3.6 and the discussion of Kepler's problem in Section 4.7). Moreover, $\boldsymbol{r}(t)$ is the only time-dependent quantity in this equation because $\boldsymbol{L}$ and $\boldsymbol{p}$ are preserved. Let us first assume that the particles do not collide, and that the closest approach occurs at some time t_c to a distance $r_c = |\boldsymbol{r}(t_c)|$. Then the vectors $\boldsymbol{r}(t_c)$ and $\boldsymbol{p}$ will be orthogonal, and $|\boldsymbol{L}| = r_c\,|\boldsymbol{p}|$. By the properties of the vector product the distance of the closest encounter will always be

$$r_c = \frac{|\boldsymbol{L}|}{|\boldsymbol{p}|} = \frac{\big|m_1\boldsymbol{q}_1(t_0) \times \dot{\boldsymbol{q}}_1(t_0) + m_2\boldsymbol{q}_2(t_0) \times \dot{\boldsymbol{q}}_2(t_0) - M\boldsymbol{Q}(t_0) \times \dot{\boldsymbol{Q}}(t_0)\big|}{m_1\big|\dot{\boldsymbol{q}}_1(t_0) - \dot{\boldsymbol{Q}}\big|}$$

and there will be no collision if $r_c > R_1 + R_2$.

5.2.3 *The collision*

Conservation of angular momentum implies that the relative motion of the particles proceeds in a plane. When they collide they approach until, at time t_c, they reach a position $\boldsymbol{r}(t_c)$ where their distance is $|\boldsymbol{r}(t_c)| = R_1 + R_2$. We denote the direction of $\boldsymbol{r}$ at this time as $\hat{\boldsymbol{\beta}}$ and augment it by an orthogonal direction $\hat{\boldsymbol{\alpha}}$ such that $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{L}} = \boldsymbol{L}/|\boldsymbol{L}|)$ form an orthonormal basis. We select the origin of the associated coordinate system such that

$$\boldsymbol{p} = (\boldsymbol{p} \cdot \hat{\boldsymbol{\alpha}})\, \hat{\boldsymbol{\alpha}} + (\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})\, \hat{\boldsymbol{\beta}}$$

At the collision there is a force $\boldsymbol{F} = F\,\hat{\boldsymbol{\beta}}$ acting on the particles, that acts in the direction of the line $\boldsymbol{r}(t_c)$ connecting the particles. Hence,

1. the momentum component in the $\hat{\boldsymbol{\alpha}}$ direction is preserved during the collision because there is no force acting in this direction
2. the collision in $\hat{\boldsymbol{\beta}}$ direction proceeds like a one-dimensional collision, Example 3.12, with the exception that one must retrace the argument using the center-of-mass frame, as discussed in Problem 4.29.

Consequently, we obtain the following momentum $\boldsymbol{p}'$ after the collision

$$\boldsymbol{p}' = (\boldsymbol{p} \cdot \hat{\boldsymbol{\alpha}})\, \hat{\boldsymbol{\alpha}} - (\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})\, \hat{\boldsymbol{\beta}} = \boldsymbol{p} - 2\,(\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})\, \hat{\boldsymbol{\beta}}$$

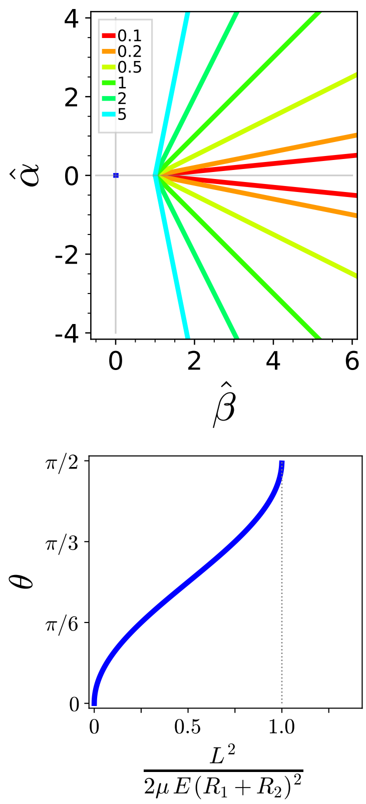


Figure 5.7: Collision of two hard-ball particles with radii R_1 and R_2: (top) Trajectory shape. The labels denote the ratios $(\boldsymbol{p} \cdot \hat{\boldsymbol{\alpha}})/(\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})$. (bottom) Scattering angle θ.

5.2.4 *Self Test*

Problem 5.2. **Scattering angle for hard-ball particles**

In Figure 5.7 we show shows the trajectory shape and the scattering angle for hard-ball scattering.

a) What is the dimensionless length scale adopted to plot the trajectory shapes?

b) What is the impact of the angular momentum on the trajectory shape?
What is the impact of the energy?

c) Verify that

$$\sin^2\theta = \frac{L^2}{2\mu\, E\, (R_1 + R_2)^2} \tag{5.2.2}$$

and that this dependence is plotted in the lower panel of Figure 5.7.

d) What happens when $L^2 > 2\mu\, E\, (R_1 + R_2)^2$?
Which angle θ will one observe in that case?

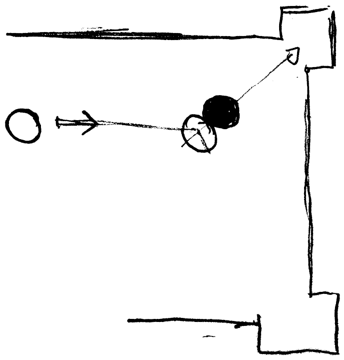
e) Show that Equations (5.1.1) and (5.2.2) agree when one identifies the length scale $R_1 + R_2$ of the hard-ball system with the distance R_{eff} of symmetry point of the cone section from the origin, i.e. with the mean value of the two intersection points with the $\hat{x}$-axis

$$R_{\text{eff}} = \frac{1}{2}\left(\frac{R_0}{1+\epsilon} + \frac{R_0}{1-\epsilon}\right) = \frac{\epsilon\, R_0}{1-\epsilon^2}$$

Can you provide a physical argument why that must be true?

Problem 5.3. **Reflection from a wall**

Show that a particle reflected at a flat wall follows the same trajectory as a particle that collides with a particle of the same mass and at a position obtained as mirror image of the particle.



Problem 5.4. **Collisions on a billiard table**

The sketch to the right shows a billiard table. The white ball should be kicked (i.e. set into motion with velocity $\boldsymbol{v}$), and hit the black ball such that it ends up in pocket to the top right.

What is tricky about the sketched track?

What might be a better alternative?

5.3 *Volume integrals — A professor falling through Earth*

The center of mass of a set of particles was defined in Equation (4.6.1) as a weighted sum of their positions. Now we consider an extended object that is characterized by a mass distribution $\rho(\boldsymbol{q})$. We will always assume that the distribution varies slowly in space in side the object. Outside it vanishes. The weighted sum over the particle positions will then be generalized to become a volume integral.

5.3.1 *Determine volume and mass by volume integrals*

In Section 3.4.2 we introduced line integrals by dividing the integration path into small steps $\{s_i\}$, and approximating the integral as a sum over the contributions of the individual pieces. The definition of a volume integrals proceeds analogously. Now, we integrate over a region $R \subset \mathbb{R}^D$, and we start by partitioning this region into small *volume elements* ΔV_i.

Definition 5.1: Partition of Space

A set $\{\Delta V_i\,,\ i \in \mathsf{I}\}$ is a *partition* of a region $R \subset \mathbb{R}^D$ iff

a) $\forall i \in \mathsf{I} \,:\, \Delta V_i \subset R$,

b) $\forall i, j \in \mathsf{I} \,:\, i \neq j \Rightarrow \Delta V_i \bigcap \Delta V_j = \emptyset$,

c) $\forall x \in R\ \exists i \in \mathsf{I} \,:\, x \in \Delta V_i$.

Definition 5.1 entails that the union of the elements of the partition amounts to the region R,

$$R = \bigcup_{i \in \mathsf{I}} \Delta V_i$$

Let now $V = \|R\|$ denote the volume of the region R. For every partition it can be written as

$$V = \|R\| = \sum_{i \in \mathsf{I}} \|\Delta V_i\|$$

In the limit of small volume elements we write this sum as a

Definition 5.2: Volume Integral

The *volume integral* F of a function $f(\boldsymbol{q})$ over a region $R \subset \mathbb{R}^D$ is defined as follows as limit of a sum over the elements of a partition,[1] $\{\Delta V_i\,,\ i \in \mathsf{I}\}$ of R and points $\boldsymbol{q}_i \in \Delta V_i$,

$$F = \int_R \mathrm{d}^D q\; f(\boldsymbol{q}) = \lim_{\|\Delta V_i\| \to 0} \sum_{i \in \mathsf{I}} \|\Delta V_i\|\, f(\boldsymbol{q}_i)$$

For Cartesian coordinates $(q_1, q_2, \dots, q_D)$ the integration volume element is $\mathrm{d}^D q = \mathrm{d}q_1 \cdots \mathrm{d}q_D$ and the integral amounts to

$$F = \int_{I_1} \mathrm{d}q_1 \int_{I_2(q_1)} \mathrm{d}q_2 \cdots \int_{I(q_1, \cdots, q_{D-1}} \mathrm{d}q_D\; f(q_1, \cdots, q_D)$$

where the boundaries of the integrals must be chosen such that $(q_1, \cdots, q_D) \in R$.

Remark 5.2. For the function $f(\boldsymbol{q}) = 1$ the volume integral provides the D-dimensional volume of the region R. 🕮

The mass $m(V)$ contained in a volume $\mathcal{V}$ can be expressed as a volume integral

$$m(V) = \int_{\mathcal{V}} \mathrm{d}^3 q\; \rho(\boldsymbol{q}) = \int_{\mathcal{V}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\; \rho(x,y,z)$$
$$= \int_{x_{\min}}^{x_{\max}} \mathrm{d}x \left[\int_{y_{\min}(x)}^{y_{\max}(x)} \mathrm{d}y \left(\int_{z_{\min}(x,y)}^{z_{\max}(x,y)} \mathrm{d}z\; \rho(x,y,z) \right) \right]$$

where the integration runs over all $\boldsymbol{q} \in \mathcal{V}$, a volume with smallest x-value $x_{\min}$ and largest x-value $x_{\max}$, its y-values between $y_{\min}(x)$ and $y_{\max}(x)$ for a given x, and z-values between $z_{\min}(x,y)$ and $y_{\max}(x,y)$ for given x and y.

Remark 5.3. We adopt the convention that the mass density is zero outside an object. As a consequence its total M mass is obtained as

$$M = \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})$$

The boundaries of the integral that define the shape of the body have been absorbed into the definition of the density. 🕮

We illustrate the steps taken to evaluate a volume integral by calculating the area and volume of some simple geometric shapes:

Example 5.1: Surface areas of rectangles and circles

a) The surface area of the rectangle $R \subset \mathbb{R}^2$ with $(x,y) \in R$ iff $0 \le x \le a$ and $-b < y < b$ is

$$\|R\| = \int_R \mathrm{d}^2 q = \int_0^a \mathrm{d}x \int_{-b}^{b} \mathrm{d}y = 2ab$$

b) The surface area of the circle C with center at the origin

[1] Considerable care is taken in calculus courses to explore under which conditions the limit exists and is well-defined. Here, we assume that the function f varies smoothly inside the region. In other words, we assume that for all partition elements the difference $|f(\boldsymbol{q}) - f(\boldsymbol{q}_i)| \lll |f(\boldsymbol{q}_i)|$ for all points $\boldsymbol{q} \in \Delta V_i$.

and radius R is

$$\|C\| = \int_C \mathrm{d}^2 q = \int_{-R}^{R} \mathrm{d}x \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}y = 2 \int_{-R}^{R} \mathrm{d}x \, \sqrt{R^2 - x^2}$$
$$= 2\,R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos\theta \, \sqrt{1 - \sin^2\theta} = 2\,R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos^2\theta$$
$$= R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \left(\cos^2\theta + \sin^2\theta\right) = \pi\,R^2$$

The choice of the integration boundaries is illustrated in Figure 5.8. Upon moving to the second line of this equation we substituted $x = R\,\sin\theta$, and in the step to the third line we made use of the π-periodicity of $\cos^2\theta$.

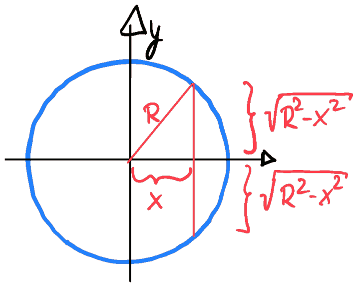


Figure 5.8: Notations adopted in the surface integral performed in Example 5.1b).

Example 5.2: Volume of a sphere

The volume of a three-dimensional sphere S with center at the origin and radius R is

$$\|S\| = \int_S \mathrm{d}^2 q = \int_{-R}^{R} \mathrm{d}x \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}y \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \mathrm{d}z$$
$$= \int_{-R}^{R} \mathrm{d}x \, \pi \left(\sqrt{R^2 - x^2}\right)^2 = \pi \int_{-R}^{R} \mathrm{d}x \, \left(R^2 - x^2\right)$$
$$= \pi \left(2\,R^3 - \frac{2}{3}\,R^3\right) = \frac{4\,\pi}{3} R^3$$

Upon moving to the second line we observed that the y and z integrals agreed with the ones performed to evaluate the are of a circle, cf. Example 5.1b).

5.3.2 *Change of variables*

The shape of a circle with center at the origin and radius R can much easier be described by polar coordinates rather than Cartesian coordinates:[2] $\{(\rho, \theta) \in \mathbb{R}^+ \times [0, 2\pi) \,:\, \rho < R\}$. To take advantage of this simplification we have to introduce a transformation of the integration coordinates from Cartesian to polar coordinates. A heuristic guess based on Figure 5.9 suggests that a volume element $\mathrm{d}x\,\mathrm{d}y$ at the position $(x, y) = (\rho\,\cos\theta, \rho\,\sin\theta)$ should be replaced by $\rho\,\mathrm{d}\theta\,\mathrm{d}\rho$. One readily verifies that this is a reasonable choice by working out the area of the circle with radius R:

[2] In order to avoid confusion with the radius of the circle the radial coordinate of the polar coordinates is here denoted as ρ.

$$\|C\| = \int_C \mathrm{d}^2 q = \int_0^R \mathrm{d}R \int_0^{2\pi} \mathrm{d}\theta \, R = \pi \int_0^R \mathrm{d}R \, 2\,R = \pi\,R^2$$

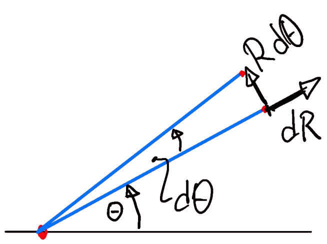


Figure 5.9: Integration volume for polar coordinates.

with a much easier calculation than in Example 5.1b).

Formally the change of the integration volume is determined by generalizing the substitution rule for integrals, as illustrated in Figure 3.13 for one dimensional integrals. In this rule the derivative $f'(x)$ account for the change of the width of the rectangles that are summed to approximate the integral. In order to generalize this idea we recall from the discussion of line integrals that

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}R}\,\mathrm{d}R = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\mathrm{d}R \quad\text{and}\quad \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\theta}\,\mathrm{d}\theta = \begin{pmatrix}-R\,\sin\theta\\ R\,\cos\theta\end{pmatrix}\mathrm{d}R$$

In general the derivatives involved in the definition of the length elements do not have unit length and they need not be orthogonal. Their length reflects the change of the length unit that we also encounter in the one-dimensional case. The angle between the vectors indicates that in two-dimensions one can also partition space by using parallelograms rather than rectangle. The unit area for the integration will always be the area spanned by the two vectors.

In D dimensions the integration volume is defined by the volume spanned by the D derivative vectors of the position vector $\boldsymbol{q}$ with respect to the new coordinates. It is commonly expressed in terms of the Jacobi determinant. We first introduce the notion of a

Definition 5.3: Determinant

The *determinant* of a matrix amounts to the volume spanned by its column vectors. For a matrix A it is denoted as $\det \mathsf{A}$.

Remark 5.4. The determinant of 2×2 and 3×3 matrices takes the form of the (sum of) products along the diagonals from left to right minus the (sum of) products of the diagonals from right to left,

$$\det\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$$

$$\det\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} \\ - a_{11}\,a_{23}\,a_{32} + a_{22}\,a_{31}\,a_{13} + a_{33}\,a_{12}\,a_{21}$$

These expressions are entailed by the geometric interpretation of the cross product in Section 2.9.2.

Without proof we provide the following general rule for calculating determinants

provide a reference

Theorem 5.1: Recursive calculation of determinants

Let A be a $D\times D$ matrix with $D \in \mathbb{N}$ and entries a_{ij} where $i,j \in \{1,\cdots,D\}$. For $D=1$ we define $\det\mathsf{A} = a_{11}$. For $D>1$ we introduce the $(D-1)\times(D-1)$ submatrices A_{ij} that are obtained from A by dropping its ith row and jth column.
The determinant of A can then be calculated by a recursion that either works along a row j or a column k of A,

$$\det\mathsf{A} = \sum_{j=1}^{D}(-1)^{j+k}\,\det\mathsf{A}_{jk} = \sum_{k=1}^{D}(-1)^{j+k}\,\det\mathsf{A}_{jk}$$

Altogether this allows us to identify the factor involved in a change of the integration variables as the Jacobi determinant.

Theorem 5.2: Jacobi matrix and determinant

We consider a change of integration variables from the coordinates $\boldsymbol{x} = (x_1, x_2, \cdots, x_D)$ to $(y_1, y_2, \cdots, y_D)$ that is defined by the functions $x_1(\boldsymbol{y})$, $x_2(\boldsymbol{y})$, $\cdots$, $x_D(\boldsymbol{y})$. Then the integration volume changes as

$$\mathrm{d}x_1 \cdots \mathrm{d}x_D = \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_D} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_D} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_D}{\partial y_1} & \frac{\partial x_D}{\partial y_2} & \cdots & \frac{\partial x_D}{\partial y_D} \end{pmatrix} \mathrm{d}y_1 \cdots \mathrm{d}y_D$$

The matrix involved in this transition is called the *Jacobi matrix* of the transition, and the determinant is called the *Jacobi determinant*.

Example 5.3: Integration volumes

a) *polar coordinates* $(x, y) = \rho\,(\cos\theta,\ \sin\theta)$

transform as

$$\mathrm{d}x\,\mathrm{d}y = \det \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \mathrm{d}\rho\,\mathrm{d}\theta = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

b) *cylindrical coordinates* $(x, y, z) = (\rho\cos\theta,\ \rho\sin\theta, z)$

transform as

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \det \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z$$

c) *spherical coordinates* $(x, y, z) = \rho\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$

transform as

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\,\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi = \rho^2\,\mathrm{d}\rho\,\mathrm{d}\cos\theta\,\mathrm{d}\phi$$

5.3.3 *The force of an extended object (Earth) on a point particle (professor)*

As a first step towards discussing extended objects we consider the force exerted by an extended object on a point particle. The force is obtained by integrating the forces originating from the mass elements of the body,

$$\boldsymbol{F}_{\text{tot}} = \int_{\mathbb{R}^3} \mathrm{d}^3 q\ \boldsymbol{F}(\boldsymbol{q})$$

where $\boldsymbol{q}$ is the vector from the position of the point particle to the mass element that is exerting the force. This expression involves the vector-valued generalization of volume integrals. It should be interpreted component-wise, stating the that the components $\boldsymbol{F}_{\text{tot},i} = \hat{\boldsymbol{e}}_i \cdot \boldsymbol{F}_{\text{tot}}$ of the total force in some orthonormal base $\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z$ amount to

$$\hat{\boldsymbol{e}}_i \cdot \boldsymbol{F}_{\text{tot}} = \int_{\mathbb{R}^3} \mathrm{d}^3 q \, (\hat{\boldsymbol{e}}_i \cdot \boldsymbol{F}(\boldsymbol{q}))$$

or explicitly
$$\boldsymbol{F}_{\text{tot},x} = \int_{\mathbb{R}^3} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, F_x(x,y,z)$$
$$\boldsymbol{F}_{\text{tot},y} = \int_{\mathbb{R}^3} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, F_y(x,y,z)$$
$$\boldsymbol{F}_{\text{tot},z} = \int_{\mathbb{R}^3} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, F_z(x,y,z)$$

The consequences can nicely be explored when an evil witch switches off electromagnetic interactions between a physics professor and its environment. In the absence of interaction with other matter the professor will freely fall towards the center of Earth, accelerated by a force that arises as sum of the mass elements constituting Earth (see Figure 5.10). For the professor of mass m at position $\boldsymbol{q}_P$ and the mass element at position $\boldsymbol{q}_e$ this force amounts to $\boldsymbol{F}(\boldsymbol{q}_P, \boldsymbol{q}_e) = -\nabla(m\,\rho(\boldsymbol{q}_e)\,G)/\,|\boldsymbol{q}_P - \boldsymbol{q}_e|$. For simplicity we assume that Earth is spherical and that its mass density takes a uniform value $\rho = 3\,M_E/4\pi\,R^3$. Then, the force on the professor takes the form

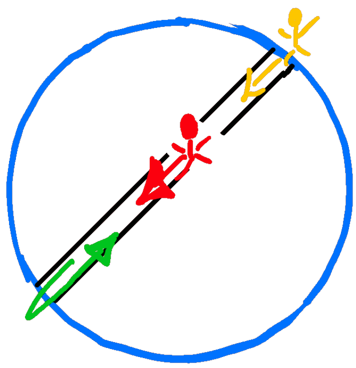


Figure 5.10: Initially positioned at the upper right (yellow), the professor will fall down (red), and eventually pop out at the other side and return (green).

$$\boldsymbol{F}_{\text{tot}} = -\int_{\mathbb{R}^3} \mathrm{d}^3 q \, \nabla \frac{m\,\rho(\boldsymbol{q}_e)\,G}{|\boldsymbol{q}_P - \boldsymbol{q}_e|} \tag{5.3.1}$$
$$= -m\,\rho\,G\,\nabla \int_{\text{Earth}} \mathrm{d}^3 q \, \frac{1}{\sqrt{q_P^2 + q_e^2 - 2\,q_P\,q_e\,\cos\theta}} \tag{5.3.2}$$

where θ is the angle between the two vectors $|\boldsymbol{q}_P|$ and $|\boldsymbol{q}_e|$, while q_P and q_e denote their respective length.

The integral is best evaluated by adopting a spherical coordinates (r, θ, ϕ) for the integration where r runs from zero to the Earth radius R, the angle θ from zero to π, and ϕ all around from zero to 2π,

$$\begin{aligned}
\boldsymbol{F}_{\text{tot}} &= -m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, r^2 \int_{-1} 1 \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi \, \frac{1}{\sqrt{q_P^2 + r^2 - 2\,q_P\,r\,\cos\theta}} \\
&= -2\pi\,m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, r^2 \left[\frac{-1}{q_P\,r} \sqrt{q_P^2 + r^2 - 2\,q_P\,r\,\cos\theta} \right]_{\cos\theta=-1}^{\cos\theta=1} \\
&= -2\pi\,m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, \frac{r}{q_P} \left(|q_P + r| - |q_P - r| \right) \\
&= -4\pi\,m\,\rho\,G\,\nabla \left[\frac{1}{q_P} \int_0^{q_P} \mathrm{d}r\, r^2 + \int_{q_P}^R \mathrm{d}r\, r \right] \\
&= m\,\rho\,G\,\nabla \left[2\pi R^2 - \frac{2\pi}{3} \boldsymbol{q}_P \cdot \boldsymbol{q}_P \right] = -\frac{m\,g}{R}\,\boldsymbol{q}_P
\end{aligned}$$

In the last step we used that the acceleration on the Earth surface is $g = MG/R = 4\pi\,\rho\,R^2\,G/3$. The professor moves under the influence of a *harmonic* central force, as studied in Problems 4.19, 4.21 and 4.30! After a while (cf. Problem 5.12) he reappears at the very same spot where he started, except that Earth moved on while he was under way.

5.3.4 *Self Test*

Problem 5.5. **Area of a parallelogram**

Determine the area of the parallelogram defined by the points $(0,0), (1,3), (4,4), (2,1)$ by

a) performing the volume integral,

b) determining the area spanned by the two vectors that define the sides starting at the corner $(0,0)$.

Problem 5.6. **Volume of a solid of revolution**

A solid of revolution is obtained by rotating some function $f(x)$ around the x axis. For instance, the function $\sqrt{R^2 - x^2}$ with $-R \leq x \leq R$ describes a sphere of radius R. The volume V of a solid of revolution are given by the integral

$$V = \pi \int \mathrm{d}x \; \big(f(x)\big)^2 \tag{5.3.3}$$

a) Sketch the function $f(x) = \sqrt{R^2 - x^2}$ and verify that the solid of revolution is indeed a sphere.

b) Determine the volume of the sphere based on the given equation. Compare you calculation and the result to the calculation given in Example 5.2.

c) Show that the volume integral for a solid of revolution provides Equation (5.3.3) when one adopts cylindrical coordinates.

Problem 5.7. **Volume of a cone**

Determine the volume of a cone with symmetry axis along the z-axis, that stands on the (x,y)-plane where it traces a circle of radius R, while its vertex is at $(0,0,H)$.

a) Perform the volume integral with Cartesian coordinates.

b) Perform the volume integral with cylindrical coordinates.

Problem 5.8. **Coordinate transformation to cylindrical coordinates**

Determine the Jacobi matrix and its determinant for the transformation from Cartesian to spherical coordinates, cf. Example 5.3c).

5.4 *Center of mass and spin of extended objects*

We consider a setting where there are only long distance force like gravity and no collisions between objects. The explicit calculation for the case of gravity in the previous section entails that in such a setting the force exerted by a planet on a point particle is identical to the one exerted by a mass point of identical mass that is located at the center of the planet (see also Problem 5.12e). In the present section we therefore explore which effects the force of a point particle exerts on an extended body.

5.4.1 *Evolution of the center of mass*

The force on the body is described by an integral that takes exactly the same form as Equation (5.3.1), where now $\boldsymbol{q}$ is a vector from the point particle to a volume element of the body.

The integral is best evaluated by introducing a coordinate frame $\hat{\boldsymbol{e}}_1(t), \dots, \hat{\boldsymbol{e}}_3(t)$ with orientation fixed in the rotating body and origin in the body's center of mass $\boldsymbol{Q} = (Q_x, Q_y, Q_z)$. In immediate generalization of Equation (4.6.1) it is located at

$$\boldsymbol{Q} = \frac{1}{M} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, \boldsymbol{q} \quad \Leftrightarrow \quad \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \begin{pmatrix} M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_x \\ M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_y \\ M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_z \end{pmatrix}$$

A given mass element will always have the same coordinates (r_1, r_2, r_3) with respect to the body-fixed basis, and in a stationary coordinate frame this position can be specified as

$$\boldsymbol{q}(t) = \boldsymbol{Q}(t) + \sum_{i=1}^{3} r_i\, \hat{\boldsymbol{e}}_i(t)$$

Remark 5.5. The vector $\boldsymbol{r}$ describes the position $(r_1, \dots, r_3)$ in the body with respect to its center of mass. When the body rotates $\boldsymbol{r}$ will evolve in time. However, the coordinates $(r_1, \dots, r_3)$ are constant numbers describing the shape of the body when they are calculated in a coordinate system with base vectors $\{\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_3\}$ fixed in the body and origin in its center of mass. Hence,

$$\boldsymbol{r} = \sum_{i=1}^{3} r_i\, \hat{\boldsymbol{e}}_i(t) \quad \text{and} \quad \dot{\boldsymbol{r}} = \sum_{i=1}^{3} r_i\, \dot{\hat{\boldsymbol{e}}}_i(t) \qquad (5.4.2)\ (5.4.1)$$

We note that this choice of coordinates entails

$$\begin{aligned} M\, \boldsymbol{Q} &= \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, \boldsymbol{q} = \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, \Big(\boldsymbol{Q}(t) + \sum_{i=1}^{3} r_i\, \hat{\boldsymbol{e}}_i(t) \Big) \\ &= \boldsymbol{Q}(t) \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q}) + \sum_{i=1}^{3} \hat{\boldsymbol{e}}_i(t) \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, r_i \\ \Rightarrow \quad 0 &= \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, r_i = \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(\boldsymbol{r})\, r_i \end{aligned} \qquad (5.4.3)$$

The latter equality holds because a shift of the origin by $\boldsymbol{Q}$ and rotation of the coordinate axes do not affect the integration volume (i. e. the Jacobi determinant of the transformation is one).

The acceleration $\ddot{\boldsymbol{q}}(t)$ takes the form

$$\ddot{\boldsymbol{q}}(t) = \ddot{\boldsymbol{Q}}(t) + \sum_{i=1}^{3} r_i \, \ddot{\hat{\boldsymbol{e}}}_i(t)$$

and the force on the spatially extended body results in

$$\begin{aligned}
\boldsymbol{F}_{\text{tot}} &= \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, \ddot{\boldsymbol{q}}(t) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, \left(\ddot{\boldsymbol{Q}}(t) + \sum_{i=1}^{3} r_i \, \ddot{\hat{\boldsymbol{e}}}_i(t) \right) \\
&= M \, \ddot{\boldsymbol{Q}} + \sum_{i=1}^{3} \ddot{\hat{\boldsymbol{e}}}_i(t) \int_{\mathbb{R}^3} \mathrm{d}^3 r \, \rho(\boldsymbol{r}) \, r_i = M \, \ddot{\boldsymbol{Q}}
\end{aligned} \tag{5.4.4}$$

The overall force $\boldsymbol{F}_{\text{tot}}$ results in an acceleration of the center of mass that behaves exactly as for a point particle described in the previous chapter. Thus, we have justified the assumption of point particles adopted in Chapter 4.

5.4.2 *Angular momentum and particle spin*

Let us now explore the angular momentum of a spatially extended particles. To this end we introduce the decomposition $\boldsymbol{q} = \boldsymbol{Q} + \boldsymbol{r}$ into the definition

$$\begin{aligned}
\boldsymbol{L}_{\text{tot}} &= \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, (\boldsymbol{q} \times \dot{\boldsymbol{q}}) = \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, ((\boldsymbol{Q} + \boldsymbol{R}) \times (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{r}})) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, (\boldsymbol{Q} \times \dot{\boldsymbol{Q}}) + \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, (\boldsymbol{Q} \times \dot{\boldsymbol{r}}) \\
&+ \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, (\boldsymbol{r} \times \dot{\boldsymbol{Q}}) + \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \\
&= M \, \boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \boldsymbol{Q} \times \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \boldsymbol{r}(t) - \dot{\boldsymbol{Q}} \times \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \boldsymbol{r}(t) \\
&+ \int_{\mathbb{R}^3} \mathrm{d}^3 q \, \rho(\boldsymbol{q}) \, (\boldsymbol{r} \times \dot{\boldsymbol{r}})
\end{aligned}$$

The first summand amounts to the angular momentum of the center of mass, $\boldsymbol{L}_{CM} = M \, \boldsymbol{Q} \times \dot{\boldsymbol{Q}}$. The second and the third term vanish due to Equation (5.4.3). The forth term can be simplified by performing the integration in the comoving coordinate frame. The coordinate transformation involves a translation by $\boldsymbol{Q}$ and rotation. Hence, the Jacobi determinant is one, and the term only depends on the local coordinates $\boldsymbol{r}$. It is denoted as particle spin.

Definition 5.4: Particle Spin

The *total angular momentum* $\boldsymbol{L}_{\text{tot}}$ of a particle can be decomposed into the angular momentum $\boldsymbol{L}_{CM}$ of its center-of-mass motion, and its *spin* $\boldsymbol{S}$ around the center of mass, $\boldsymbol{Q}$,

$$\boldsymbol{L}_{\text{tot}} = \boldsymbol{L}_{CM} + \boldsymbol{S} \tag{5.4.5a}$$

$$\text{with} \quad \boldsymbol{L}_{CM} = M \, \boldsymbol{Q} \times \dot{\boldsymbol{Q}} \tag{5.4.5b}$$

$$\boldsymbol{S} = \int_{\mathbb{R}^3} \mathrm{d}^3 r \, \rho(r_1 . r_2 . r_3) \, \boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{5.4.5c}$$

Remark 5.6. The decomposition of the total angular momentum has important consequences in collisions. For spatially extended objects the conservation of angular momentum only implies that the sum of the spin and the angular momentum of the center-of-mass motion are conserved. As a consequence, the incoming and outgoing angle can differ for a reflection at a wall, and the center of mass of the particle can even move in different planes before and after the collision. This will be demonstrated in the worked example in Section 5.6.

The discussion of particle spin can further be simplified by expressing the rotation of the body by the vector $\boldsymbol{\Omega}$ that indicates the rotation axis and angular speed $|\boldsymbol{\Omega}|$, and exploring that the relative positions of the mass elements in the body do not change upon rotation. Due to Equation (5.4.1) we have

$$\begin{aligned} \boldsymbol{S} &= \int_{\mathbb{R}^3} \mathrm{d}^3 r \, \rho(\boldsymbol{r}) \, \boldsymbol{r} \times \dot{\boldsymbol{r}} \\ &= \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_i \times \dot{\hat{\boldsymbol{r}}}_j \int \mathrm{d}^3 r \, r_i \, r_j \, \rho(\boldsymbol{r}) = \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_i \times \dot{\hat{\boldsymbol{r}}}_j \, t_{ij} \\ \text{with} \quad t_{ij} &= \int_{\mathbb{R}^3} \mathrm{d}r_1 \, \mathrm{d}r_2 \, \mathrm{d}r_3 \, r_i \, r_j \, \rho(r_1, r_2, r_3) \end{aligned}$$

Note that the coefficients t_{ij} are properties of the body. They characterize the mass distribution of the body, and do not depend on the motion. The situation simplifies further when one observes that the velocities $\dot{\hat{\boldsymbol{r}}}_k$ are unit vectors that must be orthogonal to $\hat{\boldsymbol{r}}_k$ and to Ω.[3] Hence, the velocities can be expressed as

$$\dot{\hat{\boldsymbol{r}}}_k = \hat{\boldsymbol{r}}_k \times \boldsymbol{\Omega}$$

[3] Recall that $\hat{\boldsymbol{r}}_k \cdot \hat{\boldsymbol{r}}_k = 1$ such that $2 \, \hat{\boldsymbol{r}}_k \cdot \dot{\hat{\boldsymbol{r}}}_k = 0$, and by construction the motion of mass elements is orthogonal to the axis of rotation.

With this notations the kth component of $\boldsymbol{S}$ can be expressed as

check signs!

$$\begin{aligned} S_k = \hat{\boldsymbol{r}}_k \cdot \boldsymbol{S} &= \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_k \cdot \left(\hat{\boldsymbol{r}}_i \times (\hat{\boldsymbol{r}}_j \times \boldsymbol{\Omega}) \right) t_{ij} \\ &= \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_k \cdot \left(\hat{\boldsymbol{r}}_j \, (\boldsymbol{\Omega} \cdot \hat{\boldsymbol{r}}_i) - \boldsymbol{\Omega} \, (\hat{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{r}}_j) \right) t_{ij} \\ &= \sum_{ij=1}^{3} \left(\delta_{jk} \, \Omega_i - \Omega_k \, \delta_{ij} \right) t_{ij} \\ &= \sum_{i=1}^{3} \Omega_i \sum_{j=1}^{3} \delta_{jk} \, t_{ij} - \Omega_k \sum_{ij=1}^{3} \delta_{ij} t_{ij} \\ &= \sum_{i=1}^{3} \Omega_i \left(t_{ik} - \delta_{ik} \sum_{j} t_{jj} \right) \end{aligned}$$

This amounts to a multiplication of the vector $\boldsymbol{\Omega}$ written in terms of its components Ω_i. We summarize this observation in the following definition

Definition 5.5: Tensor of Inertia

The rotation of a solid body with a fixed mass distribution $\rho(\boldsymbol{r})$ can be described by a vector Ω that defines the rotation axis and speed. It is related to the spin $\boldsymbol{S}$ of the body by multiplication with the *tensor or inertia*

$$\boldsymbol{S} = \Theta\,\boldsymbol{\Omega}\,,$$

i. e. a symmetric matrix with components

$$\Theta_{ij} = \int_{\mathbb{R}^3} \mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3\,\left(r_i\,r_j - \sum_{k=1}^{3} r_k\,r_k \right)\,\rho(r_1, r_2, r_3)$$

Example 5.4: Intertial tensor for a solid ball

For a ball of radius R with uniform mass density ρ the tensor of inertia has the following entries for its diagonal elements

$$\Theta_{ii} = \int_{\mathbb{R}^3} \mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3\,\left(r_i\,r_i - \sum_{k=1}^{3} r_k\,r_k \right)\,\rho(r_1, r_2, r_3)$$

We evaluate the integral in spherical coordinates with $r = |\boldsymbol{r}|$ and θ denoting the angle with respect to the i-axis, which we denote as z-axis in the following. Hence,

$$(r_x, r_y, r_z) = r\,(\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta)\,,$$

and

$$\begin{aligned}
\Theta_{ii} &= \rho \int_0^R \mathrm{d}r\,r^2 \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi\,(r_x^2 + r_y^2) \\
&= 2\,\pi\,\rho \int_0^R \mathrm{d}\rho\,r^2 \int_{-1}^{1} \mathrm{d}\cos\theta\,r^2\,\sin^2\theta \\
&= 2\,\pi\,\rho \left(\int_0^R \mathrm{d}\rho\,r^4 \right) \left(\int_{-1}^{1} \mathrm{d}\cos\theta\,\left(1 - \cos^2\theta\right) \right) \\
&= 2\,\pi\,\rho\,\frac{R^5}{5}\,\left(2 - \frac{2}{3}\right) = \frac{2}{5}\,M\,R^2
\end{aligned}$$

Moreover, for the off-diagonal element θ_{ik} we align the k-axis with $\phi = 0$ and find

$$\begin{aligned}
\Theta_{ik} &= \rho \int_0^R \mathrm{d}r\,r^2 \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi\,r_x\,r_z \\
&= \rho \int_0^R \mathrm{d}r\,r^2 \int_0^{\pi} \mathrm{d}\theta\,\sin\theta \int_0^{2\pi} \mathrm{d}\phi\,r^2\,\sin\theta\,\cos\theta \\
&= 2\pi\,\rho\,\frac{R^5}{5} \int_0^{\pi} \mathrm{d}\theta\,\sin^2\theta\,\cos\theta = 0
\end{aligned}$$

since $\sin^2\theta\,\cos\theta$ is antisymmetric with respect to $\pi/2$.

The finding that the off-diagonal elements of the tensor of inertia vanish is no coincidence. In **??** we will show that this happens whenever the mass distribution features a symmetry in the ik plane. Moreover, the ...theorem of linear algebra states that one can always choose coordinates where all off-diagonal elements of the tensor of inertia vanish.[4] The particular axes where this happens are called the axis of inertia of a body.

fill in name and reference

[4] For a general matrix this is not true. It is a consequence of the fact that Θ is symmetric, i. e. $\Theta_{ij} = \Theta_{ji}$ for all its entries.

check wording

Definition 5.6: Axis of inertia

For each solid body there is a choice of internal coordinate axes $\hat{\boldsymbol{r}}_i$, $i = 1, \cdots, 3$ where the tensor of inertia takes a diagonal form. The directions selected by the axis are called *axes of inertia*, and the related diagonal entry of the matrix of inertial is denoted as *moment of inertia*.

Remark 5.7. If the mass distribution of the body obeys reflection or rotation symmetry, the axes of inertia are invariant under the symmetry transformation.

5.4.3 *Time evolution of angular momentum and particle spin*

The angular momentum $\boldsymbol{L}_{CM}$ of its center-of-mass motion behaves in exactly the same way as for point particles.

The spin changes in time according to the differential equation

$$\begin{aligned}\dot{\boldsymbol{S}} &= \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \times \ddot{\boldsymbol{r}} \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \times \ddot{\boldsymbol{q}} = \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \boldsymbol{r} \times \boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{r})\end{aligned}$$

In order to arrive at the second line, we noted that $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{q}} - \ddot{\boldsymbol{Q}}$, and that the integral for the $\ddot{\boldsymbol{Q}}$ contribution vanishes because $\int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} = 0$. Moreover, it is understood that the force $\boldsymbol{F}$ is zero for coordinates $\boldsymbol{r}$ outside the body.

Definition 5.7: Particle Torque

When the part $\boldsymbol{r}$ of a body is subjected to force $\boldsymbol{F}$ then its spin $\boldsymbol{S}$ is changing due to a torque $\boldsymbol{M}$

$$\dot{\boldsymbol{S}} = \boldsymbol{M} = \int_{\text{body}} \mathrm{d}^3 r\, \boldsymbol{r} \times \boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{r}) \tag{5.4.6}$$

Remark 5.8. Note that the torque is denoted by the letter capital M that is also frequently used for the mass. Nevertheless, there is no immediate risk that they are mixed up: The torque, $\boldsymbol{M}$, is a vector, while the mass, M, is a scalar. To further reduce the risk we will denote masses by a small letter m, when mass and torque appear in a problem.

In general the force $\boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{r})$ can only be evaluated after the CM motion has been determined. From the point of view of the rotating body it is a time-dependent force. This renders the motion of a particle in an inhomogeneous force field to be a very complex problem. However, the gravitational force on small spatial distances, where the gravitational acceleration $\boldsymbol{g}$ takes a constant value, forms a noticeable exception.

Theorem 5.3: Spinning motion and gravity

When an extended body moves subject to a spatially uniform acceleration $\boldsymbol{g}$, then its center of mass follows a free-flight parabola and its spin is preserved.

Proof. The statement about the center-of-mass motion follows from Equation (5.4.4).

Conservation of the spin is due to

$$\int d^3r\, \boldsymbol{r} \times (\rho(r_1, r_2, r_3)\, \boldsymbol{g}) = \left(\int d^3r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \right) \times \boldsymbol{g}$$
$$= \boldsymbol{0} \times \boldsymbol{g} = \boldsymbol{0} \qquad \square$$

Rather than in the free flight of a body, one also often encounters a spinning body that is fixed at some point. Besides gravity there is one additional force acting on the body that is constraining its motion. When this force acts *only* on the center of mass, then it has no effect on the spin and only changes the evolution of the center of mass. When it acts on another point on the body, then the total angular momentum is no longer conserved. This happens for instance for a spinning top where one fixes a point on its axis.

add:
discussion of motion with additional reference point
Euler angles

5.4.4 *Self Test*

add problems:
moments of inertia
ruler pendulum
suspension bridge
torque on triangle/tetraeder

5.5 *Bodies with internal degrees of freedom: Revisiting mobiles*

In Section 2.10 we worked out the positions of masses for a mobile where all masses are the same and where all sticks are straight. It is worth while to revisit this problem from a more advanced mathematical perspective.

5.5.1 *Mobile at rest*

The mobile is at rest when its center of mass does not move, $\dot{\boldsymbol{Q}} = \boldsymbol{0}$, and when it has not spin. It remains at rest, when it does not experience a total force that will induce a motion of the center of mass, and no torque that induces spin.

According to Equation (5.4.4) the center of mass can remain at rest when the total force $\boldsymbol{F}_{\text{tot}}$ vanishes. This implies that the force $\boldsymbol{F}_s$ at the suspension point of the mobile must balance the total weight of the mobile $M\,\boldsymbol{g}$,

$$\mathbf{0} = \boldsymbol{F}_{\text{tot}} = \boldsymbol{F}_s + M\,\boldsymbol{g} \quad \Rightarrow \quad \boldsymbol{F}_s = -M\,\boldsymbol{g}$$

According to Equation (5.4.6) the mobile will not topple (i. e. pick up or change its spin) when $\boldsymbol{M} = \mathbf{0}$, and in Section 5.4.3 we pointed out that the gravitational force does can not change the spin. Hence, we are left with the force $\boldsymbol{F}_s$ at the suspension point $\boldsymbol{q}_s = \boldsymbol{Q} + \boldsymbol{r}_s$,

$$\boldsymbol{M} = \boldsymbol{r}_s \times \boldsymbol{F}_s$$

It vanishes iff the force $\boldsymbol{F}_s$ acts parallel to the direction $\boldsymbol{r}_s$ from the position of the center of mass to the suspension point. Since $\boldsymbol{F}_s$ acts antiparallel to gravity this entails that the center of mass of the mobile must either be located directly below or above the suspension point, irrespective of the shape of the arms or distribution of the masses.

The mobile is not a stiff body. Rather its arms can move with respect to each other. We assume again that the mass of the arms may be neglected. The mass of the mobile is concentrated in its N weights that reside at the positions $\boldsymbol{q}_\nu$, $\nu = 1, \cdots, N$. The position of the suspension will be denoted as $\boldsymbol{q}_0$. Let us attach the mobile to a spring so that we can explicitly measure the suspension force. Clearly the forces on the mobile are conservative, such that there is a potential $\Phi(\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_N)$. The force $\boldsymbol{F}^{(\nu)}$ acting on particle ν (or on the suspension) can then be calculated by taking the derivatives with respect to the coordinates $\boldsymbol{q}_\nu = (q_{\nu,x}, q_{\nu,y}, q_{\nu,z})$, of the particle

$$\boldsymbol{F}^{(\nu)} = -\nabla_{\boldsymbol{q}_\nu} \Phi = \begin{pmatrix} -\partial q_{\nu,x}\Phi \\ -\partial q_{\nu,y}\Phi \\ -\partial q_{\nu,z}\Phi \end{pmatrix}$$

When the coordinates are collected into a single vector $\boldsymbol{q} = (\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_N)$ then the mobile is in equilibrium when the $\boldsymbol{q}$-gradient of $\Phi(\boldsymbol{q})$ vanishes, $\mathbf{0} = \nabla_{\boldsymbol{q}} \Phi(\boldsymbol{q})$. However, when taking the partial derivatives one has to keep in mind that one must not fix the values of the other coordinates but rather keep in mind the constraints of motion of the mobile (recall Example 3.7). Alternatively, one can account for the elasiticity of the cords and bars in the mobile, and the resulting restoring forces to pulling and bending. When also all these forces are accounted for the stationary point can be found by a variation principle.

move theory of variations from Chap 6.2 to this point

An more elegant way to deal with this problem will be presented in Chapter 6. Here we already note that the condition on Φ can be interpreted as a multi-dimensional requirement for a stationary point. The force will be zero, even when $\Phi(\boldsymbol{q})$ takes a maximum. However, in that case small fluctuations will induce forces that drive the system away from the stationary point. The mobile will stay put when $\Phi(\boldsymbol{q})$ takes a minimum. Small perturbations will then only lead to some wiggling close to the minimum. The mobile can slowly move because there are perturbations to its shape where all masses stay exactly at the same height. In terms of the potential this amounts to neutral directions where the potential is flat. In order to formally underpin this intuition we introduce multidimensional Taylor expansions.

5.5.2 *Multidimensional Taylor expansions*

We consider a scalar function $\Phi : \mathbb{R}^D \to \mathbb{R}$ that assigns a real value to its arguments $\boldsymbol{x} \in R^D$. For instance this may be the potential energy assigned to a configuration of masses characterized by a state vector $\boldsymbol{x}$. We select a reference point $\boldsymbol{x}_0$ and explore how $\Phi(\boldsymbol{x})$ deviates from $\Phi(\boldsymbol{x})$ for a small changes of the configuration, $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\epsilon}$, i. e. for a small change $\epsilon \in \mathbb{R}^D$ of the configuration. The multidimensional Taylor expansions states that

$$\Phi(\boldsymbol{x}) = \Phi(\boldsymbol{x}_0) + (\epsilon_i\,\partial_i)\Phi(\boldsymbol{x}_0) + \frac{1}{2}\,(\epsilon_i\,\partial_i)\,(\epsilon_j\,\partial_j)\Phi(\boldsymbol{x}_0) + \frac{1}{3!}\,(\epsilon_i\,\partial_i)\,(\epsilon_j\,\partial_j)\,(\epsilon_k\,\partial_k)\Phi(\boldsymbol{x}_0) +$$

Here, ϵ_i denotes the i-component of the vector $\boldsymbol{\varepsilon}$ with respect to an orthonormal basis $\hat{\boldsymbol{e}}_i$, and ∂_i is the partial derivative with respect to the according coordinate x_i of $\boldsymbol{x}$. Moreover, we use the Einstein convention that requires summation over repeated indices, i. e. $\epsilon_i\,\partial_i$ is an abbreviation for $\epsilon_i\,\partial_i = \sum_i \epsilon_i\,\partial_i$ where i runs of the set of indices labeling the base vectors, and analogous statement hold for $(\epsilon_j\,\partial_j)$ and $(\epsilon_k\,\partial_k)$.

Remark 5.9. $\partial_j\Phi(\boldsymbol{x}_0)$ should be interpreted as

$$\partial_j\Phi(\boldsymbol{x}_0) = \frac{\partial}{\partial x_j}\Phi(x_1,\dots,x_j,\dots)\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} .$$

For scalar arguments $x \in \mathbb{R}$ the expression for the multidimensional Taylor expansion reduces to the one for real functions that we have discussed before.

Proof. For a one-dimensional function $f(x)$ the Taylor expansion around x_0 with $x = x_0 + \epsilon$ the expression $(\epsilon_j\,\partial_j)$ reduces to $\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}$. Consequently,

$$\begin{aligned} f(x) &= f(\boldsymbol{x}_0) + \left(\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\right) f(\boldsymbol{x}_0) + \frac{1}{2}\,\left(\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\right)\,\left(\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\right)\,f(x_0) \\ &\quad + \frac{1}{3!}\,\left(\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\right)\,\left(\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\right)\,\left(\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\right)\,f(x_0) + \dots \\ &= f(\boldsymbol{x}_0) + \epsilon\,f'(\boldsymbol{x}_0) + \frac{1}{2}\,\epsilon^2\,f''(x_0) + \frac{1}{3!}\,\epsilon^3\,f'''(x_0) + \dots \end{aligned}$$

These are the first terms of the 1D Taylor expansion. □

The first terms of the Taylor expansion can also be written in the form

$$\Phi(\boldsymbol{x}) = \Phi(\boldsymbol{x}_0) + (\boldsymbol{\varepsilon} \cdot \nabla)\Phi(\boldsymbol{x}_0) + \frac{1}{2}\, \boldsymbol{\varepsilon}^T\, \mathsf{C}(\boldsymbol{x}_0)\, \boldsymbol{\varepsilon} + \dots$$

where the matrix $\mathsf{C}(\boldsymbol{x}_0)$ has the components $c_{ij}(\boldsymbol{x}_0) = \partial_i \partial_j \Phi(\boldsymbol{x}_0)$.

Proof. For the first-order term we have

$$(\boldsymbol{\varepsilon} \cdot \nabla)\Phi(\boldsymbol{x}_0) = \left(\sum_j \epsilon_j\, \partial_j \right) \Phi(\boldsymbol{x}_0) = \left(\epsilon_j\, \partial_j\right)\, \Phi(\boldsymbol{x}_0)$$

where the second equality amounts to the simplification of notation achieved by the Einstein convention.

For the second-order term we have

$$\begin{aligned}
\boldsymbol{\varepsilon}^T\, \mathsf{C}(\boldsymbol{x}_0)\, \boldsymbol{\varepsilon} &= \begin{pmatrix} \epsilon_1\,, \epsilon_2\,, \epsilon_3\,, \dots, \end{pmatrix} \begin{pmatrix} \partial_1^2\Phi(x_0) & \partial_1\,\partial_2\Phi(x_0) & \partial_1\,\partial_3\Phi(x_0) & \dots \\ \partial_2\,\partial_1\Phi(x_0) & \partial_2^2\Phi(x_0) & \partial_2\,\partial_3\Phi(x_0) & \dots \\ \partial_3\,\partial_1\Phi(x_0) & \partial_3\,\partial_2\Phi(x_0) & \partial_3^2\Phi(x_0) & \dots \\ \vdots & \ddots & & \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \end{pmatrix} \\
&= \sum_{jk} \epsilon_j\, \partial_j\, \partial_k \Phi(x_0)\, \epsilon_k = \sum_{jk} (\epsilon_j\, \partial_j)\, (\epsilon_k\, \partial_k)\, \Phi(x_0) \\
&= \left(\sum_j \epsilon_j\, \partial_j \right) \left(\sum_k \epsilon_k\, \partial_k \right) \Phi(x_0) = \left(\epsilon_j\, \partial_j\right)\, \left(\epsilon_k\, \partial_k\right)\, \Phi(x_0)
\end{aligned}$$

where the last equality amounts to the simplification of notation achieved by the Einstein convention. □

For scalar arguments the condition that $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$ amount to the requirement that the slope vanishes at an extremum.

When $\Phi(\boldsymbol{x})$ is a potential then the requirement $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$ amounts to the requirement that the force $\boldsymbol{F}(\boldsymbol{x})$ vanishes at the position x_0,

$$\boldsymbol{F}(\boldsymbol{x}_0) = -\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$$

Hence, we say that the function $\Phi(\boldsymbol{x})$ has a stationary point at $\boldsymbol{x}_0$ when $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$. This underpins the heuristic discussion of the potential energy of the mobile that we gave above.

In particular $\Phi(\boldsymbol{x})$ has a minimum at $\boldsymbol{x}_0$ iff

- $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$, and
- all eigenvalues of $\mathsf{C}(\boldsymbol{x}_0)$ are positive.

Proof. We explore how $\Phi(\boldsymbol{x}_0)$ changes when one considers a point $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\varepsilon}$ in the vicinity of $\boldsymbol{x}_0$, where we express the deviation in the orthonormal basis spanned by the eigenvectors $\hat{\boldsymbol{e}}_i$ of C. Adopting Einstein notation we have

$$\begin{aligned}
&\boldsymbol{\varepsilon} = \epsilon_i\, \hat{\boldsymbol{e}}_i \\
\Rightarrow \quad &\boldsymbol{\varepsilon}^T\, \mathsf{C}\, \boldsymbol{\varepsilon} = \epsilon_i\, \hat{\boldsymbol{e}}_i \cdot \left(\mathsf{C}\epsilon_k\, \hat{\boldsymbol{e}}_k\right) = \epsilon_i\, \hat{\boldsymbol{e}}_i \cdot \left(\lambda_k \epsilon_k\, \hat{\boldsymbol{e}}_k\right) = \lambda_k \epsilon_k\, \epsilon_i\, \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_k = \lambda_k \epsilon_k\, \epsilon_i\, \delta_{ik} = \lambda_k \epsilon_k\, \epsilon_k
\end{aligned}$$

such that

$$\Phi(\boldsymbol{x}_0 + \boldsymbol{\epsilon}) = \Phi(\boldsymbol{x}_0) + \epsilon_k \cdot \partial_k \Phi(\boldsymbol{x}_0) + \frac{1}{2} \lambda_k \epsilon_k\, \epsilon_k$$

1. Assume that $\partial_k \Phi(\boldsymbol{x}_0) \neq 0$ for some coordinate k. We will then choose the orientation of the associated unit vector such that $\partial_k \Phi(\boldsymbol{x}_0) = m > 0$ and consider a displacement $\boldsymbol{\varepsilon} = \epsilon\, \hat{\boldsymbol{e}}_k$. The change of the value of $\Phi(\boldsymbol{x}_0)$ amounts then to

$$\Phi(\boldsymbol{x}_0 + \epsilon\, \hat{\boldsymbol{e}}_k) - \Phi(\boldsymbol{x}_0) = m\,\epsilon + \frac{\lambda_k}{2}\epsilon^2 + \cdots = \epsilon\, \left(m + \frac{\lambda_k}{2}\epsilon + \dots \right)$$

For $|\epsilon| < 2m/|\lambda_k|$ the expression in the bracket takes a positive value, such that $\Phi(\boldsymbol{x}_0 + \epsilon\, \hat{\boldsymbol{e}}_k) < \Phi(\boldsymbol{x}_0)$ for small negative values of ϵ. Consequently, $\Phi(\boldsymbol{x}_0)$ can only be a minimum when $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$.

2. Assume that $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$ and that all eigenvalue $\lambda_k > 0$. For small $\boldsymbol{\varepsilon}$ the change of the value of $\Phi(\boldsymbol{x}_0)$ amounts then to

$$\Phi(\boldsymbol{x}_0 + \epsilon\, \hat{\boldsymbol{e}}_k) - \Phi(\boldsymbol{x}_0) \simeq \frac{1}{2}\,\lambda_k\,\epsilon_k^2 > 0$$

such that that the function takes values larger than $\Phi(\boldsymbol{x}_0)$ for all positions $\boldsymbol{x}$ in the vicinity of $\boldsymbol{x}_0$. □

Analogously, to the discussion of the minimum one shows that $\Phi(\boldsymbol{x}_0)$ takes a maximum when the gradient vanishes, $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$ and when all eigenvalue λ_k take negative values.

The function Φ takes a saddle at $\boldsymbol{x}_0$ when there are positive and negative eigenvalues and when $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$.

When some eigenvalues vanish and all others are positive (negative), then higher-order contributions of the Taylor expansion must be considered to determine if Φ takes a minimum (maximum).

5.5.3 *Self Test*

Problem 5.9. **Symmetry properties of the second-order contributions**

Verify that the left and the right eigenvectors of C are identical, up to transposition.

Why does this imply that the normalized eigenvectors span an orthonormal basis?

Problem 5.10. **Equipotential lines for a 2D potential**

Consider a potential $\Phi(\boldsymbol{x})$ with $\boldsymbol{x} \in \mathbb{R}^2$. Sketch the contour lines of the potential for the following situations

- $\nabla\Phi(\boldsymbol{x}) = (1,1)$ and $\mathsf{C}(\boldsymbol{x}) = 0$ for all positions $\boldsymbol{x}$.
- $\nabla\Phi(1,2) = \mathbf{0}$ and $\mathsf{C}(1,2) = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ with
  1. $b > 1$,
  2. $1 > b > -1$,
  3. $b < -1$,
  4. $b = 1$.

5.6 *Worked example: Reflection of balls*

We consider the reflection of a ball from the ground, the lower side of a table, and back. The ball is considered to be a sphere with radius R, mass m, and moments of inertia $m\alpha R^2$ (by symmetry they all agree). Its velocity at time t_0 will be denoted as $\dot{\boldsymbol{z}}_0$. It has no spin initially. $\boldsymbol{\omega}_0 = \boldsymbol{0}$. The velocity and the spin after the n^{th} collision will be denoted as $\dot{\boldsymbol{z}}_n$ and $\boldsymbol{\omega}_n$. We will disregard gravity such that the ball travels on a straight path in between collisions.

a) Sketch the setup, and the parameters adopted for the first collision: The positive x axis will be parallel to the floor and the origin will be put into the location of the collision. Its direction will be chosen such that the ball moves in the x-z plane. Take note of all quantities needed to discuss the angular momentum with respect to the origin.

b) Upon collision there is a force normal to the floor, $\boldsymbol{F}_\perp$, and a force tangential to the floor, $\boldsymbol{F}_\parallel$. The spin of the ball will *only* change due to the tangential force. The normal force $\boldsymbol{F}_\perp$ acts in the same way as for point particles. The velocity in vertical direction reverses direction and preserved its modulus. Denote the velocity component in horizontal direction as $v_n = \hat{x} \cdot \dot{\boldsymbol{z}}$, and demonstrate that conservation of energy and angular momentum imply that

$$v_n^2 + \alpha R^2 \omega_n^2 = v_{n+1}^2 + \alpha R^2 \omega_{n+1}^2$$

$$v_n - \alpha R\, \omega_n = v_{n+1} - \alpha R\, \omega_{n+1}\,.$$

Show that the tangential velocity component will therefore also reverse its direction and preserves the modulus,

$$v_n + R\, \omega_n = -(v_{n+1} + R\, \omega_{n+1})\,.$$

c) Determine $v_1(v_0, \omega_0)$ and $\omega_1(v_0, \omega_0)$ for the initial conditions specified above. Now, we determine $v_2(v_1, \omega_1)$ and $\omega_2(v_1, \omega_1)$ by shifting the origin of the coordinate systems to the point where the next collision will arise, and we rotate by π to account for the fact that we collide at the lower side of the table. What does this imply for v_1 and ω_1? Continue the iteration, and plot v_1, v_2 and v_3 as function of α. Discuss the result for a sphere with uniform mass distribution (what does this imply for ω?), and a sphere with $\omega = 1/3$.
Hint: For the plot one conveniently implements the recursion, rather than explicitly calculating v_3.

d) What changes in this discussion when the ball has a spin initially?

add:
tennis racket theorem?

5.7 *Problems*

5.7.1 *Practicing Concepts*

Problem 5.11. **Determining the volume, the mass, and the center of mass**

Determine the mass M, the area or volume V, and the center of mass $\boldsymbol{Q}$ of bodies with the following mass density and shape.

a) A triangle in two dimensions with constant mass density $\rho = 1\,\text{kg/m}^2$ and side length 6 cm, 8 cm, and 10 cm.
Hint: Determine first the angles at the corners of the triangle. Decide then about a convenient choice of the coordinate system (position of the origin and direction of the coordinate axes).

b) A circle in two dimensions with center at position (a, b), radius $R = 5\,\text{cm}$, and constant mass density $\rho = 1\,\text{kg/m}^2$.
Hint: How do M, V and $\boldsymbol{Q}$ depend on the choice of the origin of the coordinate system?

c) A rectangle in two dimensions, parameterized by coordinates $0 \leq x \leq W$ and $0 \leq y \leq B$, and a mass density $\rho(x, y) = \alpha\, x$.
What is the dimension of α in this case?

d) A three-dimensional wedge with constant mass density $\rho = 1\,\text{kg/m}^3$ that is parameterized by $0 \leq x \leq W$, $0 \leq y \leq B$, and $0 \leq z \leq H - Hx/W$.
Discuss the relation to the result of part b).

e) A cube with edge length L. When its axes are aligned parallel to the axes $\hat{x}$, $\hat{y}$, $\hat{z}$, it density takes the form $\rho(x, y, z) = \beta\, z$.
What is the dimension of β in this case?

Problem 5.12. **Return time and position of the professor**

a) How long will the professor take to arrive in down-under, and when will he reappear for the first time close to home?

b) How far will Earth have moved in that time? When this happens to him in Leipzig, where will he reappear, and when will he see land again for the next time?

c) Adopt an orthonormal coordinate system (x, y, z) that is co-rotating with Earth, with origin in the Earth center, z-axis oriented towards the North pole, and x-direction towards the latitude of Leipzig. Sketch the trajectory of the professor in the (x, y)-plane when he was at rest initially.

d) Observe that the professor is initially standing on the surface of Earth. What does this imply for his initial velocity? How does the trajectory change?

e) Let him how start with zero velocity from the Moon surface. What does this imply for the force law? How does the trajectory change?

5.7.2 Proofs

5.7.3 Transfer and Bonus Problems, Riddles

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Index