Lecture Notes by Jürgen Vollmer

Theoretical Mechanics

— Working Copy — Chapter 4 —

— 2021-10-07 04:50:53+02:00—

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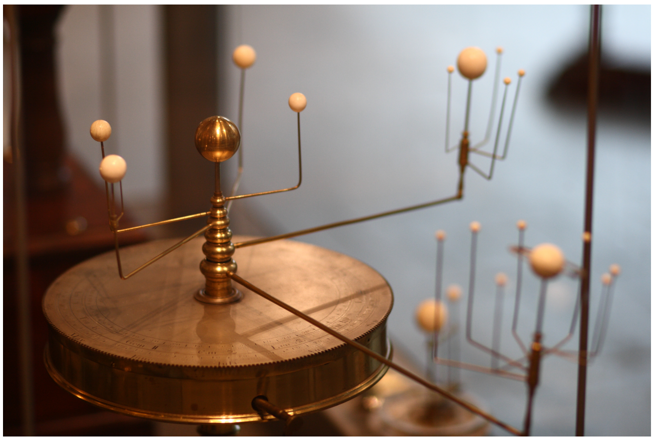
Motion of Point Particles

In Chapter 3 we learned how to set up a physical model based on finding the forces acting on a body, and thus determining the acceleration of its motion. For a particle of mass m and position $\boldsymbol{q}$ Newton's second law relates its acceleration $\ddot{\boldsymbol{q}}$ to the force that is acting on the particle. In Chapter 2 we saw that the total force $\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ acting on the particle may depend on $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and t. The resulting relation between the acceleration and the force is called equation of motion of the particle, Definition 3.3. In the present chapter we will discuss approaches that will allow us to systematically find the solutions of EOMs. Moreover, we will explore what type of behavior is encountered for different types of initial conditions.



Mechanical planetarium used to teach astronomy at Harvard

At the end of this chapter we will discuss the motion of planets around the sun, moons around their planets, and will be able to figure out which rules determine the intricate trajectory of 'Oumuamua shown in Figure 3.12.

4.1 Motivation and outline: EOM are ODEs

From the mathematical point of view the equation of motion is an *ordinary differential equation* (ODE).

Definition 4.1: Ordinary Differential Equation (ODE)

An *ordinary differential equation (ODE)* of n^{th} order for a function $f(t)$ expresses the n^{th} derivative of the function,

$$f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)$$

as a function of time and the lower derivatives of the function, $f^{(n-1)}(t), \dots, f^{(1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$, $f^{(0)}(t) = f(t)$,

$$f^{(n)}(t) = F(f^{(n-1)}(t), \dots, f(t), t) .$$

Here, f and F may be scalar or vector valued functions.

Remark 4.1. The EOM for a particle at position $\boldsymbol{q} \in \mathbb{R}^3$ is a second order ODE where the second time derivative $\ddot{\boldsymbol{q}}(t)$ of the vector valued function $\boldsymbol{q}(t)$ (the position of the particle) is related to $\boldsymbol{F}/m$, which is a vector that depends on $\dot{\boldsymbol{q}}$, $\boldsymbol{q}$ and t; cf. Definition 3.3. ◻

Remark 4.2. A differential equation is called an *ordinary differential equation*, when all derivatives are taken with respect to the same variable. When discussing the physics of waves, e. g. for the full description of Tsunami waves mentioned in Example 1.11, to deal with electromagnetic waves or gravitational waves, one has to deal with differential equations involving space and time derivatives. These type of equations are called *partial differential equations (PDE)*. In Leipzig they are addressed in the course "Theoretical Physics II". ◻

Commonly, the forces in an EOM for a particle only depend on particle positions and velocities, and not explicitly on time. The forces only depend on the particle configuration, and they will be the same irrespective of whether I measure them today or when my grand-daughter determines them with her grand-children at the dawn of the next century.

Definition 4.2: Autonomous Equations of Motion

An ODE is called *autonomous* when its right-hand side does not explicitly depend on time. In particular an autonomous EOM takes the form

$$m\, \ddot{\boldsymbol{q}}(t) = \boldsymbol{F}(\dot{\boldsymbol{q}}(t), \boldsymbol{q}(t)) .$$

The forthcoming discussion of ODEs makes use of the very important observation that every ODE can be stated as first order ODE in some abstract phase space. We introduce this idea for N particles with masses m_i, $i = 1 \cdots N$ that are moving in D dimensions. According to Definition 3.3 their motion is described by a system of $N\,D$ differential equations for the coordinates of the D dimensional vectors $\boldsymbol{q}_i = (q_{i,\alpha}, \alpha = 1 \cdots D)$

$$\ddot{q}_{i,\alpha} = \frac{1}{m_i}\, F_{i,\alpha}(\{\dot{\boldsymbol{q}}_i, \boldsymbol{q}_i\}_{i=1\cdots N}, t)\,, \quad i = 1 \cdots N\,, \;\; \alpha = 1 \cdots D$$

To avoid clutter in the equations we did not explicitly state here the time dependence of $\ddot{q}_{i,\alpha}(t)$, $\dot{q}_{i,\alpha}(t)$, and $q_{i,\alpha}(t)$.

By introducing the variables $\boldsymbol{v}_i = \dot{\boldsymbol{q}}_i$ the EOMs can be written as a set of $2\,D\,N$ first order ODEs

$$\begin{aligned} \dot{q}_{i,\alpha} &= v_{i,\alpha} \\ \dot{v}_{i,\alpha} &= \frac{1}{m_i}\, F_{i,\alpha}(\{\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i\}_{i=1\cdots N}, t) \end{aligned}$$

For an autonomous system this can be written in a more compact form by introducing the $2DN$ dimensional phase-space coordinate $\boldsymbol{\Gamma}$ and the flow $\boldsymbol{\mathcal{V}}$ as follows

$$\begin{aligned} \boldsymbol{\Gamma} &= \left(q_{1,1} \cdots q_{1,D}\,,\; q_{2,1} \cdots q_{N,D}\,,\; \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\; \dot{q}_{2,1} \cdots \dot{q}_{N,D}\right) \\ \boldsymbol{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\; v_{2,1} \cdots v_{N,D}\,,\; \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\; \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\right) \\ \dot{\boldsymbol{\Gamma}} &= \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}) \qquad \text{for autonomous systems.} \end{aligned}$$

Moreover, a non-autonomous system can always be expressed as an autonomous, first order ODE where $\boldsymbol{\Gamma}$ and $\boldsymbol{\mathcal{V}}$ denote points in a $2DN + 1$ dimensional phase space,

$$\begin{aligned} \boldsymbol{\Gamma} &= \left(q_{1,1} \cdots q_{1,D}\,,\; q_{2,1} \cdots q_{N,D}\,,\; \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\; \dot{q}_{2,1} \cdots \dot{q}_{N,D}\,,\; t\right) \\ \boldsymbol{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\; v_{2,1} \cdots v_{N,D}\,,\; \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\; \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\,,\; 1\right) \\ \dot{\boldsymbol{\Gamma}} &= \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}) \qquad \text{for non-autonomous systems.} \end{aligned}$$

In phase space, $\boldsymbol{\Gamma}$ denotes a point that characterizes the state of our system, and $\boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma})$ provides the *unique* direction and velocity of the temporal change of this state. In an approximation, that is accurate for sufficiently small Δt, we have

$$\boldsymbol{\Gamma}(t + \Delta t) \simeq \boldsymbol{\Gamma}(t) + \Delta t\; \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}(t))$$

In phase space the ODE therefore can be represented as a field of vectors $\boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma})$ that represent signposts signifying which direction a trajectory will take when it continues from this point, and how fast it will proceed.

Definition 4.3: Phase-Space Plot

A *phase-space plot* provides an overview of all solutions of an ODE by marking the direction of motion of the trajectories in phase space by arrows, and showing the evolution of a representative set of trajectories by solid lines. At times such a plot is therefore also denoted as the *phase-space portrait* of the solutions of an ODE.

Remark 4.3. For an autonomous system with a single DOF

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = m^{-1} F\big(v(t), x(t)\big)$$

the phase-space portrait is a two-dimensional plot with arrows $\big(v, F(v,x)/m\big)$ at the positions (x, v) in the plane, and trajectories $v(x)$. One can only see the shape of the trajectories, and not their time dependence.

Example 4.1: Phase-space plot for the harmonic oscillator

The EOM of the harmonic oscillator is $\ddot{x}(t) = -\omega^2 x(t)$ where ω can be absorbed into the time scale by adopting the dimensionless time $\tau = \omega t$. Its dimensionless energy is then given by

$$E = \frac{v^2}{2} + \frac{x^2}{2} \quad \text{with} \quad \begin{cases} \dot{x} &= v \\ \dot{v} &= -x \end{cases}$$

The energy is conserved because

$$\frac{\mathrm{d}}{\mathrm{d}t} E = x\,\dot{x} + v\,\dot{v} = x\,v - v\,x = 0$$

Therefore trajectories in phase space amount to contour lines of the the energy function $E(x, v)$. This is shown in Figure 4.1 where the energy is marked by color coding and the direction of the flow is provided by arrows.

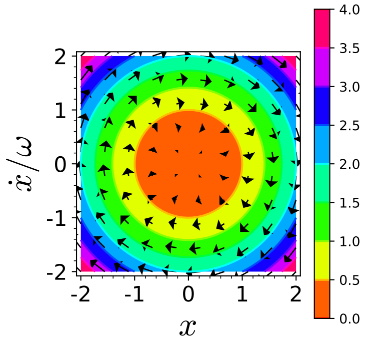


Figure 4.1: Color plot of contour lines of the energy of the harmonic oscillator, and the flow field of its EOM.

Outline

The forthcoming discussion in the present chapter will provide

i. a classification of ODEs with an emphasis on strategies to find solutions for specific initial conditions, and
ii. further discussion of phase-space plots used to characterize sets of solutions.

The methods will be introduced and motivated based on elementary physical problems that will serve as examples of particular relevance in physics.

4.2 *Integrating ODEs — Free flight*

We first discuss the motion of a single particle moving in a gravitational field that gives rise to the constant gravitational acceleration $\boldsymbol{g}$. Hence, the particle position $\boldsymbol{q}(t)$ obeys the EOM

$$\ddot{\boldsymbol{q}} = \boldsymbol{g} \tag{4.2.1}$$

The right hand side of this equation is constant. It neither depends on $\dot{\boldsymbol{q}}$, $\boldsymbol{q}$, nor explicitly on t. This has two remarkable consequences that we will exploit whenever possible.

4.2.1 *Decoupling of the motion of different DOF*

Each component q_α of $\boldsymbol{q}$ can be solved independently of the other DOF

$$\dot{q}_\alpha = g_\alpha$$

Rather than dealing with a vector-valued ODE, one can therefore solve D scalar ODEs which turns out to be a much simpler task. Indeed, we will see in our further discussion that the solution of vector-valued ODEs will often proceed via a coordinate transformation that decouples the different DOF.

4.2.2 *Solving ODEs by integration*

The ODE, Equation (4.2.1), can be solved by integration

Algorithm 4.1: Integrating ODEs

An ODE for $\boldsymbol{f}(t)$ can be solved by *integration* when its right-hand side does not depend on on $\boldsymbol{f}(t)$ and its derivatives, i. e. when it takes the form

$$\dot{\boldsymbol{f}}(t) = \boldsymbol{g}(t)$$

For the initial condition $\boldsymbol{f}(t_0) = \boldsymbol{f}_0$ one can then expresses the solution of the ODE in terms of an integral,[1]

$$\boldsymbol{f}(t) = \boldsymbol{f}_0 + \int_{t_0}^{t} \mathrm{d}t'\, \dot{\boldsymbol{f}}(t') = \boldsymbol{f}_0 + \int_{t_0}^{t} \mathrm{d}t'\, \boldsymbol{g}(t')$$

For an autonomous ODE, where $\boldsymbol{g}(t) = \boldsymbol{c} = \text{const}$, one thus obtains

$$\boldsymbol{f}(t) = \boldsymbol{f}_0 + \boldsymbol{c} \cdot (t - t_0)$$

[1] The prime of the integration variable t' indicates here that the integration variable t' must not be confused with the boundary t of the integral. Physicists often drop the prime with the understanding that this simplifies notation and there is no danger of confusion (after this note has been made).

The idea underlying the algorithm can be understood by reading the equations in reverse order and taking into account the substitution rule for integration,

$$\int_{t_0}^{t} \mathrm{d}t'\, \boldsymbol{g}(t') = \int_{t_0}^{t} \mathrm{d}t'\, \dot{\boldsymbol{f}}(t') = \int_{\boldsymbol{f}(t_0)}^{\boldsymbol{f}(t)} \mathrm{d}\boldsymbol{f} = \boldsymbol{f}(t) - \boldsymbol{f}(t_0)$$

4.2.3 *Integrating the EOM for free flight*

For the free flight only the constant acceleration $\boldsymbol{g}$ due to gravity is acting on the particle such that $\ddot{\boldsymbol{q}}(t) = \boldsymbol{g}$. For the initial conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}(t_0) = \boldsymbol{v}_0$ Algorithm 4.1 provides the velocity

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{v}_0 + \int_{t_0}^{t} \mathrm{d}t' \boldsymbol{g} = \boldsymbol{v}_0 + \boldsymbol{g} \cdot (t - t_0)$$

This equation can be integrated again, providing the position of the particle

$$
\begin{aligned}
\boldsymbol{q}(t) &= \boldsymbol{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \dot{\boldsymbol{q}}(t) = \boldsymbol{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \left(\boldsymbol{v}_0 + \boldsymbol{g}\,(t - t_0)\right) \\
&= \boldsymbol{q}_0 + \boldsymbol{v}_0 \int_{t_0}^{t} \mathrm{d}t' + \boldsymbol{g} \int_{t_0}^{t} \mathrm{d}t'\ (t - t_0) \\
&= \boldsymbol{q}_0 + \boldsymbol{v}_0\,(t - t_0) + \boldsymbol{g} \int_{0}^{t-t_0} \mathrm{d}t''\, t'' \\
&= \boldsymbol{q}_0 + \boldsymbol{v}_0\,(t - t_0) + \frac{1}{2}\,\boldsymbol{g}\,(t - t_0)^2
\end{aligned}
$$

When we introduce the components of the vectors $\boldsymbol{q}$ and $\boldsymbol{v} = \dot{\boldsymbol{q}}$ as $\boldsymbol{q} = (q_1, q_2, \dots)$ and $\boldsymbol{v} = (v_1, v_2, \dots)$, and choose the component direction anti-parallel to $\boldsymbol{g}$ as $z = q_1$, then

$$z(t) = q_1(t) = z(t_0) + v_1(t_0)\,(t - t_0) - \frac{g}{2}\,(t - t_0)^2 \tag{4.2.2a}$$

$$q_i(t) = q_i(t_0) + v_i(t_0)\,(t - t_0)\,, \quad \text{for } i > 1 \tag{4.2.2b}$$

It is illuminating to discuss these solutions form the perspective of non-dimensionalization and the evolution in phase space.

For $i > 1$ the EOM is $\ddot{q}_i = 0$. In phase space the direction field at (q_i, v_i) is then given by the vectors $(v_i(t_0), 0)$ pointing in horizontal direction, as shown in Figure 4.2(left). Moreover, for Equation (4.2.2b) we have $\dot{q}_i(t) = v_i(t_0) = \text{const}$ irrespective of $q_i(t)$. Therefore, the solutions take the form of horizontal lines. When introducing dimensionless units by adopting the velocity scale $v_i(t_0)$ one obtains

$$\hat{v}_i(t) = \frac{\dot{q}_i(t)}{\dot{q}_i(t_0)} = 1$$

For this problem all trajectories are identical up to a rescaling of the length and time units. By rescaling, all horizontal lines in the phase space can be mapped into the same dimensionless solution. From the point of view of the Buckingham-Pi Theorem 1.1 this is due to the fact that there are no dimensionless parameters in the solutions—not even due to the choice of initial conditions.

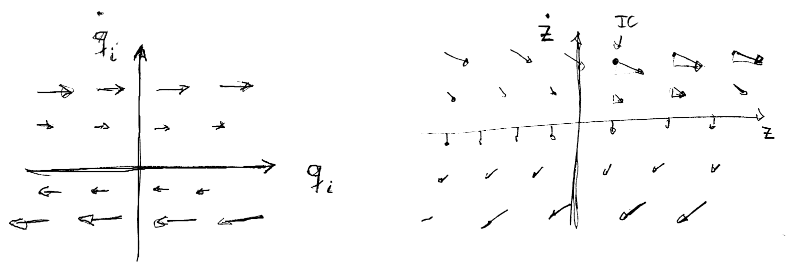


Figure 4.2: Phase-space flows for motion for free flight. (left) For direction perpendicular to g where there is no acceleration. The trajectories are horizontal lines. (right) For z anti parallel to g there is a constant acceleration $-g$. The trajectories take the form of parabola that are open to the left.

For Equation (4.2.2a) the arrows at position (z, v_z) in the phase space are directed to $(v, -g)$. For $v = 0$ they point straight down, for large v they point right and only a little bit down, and for large negative v they point left and only a little bit down, as marked in

Figure 4.2(right). The phase-space trajectories are found by observing that $\dot{z} = v_z(t_0) - g\,(t - t_0)$ implies $t - t_0 = \left[v_z(t_0) - v_z\right]/g$ such that

$$\begin{aligned} z &= z(t_0) + v_z(t_0)\,\frac{v_z(t_0) - v_z}{g} - \frac{\left[v_z(t_0) - v_z\right]^2}{2\,g} \\ &= z(t_0) + \frac{v_z(t_0) - v_z}{2\,g}\,\left[2\,v_z(t_0) - (v_z(t_0) - v_z)\right] \\ &= z(t_0) + \frac{v_z^2(t_0) - v_z^2}{2\,g} \end{aligned} \tag{4.2.3}$$

As function of v_z these are parabola with a maximum at $v_z = 0$ and height $z_{\max} = z(t_0) + v_z^2(t_0)/2g$, as shown in Figure 4.2(right). In this case the EOM involves the constant g such that only one of the initial conditions can be absorbed into dimensionless units. For dimensionless units based on the velocity scale $v_z(t_0)$ and the length scale $v_z^2(t_0)/2\,g$ we have

$$\frac{z}{v_z^2(t_0)/2\,g} = I - \left(\frac{v_z}{v_z(t_0)}\right)^2 \quad \text{with} \quad I = 1 + \frac{2\,g\,z(t_0)}{v_z^2(t_0)}$$

The trajectories in this dimensionless representation are shown in Figure 4.3. They all have the shape of a normal parabola, but the parabolas are shifted by the dimensionless constant I that is formed by the gravitational acceleration g, and the initial conditions $z(t_0)$ and $v_z(t_0)$.

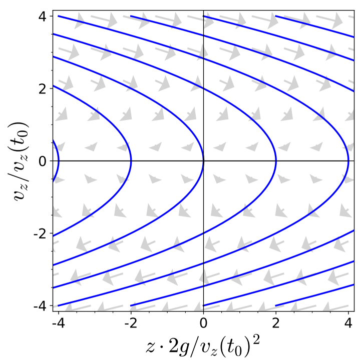


Figure 4.3: Dimensionless phase-space trajectories of a particle falling in the gravitational field without friction.

4.2.4 *Self Test*

Problem 4.1. **Estimating the depth of a pond**

You drop a stone into a pond and count n seconds till you hear it hit the water. How long a chord do you have to attach to your bucket to get up some water.

Problem 4.2. **Integrating the EOM for the flight of an Earth-bound rocket**

Integrate the EOM for rocket flight derived in Section 3.5,

$$\begin{aligned} \dot{V}_R(t) &= -g + \frac{a\,\rho\,v_f^2}{m + M_0 - a\,\rho\,v_f\,t} \\ \dot{z}(t) &= V_R(t) \end{aligned}$$

for a rocket that is launched with velocity v_0 at a height H_0, i.e. for the ICs

$$V_R(t_0) = v_0 \qquad \text{and} \qquad z(t_0) = H_0$$

How do the solutions Equations (3.5.1a) and (3.5.2a) change? Was there a way to anticipate the impact of the changing the initial height H_0?

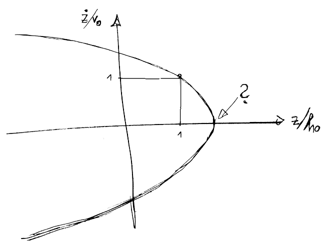


Figure 4.4: Sketch of the universal form of the free-flight trajectories in phase space, Equation (4.2.3).

Problem 4.3. **Alternative dimensionless units for trajectories with constant acceleration**

Discuss the shape of the trajectories that emerges when introducing dimensionless units based on the velocity scale $v_z(t_0)$ and the length scale $z(t_0)$ into Equation (4.2.3).

Hint: You will find parabolas as shown to in the margin. Discuss their shape and the position of their maximum.

4.3 *Separation of variables — Settling with Stokes drag*

The settling of a ball in a viscous medium can be described by the equations of motion

$$m\,\ddot{h}(t) = -m\,g - \mu\,\dot{h}(t)\,. \tag{4.3.1a}$$

Here $h(t)$ is the vertical position of the ball (height), g is the acceleration due to gravity, and the contribution $-\mu\,\dot{h}(t)$ describes Stokes friction, i. e. the viscous drag on the ball. It has the same form as the friction opposing the motion of the mine cart in Example 3.6.

The Stokes friction coefficient μ depends the viscosity of the fluid η and the geometry of the body. The viscosity $[\eta]$ of a fluid is measured in terms of Pa $=$ kg/m s. For air and water it takes values of about $\eta_{\text{air}} \simeq 2\times 10^{-5}$ kg/m s, and $\eta_{\text{water}} \simeq 1\times 10^{-3}$ kg/m s, respectively. The size of the ball will be given by its radius R. Hence, dimensional analysis implies that

$$\mu \propto R\,\eta$$

For a sphere of radius R the proportionality constant takes the value of 6π.

This problems involves the parameters g, μ and m that will absorbed into dimensionless units by introducing the dimensionless units for height $\hat{h} = h\,\mu^2/m^2\,g$, velocity $\hat{v} = \dot{h}\,\mu/m\,g$, and time $\tau = (t-t_0)\,\mu/m$. In these units the EOM takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\hat{h}(\tau) = -1 - \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{h}(\tau) \quad\Leftrightarrow\quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{h} & = \hat{v} \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{v} & = -1-\hat{v} \end{cases} \tag{4.3.1b}$$

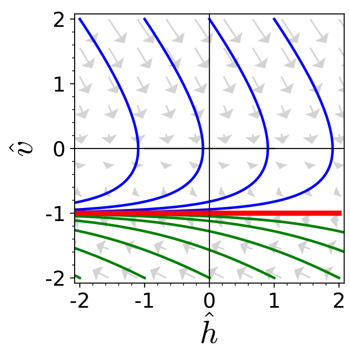


Figure 4.5: Dimensionless phase-space trajectories of a particle subjected to a constant acceleration g and Stokes drag.

The corresponding phase-space plot is shown in Figure 4.5. For positive (i. e. upwards) velocities the resulting direction field in phase space point to the lower right, and for $\hat{v}$ it points straight down. However, the arrows are steeper than for the case without friction, Figure 4.3. For $\hat{v} > 0$ the trajectories in the two cases look similar, but with friction they follow curves that are broader than the parabola for the frictionless fall. For downwards, the flows differ qualitatively: Trajectories started with zero velocity never cross the $\hat{v} = -1$ line, and trajectories that are started with a speed larger than 1 are no longer accelerated by gravity, but slowed down by friction until they also reach their terminal velocity -1 that is marked by a red line.

After having reached this qualitative insight into the dynamics, we will look now for the explicit solution of the EOM. Equation (4.3.1) can be integrated once, yielding an ODE for the settling velocity starting from a height $\hat{h}_0$ with velocity $\hat{v}_0$,

$$\dot{h}(\tau) = \hat{v}_0 - \tau - \big(\hat{h}(\tau) - \hat{h}_0\big)$$

This equation can not be solved by integration, employing Algorithm 4.1, because its right-hand side explicitly depends on the function $h(t)$ that must still be determined as solution of the ODE. It is a better strategy in this case to adopt another solution strategy.

4.3.1 *Solving ODEs by separation of variables*

In the case at hand the ODE (4.3.1a) can be interpreted as a first order ODE for the settling velocity $v = \dot{h}$ where $\dot{v}$ is provided as a function of only v. Such an ODE is best solved by separation of variables.

Algorithm 4.2: Separation of variables

A one-dimensional first-order ODE of the form

$$\dot{f}(t) = g(f(t))\, h(t)$$

can be solved by *separation of variables*. For the initial condition $f(t_0) = f_0$ one obtains then

$$\int_{t_0}^{t} \mathrm{d}t'\, h(t') = \int_{t_0}^{t} \mathrm{d}t' \,\frac{\dot{f}(t')}{g(f(t'))} = \int_{f_0}^{f(t)} \mathrm{d}f \,\frac{1}{g(f)}$$

which provides the solution in terms of two integrals.

Remark 4.4. Let us assume that we find the antiderivatives $H(t)$ with $\mathrm{d}H/\mathrm{d}t = h(t)$ as well as $A(f)$ with $\mathrm{d}A/\mathrm{d}f = 1/g(f)$ and inverse $I(f)$, i.e. $I(A(f)) = f$. Then separation of variables provides the explicit solution

$$\begin{aligned} H(t) - H(t_0) &= A(f(t)) - A(f(t_0)) \\ \Rightarrow \quad f(t) &= I\big(H(t) - H(t_0) + A(f(t_0))\big) \end{aligned}$$

We will see an example of this type when we resume the discussion of Stokes drag in Section 4.3.2.

Remark 4.5. Often the integrals can be performed but the inverse $I(f)$ can not be given in a closed form. If one can find the inverse of $H(t)$, i.e. a function $J(H)$ with $J(H(t)) = t$ then the solution can still be given in the (rather unusual) explicit form

$$t = J\big(A(f) - A(f(t_0)) + H(t_0)\big)$$

This is always possible for autonomous ODEs, i.e. in particular for Equation (4.3.1a) with $f(t) = \dot{h}(t)$.

Example 4.2: Separation of variable

The solution of the differential equation

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{\mathrm{e}^{y^2(x)}}{1 - 2\,y^2(x)} \qquad \text{with} \quad y(0) = 0$$

obeys

$$x = \int_0^x \mathrm{d}x' = \int_0^{y(x)} \mathrm{d}y\,(1 + 2\,y^2)\,\mathrm{e}^{y^2} = y(x)\,\mathrm{e}^{y^2(x)}$$

One can not solve this equation to specify $y(x)$, but it is easy to plot $y\,\mathrm{e}^{y^2}$ and swap the axes (see Figure 4.6).

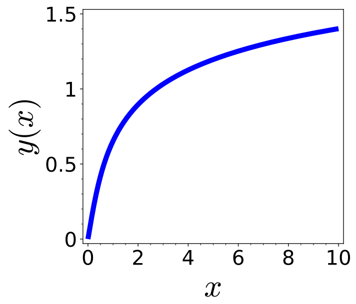


Figure 4.6: Solution of the ODE discussed in Example 4.2.

Remark 4.6. When neither of the inverse functions are known, then the solution can only be stated as an implicit equation

$$H(t) - A(f) = H(t_0) - A(f(t_0)) = \text{const}$$

Hence, the solutions amount to the contour lines of the function $G(t, f) = H(t) - A(f)$ that is plotted to this end as function of the two variables (t, f).[2]

[2] Plotting of contour lines is supported by all scientific plot programs. In Gnuplot it is facilitated via the "set implicit" option for a 2d plot command "plot", or by using "set contour" together with a 3d plot called by "splot". In Sage there are the commands 'plot_implicit()' and 'contour_plot()'. In Python with Matplotlib there is 'matplotlib.pyplot.contour()'.

Example 4.3: Separation of variable

The solution of the differential equation

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{y(x)\;(1 + \cos x)}{1 + y(x)} \qquad \text{with} \quad y(0) = 1$$

obeys

$$x + \sin x - 1 = \int_0^x \mathrm{d}x'\,(1 + \cos x')$$

$$= \int_1^{y(x)} \mathrm{d}y\,\left(\frac{1}{y} + 1\right) = \ln y + y - 1$$

One can not solve this equation to specify $y(x)$ or $x(y)$. Hence, the solution is given as implicit equation

$$\Leftrightarrow \quad G(x, y) = y + \ln y - x - \sin x = 0$$

whose solution is plotted in Figure 4.7.

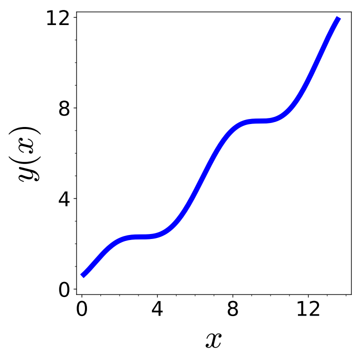


Figure 4.7: Solution of the ODE discussed in Example 4.3.

4.3.2 *Solving the EOM for settling with Stokes drag*

For Equation (4.3.1a) we will now derive the velocity $v(t) = \dot{h}(t)$ for an initial velocity v_0 by applying Algorithm 4.2. In order to simplify notations we perform the derivation in dimensionless units, Equation (4.3.1b), and introducing the physical variables in the end. Separation of variables provides that for a particle with initial velocity $\hat{v}_0$

$$\tau = \int_0^\tau \mathrm{d}\tau' = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w \, \frac{1}{1+w} = -\ln \frac{1+\hat{v}(\tau)}{1+\hat{v}_0}$$

$$\Leftrightarrow \quad \hat{v}(\tau) = -1 + (\hat{v}_0 + 1)\, \mathrm{e}^{-\tau} \tag{4.3.2}$$

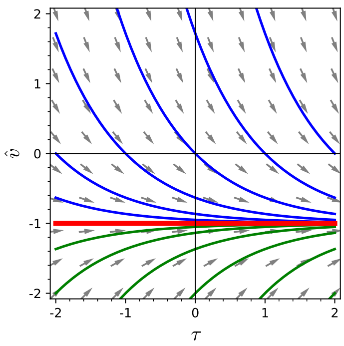


Figure 4.8: Sketch of $w(\tau)$ as obtained in Equation (4.3.2).

The solutions are shown in Figure 4.8. Stokes drag entails that for large times, $\tau \gg 1$, the ball is sinking with the constant Stokes velocity that takes the value -1 in our dimensionless units. Due to $-1 = \hat{v}_\infty = \mu v_\infty / m\,g$ this implies $v_\infty = -m\,g/\mu$ in terms of the physical units.

The position of the sphere can be obtained by integrating Equation (4.3.2) for $\hat{v}(\tau) = \mathrm{d}\hat{h}/\mathrm{d}\tau$ with initial condition $\hat{h}_0$,

$$\begin{aligned}\hat{h}(\tau) &= \hat{h}_0 + \int_0^\tau \mathrm{d}\tau \, \frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = \hat{h}_0 + \int_0^\tau \mathrm{d}\tau \left(-1 + (\hat{v}_0+1)\, \mathrm{e}^{-\tau}\right) \\ &= \hat{h}_0 - \tau + (\hat{v}_0 + 1)\, \left(1 - \mathrm{e}^{-\tau}\right) \end{aligned} \tag{4.3.3a}$$

or in terms of physical units

$$h(t) = h_0 - v_\infty\,(t - t_0) + \frac{m}{\mu}\,(v_0 - v_\infty)\, \left[1 - \exp\left(-\frac{\mu}{m}(t - t_0)\right)\right] \tag{4.3.3b}$$

4.3.3 *Relation to free fall*

It is instructive to explore how the evolution with Stokes friction is related to the free flight $h_f(t) = h_0 + v_0(t - t_0) - g\,(t - t_0)^2$ obtained in Section 4.2. This can most effectively be done by Taylor expansion of Equation (4.3.3) for small τ, and subsequently expressing the result in physical units.

Definition 4.4: Taylor expansion

The *Taylor expansion* to order N provides an approximation of a function $f(x)$ at a position x_0. It is obtained by matching the first N derivatives of the function and of a polynomial of order N that represents the Taylor approximation (or *Taylor approximation*),

$$f(x) \simeq \sum_{n=0}^{N} \left. \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} \right|_{x=x_0} \frac{(x - x_0)^n}{n!}$$

Remark 4.7 (Leading-order Taylor expansion). The first-order, or leading-order Taylor expansion is a linear function $t(x) = t_0 + t_1 x$ with coefficients $t_0 = f(x_0)$ and $t_1 = f'(x_0)$. Hence, we have $t(x_0) = t_0 = f(x_0)$ and $t'(x) = t_1 = f'(x_0)$. This is a tangent to the function $f(x)$ that approximates f at x_0 by having the same functional value and slope. Examples for the sine function are shown in Figure 4.9.

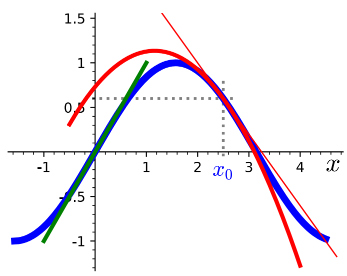


Figure 4.9: The leading order and second order Taylor approximations of the sine function at the origin (green) and at the position $x_0 = 2.5$.

Remark 4.8 (Second-order Taylor expansion). The second-order Taylor expansion is a quadratic function $t(x) = t_0 + t_1 x + t_2 x^2$ with coefficients $t_0 = f(x_0)$, $t_1 = f'(x_0)$, and $t_2 = f''(x_0)/2$. As for the first-order approximation, we have $t(x_0) = t_0 = f(x_0)$ and $t'(x) = t_1 = f'(x_0)$. Moreover, in this case we also have $t''(x) = 2\,t_2 = f'(x_0)$. Examples for the sine function are shown in Figure 4.9.

Example 4.4: Taylor approximations of the sine function

For $\sin x$ the even derivatives vanish at the origin, and the odd $2n-1$ derivative amounts to -1^n. Hence, the first few terms of the Taylor expansion at the origin are given by

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

At the origin the first and second order Taylor approximation agree, as shown by the green line in Figure 4.9.
For the sine-function the expansion at a position x_0 is given by

$$\sin x \simeq \sin(x_0) \left[1 - \frac{(x-x_0)^2}{2!} + \frac{(x-x_0)^4}{4!} - \frac{(x-x_0)^6}{6!} + \cdots\right] + \cos(x_0) \left[(x-x_0) - \frac{(x-x_0)^3}{3!} + \frac{(x-x_0)^5}{5!} - \cdots\right]$$

The red lines in Figure 4.9 show the first-order (thin red line) and the second order (thick red line) approximation for $x_0 = 2.5$. The second order approximation remains closer to the sine-function for a bit longer than the linear first-order approximation.

Example 4.5: Taylor approximations of the exponential function

The derivatives of the exponential function $f(x) = \exp(ax)$ amount to $f^{(n)}(x) = a^n f(x)$ such that its expansion at a position x_0 is given by

$$e^{a\,x} \simeq e^{a\,x_0} \left[1 + a\,(x-x_0) + \frac{a^2\,(x-x_0)^2}{2} + \cdots \cdots + \frac{a^n\,(x-x_0)^n}{n!} + \cdots\right]$$

For $x_0 = 0$ this simplifies to

$$e^{a\,x} = \sum_{n=0}^{\infty} \frac{(a\,x)^n}{n!} \simeq 1 + a\,x + \frac{(a\,x)^2}{2} + \cdots$$

Based on the Taylor expansion of the exponential function $e^{-\tau} = \sum_{n=0}^{\infty}(-\tau)^n/n!$ we find for Equation (4.3.3)

$$\begin{aligned}\hat{h}(\tau) &= \hat{h}_0 - \tau + (\hat{v}_0 + 1)\left(\tau - \frac{\tau^2}{2} + \frac{\tau^3}{6} - \cdots\right)\\ &= \hat{h}_0 + \hat{v}_0\,\tau - (\hat{v}_0 + 1)\,\frac{\tau^2}{2}\,\left(1 - \frac{\tau}{3} + \cdots\right)\end{aligned}$$

The solution with physical units is obtained by substituting $\hat{h} = \mu^2\, h/m^2\, g$, $\hat{h}_0 = \mu^2\, h_0/m^2\, g$, $\tau = \mu\,(t - t_0)/m$, and $\hat{v}_0 = \mu\, \dot{h}(t_0)/m\, g$. Hence,

$$h(t) = h_0 + v_0\,(t-t_0) - \frac{\mu}{m}\left(v_0 + \frac{m\,g}{\mu}\right)\frac{(t-t_0)^2}{2}\left(1 - \frac{\mu\,(t-t_0)}{3\,m} + \cdots\right)$$
$$= h_0 + v_0\,(t-t_0) - \frac{g}{2}\,(t-t_0)^2\left(1 - \frac{v_0}{v_\infty}\right)\left(1 - \frac{\mu\,(t-t_0)}{3\,m} + \cdots\right)$$

This implies that Stokes friction provides a small corrections to the free flight if the initial velocity is small as compared to the asymptotic velocity of free flight, $|v_0| \ll v_\infty = m\,g/\mu$. Further, one must restrict the attention to times that are small as compared to the time scale m/μ where the asymptotic velocity is reached. Equation (4.3.2) implies that this amounts to situations where the velocity $|v(t)|$ is small as compared to the Stokes settling speed v_∞. This is discussed now for two concrete cases:

Example 4.6: Stokes friction for a steel ball

A steel ball with a diameter of 1 cm has a mass of about

$$m = \frac{4\pi}{3}\, 2\times 10^3\,\mathrm{kg/m^3}\,\frac{1\times 10^{-6}\,\mathrm{m^3}}{8} \simeq 1\times 10^{-3}\,\mathrm{kg}$$

In air it will reach a terminal velocity of about

$$v_{\mathrm{air}} = \frac{m\,g}{\mu_{\mathrm{air}}} = \frac{3\,m\,g}{2\,\eta_{\mathrm{air}}\,R} = \frac{3\times 1\times 10^{-3}\,\mathrm{kg}\,10\,\mathrm{m/s^2}}{2\times 2\times 10^{-5}\,\mathrm{kg/m\,s}\,1\times 10^{-2}\,\mathrm{m}}$$
$$\simeq 7.5\times 10^4\,\mathrm{m/s}$$

Saturation to this velocity occurs on time scales

$$t_{\mathrm{air}} = \frac{m}{\mu_{\mathrm{air}}} = \frac{m}{\eta_{\mathrm{air}}\,R} = \frac{1\times 10^{-3}\,\mathrm{kg}}{2\times 10^{-5}\,\mathrm{kg/m\,s}\,1\times 10^{-2}\,\mathrm{m}} = 5\times 10^3\,\mathrm{s}$$

and this time the bullet will have dropped by a distance $g\,t_c^2/2 = 2.5\times 10^7$ m which is much more than the thickness of the atmosphere. We conclude that Stokes friction is not relevant for the motion of a bullet in air.
Even in water, where the viscosity is larger by a factor of 50, we will have

$$v_{\mathrm{water}} = \frac{3\,m\,g}{2\,\eta_{\mathrm{water}}\,R} = \frac{3\times 1\times 10^{-3}\,\mathrm{kg}\,10\,\mathrm{m/s^2}}{2\times 1\times 10^{-3}\,\mathrm{kg/m\,s}\,1\times 10^{-2}\,\mathrm{m}}$$
$$\simeq 1.5\times 10^3\,\mathrm{m/s}$$

Saturation to this velocity occurs on time scales

$$t_{\mathrm{water}} = \frac{m}{\eta_{\mathrm{water}}\,R} = \frac{1\times 10^{-3}\,\mathrm{kg}}{1\times 10^{-3}\,\mathrm{kg/m\,s}\,1\times 10^{-2}\,\mathrm{m}} = 100\,\mathrm{s}$$

and this time the bullet will have dropped by a distance $g\,t_{\mathrm{water}}^2/2 = 5\times 10^4$ m which is deeper than the deepest point in our Oceans.[3]

[3] In Section 4.4 we will see what goes wrong here.

Example 4.7: Stokes friction for sperms

Sperms are cells equipped with cilia that allow them to swim towards the egg for fertilization. They have a characteristic size L of a few micrometers and they swim in an environment that is approximated here as water. Their mass is of the order of $m_{\text{sperms}} = \rho_{\text{water}} L^3$. In this case their asymptotic speed is reached at a time scale

$$t_{\text{spermium}} = \frac{m_{\text{spermium}}}{\mu_{\text{spermium}}} = \frac{\rho_{\text{water}} L^2}{\eta_{\text{water}}} = \frac{1 \times 10^3 \,\text{kg/m}^3 \; 1 \times 10^{-12}\,\text{m}^2}{1 \times 10^{-3}\,\text{kg/m s}} = 1 \times 10^{-6}\,\text{s}$$

Stokes friction plays a major role for their swimming. See Purcell (1977) for more details.

4.3.4 *Self Test*

Problem 4.4. **Solving ODEs by separation of variables**

Determine the solutions of the following ODEs

a) $\dfrac{dy}{dx} = \dfrac{\cos^2 y}{\sin^2 x}$ such that $y(\pi/4) = 0$

b) $\dfrac{dy}{dx} = \dfrac{3\,x^2\,y}{2y^2 + 1}$ such that $y(0) = 1$

c) $\dfrac{dy}{dx} = -\dfrac{1 + y^3}{x\,y^2\,(1 + x^2)}$ such that $y(1) = 2$

Problem 4.5. **Taylor approximations of the Cosine function**

Find the Taylor approximation for the cosine function

a) analogously to the discussion in Example 4.4, and

b) based on Euler's equation $e^{ix} = \cos x + i \sin x$.
Hint: Insert $a = i$ into the expansion provided in Example 4.5, and collect real and imaginary parts.

Problem 4.6. **Taylor series**

Find the Taylor approximation for the sine function close to

a) $x = \pi/4$ b) $x = \pi/2$ c) $x = 3\,\pi/2$

Hint: Make use of the result of Problem 4.5 and the symmetries of trigonometric functions.

Problem 4.7. **Stopping distance of a yacht**

A yacht of mass 750 kg is sailing on the sailing into the harbor with a speed of 6 m/s. At this moment it is experiencing a friction force of 900 N. At time $t = 0$ the skipper switches off the motor such that only the friction is acting on the boat. Let the water resistance be proportional to the speed.

a) How long will it take till the yacht has come to rest?

b) How long will if take till the speed has been reduced to 1.5 m/s and which distance has the yacht traversed till that time?

Problem 4.8. **Free fall with viscous friction**

In Equation (4.3.3) we derived the time evolution of the height $h(t)$ of a ball that is falling a gravitational field and subjected to Stokes drag.

a) Make a plot of $\hat{h}(\tau)$ as function of τ, where you compare the evolution of trajectories that start with different initial velocity $\hat{v}_0$ from the same height $\hat{h}_0 = 0$.

b) Make a plot of $h(t)$ as function of t, where you compare the evolution of trajectories that start with the same initial velocity v_0 from the same height h_0, but are subjected to a different drag μ (for instance because they have different radius).

4.4 *Worked example: Free flight with turbulent friction*

In Example 4.6 we reached the puzzling conclusion that — for all physically relevant parameters — Stokes friction plays no role for the motion of a steel ball in air and water. On the other hand, we know from experience that friction arises to the very least for large velocities, like for gun shots. This apparent contradiction is resolved by observing that the drag is not due to Stokes drag. Rather for most settings in our daily live friction arises because the motion of the fluid around the considered object goes turbulent, as anticipated in Problem 3.3. A ball of mass m, radius R, and mass density $\rho_{\text{ball}} = 3m/4\pi R^3$ that is moving with speed v through a fluid of mass density ρ_{fluid} will experience a turbulent drag force of modulus

$$F_D = m \frac{\rho_{\text{fluid}}\, C_D}{8\, \rho_{\text{ball}}\, R}\, v^2 = m\, \kappa\, v^2 \qquad (4.4.1)$$

Here, C_D is a dimensionless number that typically takes values between 0.5 and 1. A very beautiful description of the physics of this equation has been provided in an instruction video of the NASA (click here to check it out).

To address motion affected by turbulent drag we measure time in units of $(\kappa g)^{-1/2}$ and velocity in units of $(g/\kappa)^{1/2}$. The dimen-

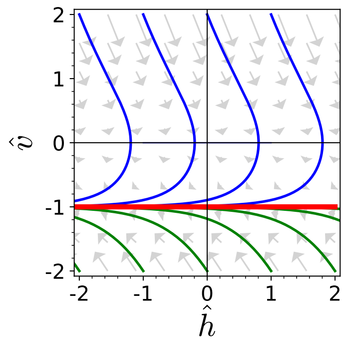


Figure 4.10: Dimensionless phase-space trajectories of a particle subjected to a constant acceleration g and turbulent drag.

sionless velocity $\hat{v}(\tau)$ will then obey the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{h}(\tau) = \hat{v} \tag{4.4.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{v}(\tau) = -1 - \hat{v}^2(\tau)\,\mathrm{sign}\big(\hat{v}(\tau)\big) \tag{4.4.2b}$$

The requirement that the friction force always acts in the direction opposing the motion has been incorporated here by the sign function

$$\mathrm{sign}(x) = \begin{cases} -1 & \text{for} \quad x < 0 \\ \;\;0 & \text{for} \quad x = 0 \\ \;\;1 & \text{for} \quad x > 0 \end{cases}$$

The phase-space flow for this EOM is shown in Figure 4.10. It looks similar to the one for Stokes drag, with the important difference that the change of velocity grows much faster for large $|\hat{v}|$. For $\hat{v} > 0$ this gives even rise to an inflection point where the curvature of the trajectories indicated by blue lines crosses from convex to concave.

The equation for the velocity can again be solved by separation of variables,

$$\tau = \int_0^\tau \mathrm{d}\tau' = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w\,\frac{1}{1 + w^2\,\mathrm{sign}(w)}$$

In order to deal with the sign-function, we deal with the integral separately for four types of initial conditions in either of the intervals $\{(-\infty,-1), [-1,-1], (-1,0], [0,\infty)\}$.

First we consider the initial condition where $\hat{v}_0 = -1$. In that case $\frac{\mathrm{d}}{\mathrm{d}\tau} w(\tau) = 0$ such that

$$\hat{v}(\tau) = -1 \qquad \text{for} \quad \hat{v}_0 = -1 \tag{4.4.3a}$$

Next we consider initial conditions where $\hat{v}_0 < 0$, but $\hat{v}_0 \neq 0$. In this case

$$\tau = \int_{\hat{v}_0}^{v(\tau)} \frac{\mathrm{d}w}{1 - w^2} = \frac{1}{2}\,\ln\left(\frac{1 - \hat{v}(\tau)}{1 + \hat{v}(\tau)} \cdot \frac{1 + \hat{v}_0}{1 - \hat{v}_0}\right)$$

$$\Leftrightarrow \quad \hat{v}(\tau) = \begin{cases} -\tanh\big(\tau - \mathrm{atanh}\,\hat{v}_0\big) & \text{for} \quad -1 < \hat{v}_0 \le 0 \\ -\mathrm{cotanh}\big(\tau - \mathrm{acotanh}\,\hat{v}_0\big) & \text{for} \quad -1 > \hat{v}_0 \end{cases} \tag{4.4.3b}$$

Finally, we consider the case $\hat{v}_0 > 0$. We expect in that case that the particle moves up, $\hat{v}(\tau) > 0$, till some time τ_c, and then it start falling due to the action of gravity. However, in that case its velocity heads down with $-1 < \hat{v}_0 \le 0$ such that it must follow the solution $\hat{v}(\tau) = -\tanh(\tau - \tau_c)$ obtained in Equation (4.4.3b). For $\tau < \tau_c$ we find

$$\tau = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w\,\frac{1}{1 + w^2} = -\arctan(\hat{v}(\tau)) + \arctan(\hat{v}_0)$$

$$\Leftrightarrow \quad \hat{v}(\tau) = -\tan\big(\tau - \arctan(\hat{v}_0)\big)$$

such that

$$\hat{v}(\tau) = \begin{cases} -\tan(\tau - \tau_c) & \text{for} \quad \hat{v}_0 > 0 \wedge \tau < \tau_c = \arctan(\hat{v}_0) \\ -\tanh(\tau - \tau_c) & \text{for} \quad \hat{v}_0 > 0 \wedge \tau \geq \tau_c = \arctan(\hat{v}_0) \end{cases} \tag{4.4.3c}$$

A solution Equation (4.4.3) that passes through the origin and another one through $(0, -2)$ are shown in Figure 4.11.

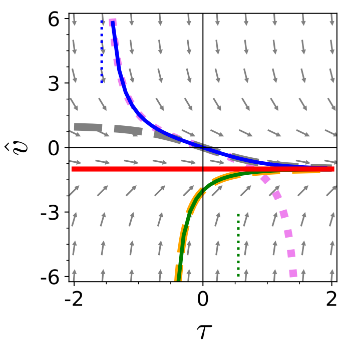


Figure 4.11: Solutions Equation (4.4.3) of Equation (4.4.2a).

4.4.1 *Range of applicability*

Turbulent friction applies whenever

$$\mu\,|v| \lesssim m\,\kappa\,v^2 \quad \Leftrightarrow \quad |v| \gtrsim v_c = \frac{\mu}{m\kappa} \simeq \frac{\eta_{\text{fluid}}}{\rho_{\text{fluid}}\,R}$$

For the 1 cm steel ball considered in Example 4.6 the cross-over velocity v_c yields

$$v_c = \begin{cases} \dfrac{2 \times 10^{-5}\,\text{kg/m\,s}}{1\,\text{kg/m}^3 \times 1 \times 10^{-2}\,\text{m}} = 2\,\text{mm/s} & \text{for air} \\ \dfrac{1 \times 10^{-3}\,\text{kg/m\,s}}{1 \times 10^{3}\,\text{kg/m}^3 \times 1 \times 10^{-2}\,\text{m}} = 0.1\,\text{mm/s} & \text{for water} \end{cases}$$

Moreover, the characteristic time for turbulent drag is

$$\begin{aligned} t_c = (\kappa g)^{-1/2} &= \sqrt{\frac{\rho_{\text{ball}}\,R}{\rho_{\text{fluid}}\,g}} \\ &= \begin{cases} \sqrt{\dfrac{2 \times 10^3\,\text{kg/m}^3 \times 1 \times 10^{-2}\,\text{m}}{1\,\text{kg/m}^3 \times 10\,\text{m/s}^2}} \simeq 1.4\,\text{s} & \text{for air} \\ \sqrt{\dfrac{2 \times 10^3\,\text{kg/m}^3 \times 1 \times 10^{-2}\,\text{m}}{1 \times 10^3\,\text{kg/m}^3 \times 10\,\text{m/s}^2}} \simeq 0.04\,\text{s} & \text{for water} \end{cases} \end{aligned}$$

As a consequence, one may safely assume that Stokes friction is always negligible for the steel ball. Either friction may be neglected or turbulent friction must be considered.

4.4.2 *Self Test*

Problem 4.9. **Turbulent friction**

Assume that the Earth atmosphere gives rise to the same turbulent drag, irrespective of height.

a) What is the maximum time after which a steel ball that is shot up with vertical velocity v_0 will hit the ground?

b) Does it make a noticeable difference when you require that v_0 must not surpass the speed of light $c = 3 \times 10^8\,\text{m/s}$?

Problem 4.10. **Free fall with turbulent friction**

In Equation (4.4.3) we derived the velocity of a ball that is accelerated by gravity and slowed down by turbulent drag.

a) How will the height $\hat{h}$ of the trajectories evolve for large times τ?

b) Determine the full time dependence of the dimensionless height $\hat{h}$ by solving the ODE $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{h} = \hat{v}$.
Hint: Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \ln \cos \tau = -\tan \tau\,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau} \ln \cosh \tau = \tanh \tau\,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau} \ln \sinh \tau = \mathrm{cotanh}\tau$$

c) Make a plot of $\hat{h}(\tau)$ as function of τ, where you compare the evolution of trajectories that start with different initial velocity $\hat{v}_0$ from the same height $\hat{h}_0 = 0$.

d) Insert the definitions of the dimensionless units in order to find the solutions for the velocity $v(t)$ and height $h(t)$ in physical units.

Problem 4.11. **Stopping length of a yacht**

For moderate speeds[4] a yacht experiences a turbulent drag force of the form given in Equation (4.4.1). In the following we assume that $\kappa^{-1} \simeq 10\,\mathrm{m}$.

[4] Recall Problem 1.7 for a discussion of large speeds.

a) The sails of the yacht are shortened at a speed of $v_0 = 10\,\mathrm{m\,s^{-1}}$ and some position x_0. Subsequently, it is running straight ahead. Determine the position $x(t)$ of the yacht.

b) You will find that the yacht never comes to rest. Is that in line with your physical intuition? What might be the origin of this finding?

c) What would happen when the yacht is rather subjected to Stokes drag, Equation (4.3.1a), with a friction coefficient of the order of $\mu \simeq R\,\eta = 3\,\mathrm{m}\,1 \times 10^{-3}\,\mathrm{kg/m\,s} = 3 \times 10^{-3}\,\mathrm{kg/s}$.

d) How does the result of c) refer to that of a). Does the comparison help to solve the issue raised in b)?

4.5 *Linear ODEs — Particle suspended from a spring*

There are two forces acting on a particle is suspended from a spring: the gravitational force $-m\,g$ and the spring force $-k\,z(t)$ where $z(t)$ measures the displacement of the spring from its rest position. Hence, the EOM of the particle takes the form

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) \tag{4.5.1}$$

This equation can neither be integrated directly, because its right hand side depends on $z(t)$, nor can it be solved by separation of variables, because its right hand side depends on $z(t)$ rather than only on $\dot{z}(t)$. It falls into the very important class of *linear ODEs*.

Definition 4.5: Linear ODEs

An ODE is called a *linear ODE* when $z(t)$ and its derivatives only appear as linear terms in the ODE. Hence, an N^{th} order linear ODEs for $z(t)$ takes the general form

$$I(t) = z^{(N)}(t) + c_{N-1}(t)\, z^{(N-1)}(t) + \cdots + c_0(t)\, z(t)$$

The functions $I(t)$, $c_\nu(t)$, $\nu = 0 \cdots N-1$, are called the coefficients of the linear ODE. When they do not depend on time we speak of a linear ODE with *constant coefficients*. In particular, $I(t)$ is called *inhomogeneity*; when it vanishes the ODE is called *homogeneous*.

Example 4.8: Particle suspended from a spring

Equation (4.5.1) is an inhomogeneous second-order linear ODE with the constant coefficients $f_0 = k$, $f_1 = 0$, and inhomogeneity $I = m\, g$.

Remark 4.9. An N^{th}-order linear ODE where the coefficient in front of the N^{th} derivative takes the value $c_N \neq 1$ can be stated in the form given in Definition 4.5 by division with c_N.

Example 4.9: Damped harmonic oscillator

The harmonic oscillator with damping γ and spring constant k

$$m\, \ddot{x}(t) = -m\, \gamma\, \dot{x}(t) - k\, x(t)$$

is described by a homogeneous second order, linear ODE with the constant coefficients $k_1 = \gamma$ and $k_0 = k/m$.

4.5.1 *Solving linear ODEs with constant coefficients*

Linear ODEs with constant coefficients are solved as follows

Algorithm 4.3: Linear ODEs with constant coefficients

An N^{th}-order linear ODE with constant coefficients,

$$I = \sum_{\nu=0}^{N} c_\nu\, f^{(\nu)}(t)$$

can be recast into a homogeneous ODE by considering $h(t) = f(t) - I/c_0$, which is a solution of the corresponding homogeneous, linear ODE

$$0 = \sum_{\nu=0}^{N} c_\nu\, h^{(\nu)}(t)$$

Its solutions can be written as

$$h(t) = \sum_{k=1}^{N} A_k \, e^{\lambda_k t}$$

where the numbers λ_k, $k = 1 \cdots N$ are the N distinct roots of the characteristic polynomial

$$0 = \sum_{\nu=0}^{N} c_\nu \, \lambda^\nu$$

and the amplitudes A_k, $k = 1 \dots N$ must be chosen such that $f(t) = I + c_0 \, h(t)$ obeys the initial conditions

$$\begin{aligned} f(t_0) &= \frac{I}{c_0} + \sum_{k=1}^{N} A_k \, e^{\lambda_k t_0} \\ f^{(1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k \, e^{\lambda_k t_0} \\ \vdots &= \vdots \\ f^{(N-1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k^{N-1} \, e^{\lambda_k t_0} \end{aligned}$$

The idea underlying this algorithm is founded on three insights:

- the solutions of a homogeneous linear ODE form a vector space,
- $\exp(\lambda t)$ is a solution of the ODE iff it is a root of the characteristic polynomial, and
- the functions $\{\exp(\lambda_i t), i = 1, \dots, N\}$ form a basis of the vector space.

The proof will be provided in Problem 4.15.

Remark 4.10. When the polynomial only has $M < N$ distinct roots the set of functions $\{\exp(\lambda_i \, t), i = 1, \dots, M\}$ is missing $N - M$ elements to form a basis for the space of solutions. The set is augmented then by functions of the form $t \exp(\kappa \, t)$ for double roots, $t^2 \exp(\kappa \, t)$ for triple roots, etc. In this course we only deal with second order ODEs, where at most double roots arise. The solution strategy for that case will be discussed in Section 4.5.3. ◻

4.5.2 *Solving the ODE for the mass suspended from a spring*

For Equation (4.5.1) this implies that $h(t) = z(t) + mg/k$ with

$$0 = \ddot{h}(t) + \frac{k}{m} \, h(t) \tag{4.5.2}$$

such that we obtain

$$\lambda_\pm = \pm\sqrt{\frac{k}{m}} = \pm\omega \qquad \text{as solution of} \quad 0 = \lambda^2 + \frac{k}{m}$$

Consequently, the motion of the spring is described by

$$z(t) = -\frac{mg}{k} + A_+ \, \mathrm{e}^{\omega\,(t-t_0)} + A_- \, \mathrm{e}^{-\omega\,(t-t_0)}$$

This is a real-valued function if and only if A_+ and A_- are canonically conjugated complex numbers, such that we can write $A_\pm = A \, \exp(\pm \mathrm{i}\, \varphi)/2$ with $A \in \mathbb{R}$. As a consequence of $\cos x = (\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x})/2$ we then obtain

$$z(t) = -\frac{mg}{k} + A \, \cos\big(\varphi + \omega\,(t - t_0)\big) \tag{4.5.3a}$$

where A and φ must be fixed based on the initial conditions

$$\begin{aligned} z(t_0) &= -\frac{mg}{k} + A \, \cos(\varphi) \\ \dot z(t_0) &= \qquad -\omega \, A \, \sin(\varphi) \end{aligned}$$

or

$$A^2 = \left(z(t_0) + \frac{mg}{k}\right)^2 + \frac{\dot z^2(t_0)}{\omega^2} \qquad \text{and} \qquad \varphi = \arcsin\left(\frac{\dot z(t_0)}{\omega\, A}\right) \tag{4.5.3b}$$

4.5.3 *Solution for the damped harmonic oscillator*

The damped harmonic oscillator is described by the linear EOM

$$0 = \ddot x(t) + \gamma\, \dot x(t) + \frac{k}{m}\, x(t) \qquad \text{with} \quad \gamma,\, k,\, m \in \mathbb{R}_+ \,. \tag{4.5.4}$$

It characteristic polynomial

$$0 = \lambda^2 + \gamma\,\lambda + \frac{k}{m}$$

has the solutions

$$\lambda_\pm = -\frac{1}{2}\,\left(\gamma \pm \sqrt{\gamma^2 - 4\,k/m}\right)$$

Here λ_+ and λ_- can either be both real, a pair of complex conjugated numbers, or we have to deal with the case $\gamma^2 = 4\,k/m$ where there only is a single root. We treat the cases one after the other.

1. Two real roots

In this case $\gamma^2 < 4\,k/m$ such that $\lambda_\pm \in \mathbb{R}_-$. The motion of the oscillator is described by

$$x(t) = A_+ \, \mathrm{e}^{\lambda_+\,(t-t_0)} + A_- \, \mathrm{e}^{\lambda_-\,(t-t_0)}$$

which is a real-valued function for amplitudes $A_\pm \in \mathbb{R}$. The solution for the initial conditions $x(t_0) = x_0$ and $\dot x(t_0) = v_0$ is then found by solving the equations

$$\left.\begin{aligned} x_0 &= A_+ + A_- \\ v_0 &= A_+\,\lambda_+ + A_-\,\lambda_- \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{cases} A_+ = m\,(x_0\,\lambda_- - v_0)/\sqrt{\gamma^2 - 4\,k/m} \\ A_- = -m\,(x_0\,\lambda_+ - v_0)/\sqrt{\gamma^2 - 4\,k/m} \end{cases}$$

Problem 4.13 instructs the reader to plot these solutions for different combinations of A_+ and A_-.

2. Two complex roots

This discussion is analogous to the one provided in Section 4.5.2. One obtains

$$x(t) = A\, e^{-\gamma\,(t-t_0)/2}\, \cos\big(\varphi + \omega_\gamma\,(t-t_0)\big) \tag{4.5.5}$$

where A and φ must be fixed based on the initial conditions

$$x(t_0) = A\, \cos(\varphi)$$
$$\dot{x}(t_0) = -\omega_\gamma\, A\, \sin(\varphi)$$

or $$A^2 = z^2(t_0) + \frac{\dot{z}^2(t_0)}{\omega_\gamma^2} \qquad \text{and} \qquad \varphi = \arcsin\left(\frac{\dot{z}(t_0)}{\omega_\gamma\, A}\right)$$

In Problem 4.14 the reader is advised to fill in the details of this derivation.

3. A single double root

For $\gamma^2 = 4\,k/m$ the characteristic polynomial has a single root $\lambda = -\gamma/2$ such that we only find a single solution $\exp(\lambda t)$ of the ODE. The ODE is solved then as follows:

Algorithm 4.4: Linear 2nd order ODEs: the degenerate case

A 2^{nd}-order linear homogeneous ODE whose characteristic polynomial has a double root at $\lambda = c$ takes the form

$$0 = \ddot{h}(t) - 2\,c\,\dot{h}(t) + c^2\,h(t) \quad \text{with} \quad c \in \mathbb{C}$$

This ODE has two independent solutions $\exp(\lambda\,t)$ and $t\,\exp(\lambda\,t)$ such that its general solutions can be written as

$$h(t) = (A + B\,(t-t_0))\, e^{c\,(t-t_0)}$$

Here the amplitudes A and B must be chosen such that the solution obeys the initial condition

$$\left.\begin{aligned} h(t_0) &= h_0 = A \\ \dot{h}(t_0) &= v_0 = c\,A + B \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} A &= h_0 \\ B &= v_0 - c\,h_0 \end{aligned}\right.$$

Remark 4.11. The function $t\,\exp(ct)$ is a solution of the ODE iff the characteristic polynomial has a double root:

Proof.

$$\begin{aligned} 0 &= \frac{d^2}{dt^2}\left(t\,e^{c\,t}\right) + a\,\frac{d}{dt}\left(t\,e^{c\,t}\right) + b\,\left(t\,e^{c\,t}\right) \\ &= e^{c\,t}\,\Big[(2\,c+a) + \left(c^2 + a\,c + b\right)\,t\Big] \\ \Rightarrow \quad & (2\,c+a) = 0 \quad \wedge \quad \left(c^2 + a\,c + b\right) = 0 \end{aligned}$$

The first equations holds iff $a = -2\,c$ and the second condition implies then that $b = c^2$. □

For the damped harmonic oscillator we have $c = -\gamma/2$ such that

$$x(t) = \left[x_0 + \left(v_0 + \frac{\gamma\, x_0}{2}\right)\, (t - t_0)\right]\, \mathrm{e}^{-\gamma\, t/2}$$

is the solution with $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$.

4.5.4 *Self Test*

Problem 4.12. **Alternative solution for the mass suspended from a spring**

In Equation (4.5.3) we provided the solution of the EOM (4.5.2) of a mass suspended from a spring. Occasionally one also finds the solution given in the form

$$z(t) = -\frac{mg}{k} + A_1\, \cos\big(\omega\, (t - t_0)\big) + A_2\, \cos\big(\omega\, (t - t_0)\big)$$

What is the relation between these two solutions? How does one find one from the other?

Hint: Start from Equation (4.5.3) and use trigonometric relations.

Problem 4.13. **Overdamped solutions of the damped harmonic oscillator: time dependence and in phase-space portrait**

In this exercise we discuss the form of the overdamped solutions of the damped harmonic oscillator.

a) Consider first ICs where A_+ and A_- are positive. Verify that there is a time t_c where the two contributions to $x(t)$ are equal. Plot $x(t)\, \exp(-\lambda_+)\, (t_c - t_0)/A_+$ as function of $t - t_0 - t_c$, choosing a log-scale for the ordinate axis. You should observe linear behavior for large negative and positive values on the mantissa $t - t_0 - t_c$. What are the slopes of these lines? What is the value where the function intersects with $t - t_0 - t_c = 0$?

b) Consider first ICs where $A_+ > 0 > A_-$ and plot $x(t)$ as function of $t - t_0$. Add the functions that describe the asymptotics of $x(t)$ for very small and for large times. You will find that this function has a root and a maximum. Find the time where this happens, and the function value at the maximum.

c) Sketch the motion in phase space! Make use to this end of the special points that you evaluated in b).

Problem 4.14. **Damped oscillations of the damped harmonic oscillator: derivation and phase-space portrait**

For $\gamma^2 > 4\, k/m$ the damped harmonic oscillator shows damped oscillations as given in Equation (4.5.5).

a) Why should the amplitudes $A_\pm$ of the two solutions be complex conjugate?

b) Choose the ansatz $A_{\pm} = A\ \exp(\pm i\varphi)/2$ and derive the result provided in Equation (4.5.5).

c) Note that there is no explicit γ dependence of IC.
Why does it drop out?

d) How does this motion look like in phase space?

Problem 4.15. **The solutions of a homogeneous linear ODE form a vector space**

The set of solutions S_N of a homogeneous N^{th}-order homogeneous linear ODE,

$$0 = \sum_{\nu=0}^{N} c_\nu(t)\, f^{(\nu)}(t)\,, \tag{4.5.6}$$

forms a vector space (cf. the Definition 2.9). Proof to this this end that

a) $(\mathsf{S}, +)$ is a commutative group. The non-trivial statement that must be checked to this end is that

$$\forall s_1(t), s_2(t) \in \mathsf{S}\ :\quad s_1(t) + s_2(t) \in \mathsf{S}$$

b) Verify that

$$\forall \alpha \in \mathbb{C}\,,\ \ s(t) \in \mathsf{S}\ :\quad \alpha\, s(t) \in \mathsf{S}$$

and show that the other properties of a vector space follow trivially from the properties of real functions.

c) Show that the vector space S_N has dimension N.

d) Show that the functions $\exp(\lambda t)$ are a solution of Equation (4.5.6) iff λ is a root of the characteristic polynomial.

e) In Algorithm 4.3 we wrote the solutions as $h(t) = \sum_k A_k\ \exp(\lambda_k\, t)$.
Show that this can be interpreted as a representation of the vector $h(t)$ as a linear combination with coordinates A_k with respect to a basis $\{\exp(\lambda_k\, t), k = 1, \dots, N\}$.
Why is it important to this end that the characteristic polynomial has distinct roots?

f) What about inhomogeneous, linear ODEs? Do their solutions form a vector space, too? If yes: proof it! If no: provide counterexamples for all properties that are violated.

add: variation of constants: fish pond, bells

4.6 *Employing constants of motion — the center of mass (CM) inertial frame*

One of the most important objectives of physics is the description of the motion of interacting particles. As a first step in this direction we discuss how to employ constants of motion to determine the motion of two point particles that interact with a conservative force depending only on the scalar distance between the particles, the interaction most commonly encountered in physical systems. The impact of spatial extension will be the topic of Chapter 5.

Definition 4.6: Point Particles

A *point particle* is an idealization of a physical object where its mass is considered to be concentrated in a single point in space $\boldsymbol{x}$. Point particles can not collide. However, their motion can be subjected to forces that depend on their position $\boldsymbol{x}$.

Example 4.10: Kepler Problem

The Kepler problem addresses the motion of a planet of mass m that orbits around a sun of mass M. The sun and the planet are so far apart that it is justified to consider their masses as concentrated in the positions $\boldsymbol{q}_P$ and $\boldsymbol{q}_S$, and to approximate their interaction as arising from the potential

$$\Phi(R) = \frac{mMG}{R}$$

where $G = 6.672\,59 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$ is the constant of gravitation and $R = |\boldsymbol{q}_P - \boldsymbol{q}_S|$ is the distance between planet and sun. Planet and sun are considered as point particles.

Remark 4.12. The approximation of point particles has been introduced by Newton upon providing the first mathematical model for the Kepler problem. Subsequently, it has extremely successfully been applied in celestial mechanics. *Celestial Mechanics* addresses the problem of discussing the motion of all planets and their moons based on pair interactions deriving from the potential provided in Example 4.10. How to the tiny interactions between the planets impact their motion over long times? Is our solar system stable, or will–at some time in the far future–some planet or moon borrow energy from the other bodies and escape into outer space? 🕮

Remark 4.13. A straightforward application of the Kepler problem is the discussion of the motion of the Moon around Earth where the predictions have been tested extremely accurately based on satellite data and the return time of light signals send to Moon and reflected by mirrors on its surface that have been left there by space missions. The measurements clearly reveal the limitations of the model: Most add references and exercises: Earth/Moon dissipation

noticeably, the Moon gives rise to tidal forces on Earth that induce a tiny amount of dissipation. Even in celestial mechanics there are small dissipative corrections to conservative interaction.

In Section 3.4 we learned that conservation laws impose constraints on the motion of bodies that can be used to simplify the description of their motion. We consider the motion of N particles of masses m_i, $i = 1, \dots, N$ at the positions $\boldsymbol{q}_i$, $i = 1, \dots, N$ that are subjected to forces $\boldsymbol{F}_{ij}$ acting between every pair (i, j) of particles. There is not self-interaction $\boldsymbol{F}_{ii} = \boldsymbol{0}$, and the forces obey Newtons 3rd law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moreover, they are conservative, and depend only on the distance of the particles, $\boldsymbol{F}_{ij} = \nabla\Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|)$. Here the indices ij indicate that the force may depend on additional scalar parameters such as the mass or charge of the particles.

4.6.1 *Center of mass motion and relative motion*

We first determine the evolution of the position of the center of mass $\boldsymbol{Q}$ of the system

$$\boldsymbol{Q} = \frac{1}{M} \sum_i m_i \, \boldsymbol{q}_i \quad \text{with total mass} \quad M = \sum_i m_i \tag{4.6.1}$$

Its evolution is not subjected to external forces

$$\ddot{\boldsymbol{Q}} = \frac{1}{M} \sum_i m_i \, \ddot{\boldsymbol{q}}_i = \frac{1}{M} \sum_i \sum_j \boldsymbol{F}_{ij} = \boldsymbol{0} \tag{4.6.2}$$

due to Newtons 3rd law.

Hence, we find for an initial position $\boldsymbol{Q}_0$ and initial velocity $\boldsymbol{V}_0$ at an initial time t_0 that

$$\boldsymbol{Q}(t) = \boldsymbol{Q}_0 + \boldsymbol{V}_0 \, (t - t_0) \tag{4.6.3}$$

Now we introduce the coordinates relative to the center of mass $\boldsymbol{r}_i = \boldsymbol{q}_i - \boldsymbol{Q}$ and we observe that

$$\begin{aligned} m_i \, \ddot{\boldsymbol{r}}_i &= m_i \, \ddot{\boldsymbol{q}}_i - m_i \, \ddot{\boldsymbol{Q}} = m_i \, \ddot{\boldsymbol{q}}_i \\ &= \sum_j \boldsymbol{F}_{ij} = -\nabla_{\boldsymbol{q}_i} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) = -\frac{\boldsymbol{q}_i - \boldsymbol{q}_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|} \, \Phi'_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) \\ &= -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \, \Phi'_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) \end{aligned} \tag{4.6.4}$$

where $\Phi'_{ij}(x)$ denotes the derivative of $\Phi_{ij}(x)$ with respect to its scalar argument x.

Hence, the EOMs for $\boldsymbol{Q}$ and for the positions $\boldsymbol{r}_i$ relative to the center of mass can be solved separately of each other, and the EOM for the CM has a trivial solution, Equation (4.6.3). We may therefore always address the motion of the particles in a setting where their center of mass is fixed at the origin of the coordinate system.

4.6.2 *Angular momentum in celestial mechanics*

The total angular momentum is conserved for systems where all forces are due to pairwise interactions between particles pairs of particles ij that obey Newtons 3rd law $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$ with forces acting along the line connecting particle i and j, i.e. in particular for the forces of the form given in Equation (4.6.4). After all,

$$\begin{aligned}
\boldsymbol{L} &= \sum_i \boldsymbol{q}_i \times m_i \, \dot{\boldsymbol{q}}_i = \sum_i (\boldsymbol{Q} + \boldsymbol{r}_i) \times m_i \, (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{r}}_i) \\
&= \sum_i (m_i \, \boldsymbol{Q} \times \boldsymbol{Q} + \boldsymbol{Q} \times m_i \, \dot{\boldsymbol{r}}_i + m_i \, \boldsymbol{r}_i \times \dot{\boldsymbol{Q}} + \boldsymbol{r}_i \times m_i \, \dot{\boldsymbol{r}}_i) \\
&= M \boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times m_i \, \dot{\boldsymbol{r}}_i
\end{aligned}$$

where the terms that contain only a single factor of $\boldsymbol{r}_i$ or $\dot{\boldsymbol{r}}_i$ vanish because $\sum_i m_i \, \boldsymbol{r}_i = \sum_i m_i \, (\boldsymbol{q}_i - \boldsymbol{Q}) = \boldsymbol{Q} - \boldsymbol{Q} = \boldsymbol{0}$. Now we have

$$M \, \ddot{\boldsymbol{Q}} = \sum_i m_i \, \ddot{\boldsymbol{q}}_i = \sum_i \sum_j \boldsymbol{F}_{ij} = \sum_{i<j} \boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = \boldsymbol{0}$$

such that we obtain for the time derivative

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{L} &= M \, \boldsymbol{Q} \times \ddot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times m_i \, \ddot{r}_i = \boldsymbol{Q} \times M \, \ddot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times \sum_j \boldsymbol{F}_{ij} \\
&= \frac{1}{2} \left(\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} \right) \\
&= \frac{1}{2} \left(\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \sum_{ij} \boldsymbol{r}_j \times \boldsymbol{F}_{ij} \right) \\
&= \frac{1}{2} \sum_{ij} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = \boldsymbol{0}
\end{aligned}$$

Upon moving to the second line we used that $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$, and the antisymmetry of the forces $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moving to the third line we swapped the names of the summation indices i and j. In the last line, we collected terms and used that $\boldsymbol{F}_{ij}$ is parallel to $\boldsymbol{r}_i - \boldsymbol{r}_j$. We summarize this important finding the following

Theorem 4.1: Angular momentum conservation

The relative angular momentum is conserved for systems with pairwise interaction forces acting parallel to the distance between particles. The total angular momentum is conserved when external forces vanish or when they give rise to a center-of-mass forces $M\ddot{\boldsymbol{Q}}$ aligned parallel to $\boldsymbol{Q}$.

Remark 4.14. An example for the latter case is a harmonic force $\boldsymbol{F}_i = c \, m_i \, \boldsymbol{q}_i$. The proof is provided as Problem 4.19c).

Conservation of the relative angular momentum implies important constraints on the motion. In celestial mechanics this is vividly displayed in the shape of galaxies, solar systems and planetary ring structures. All these systems emerge by the gravitational collapse of large stellar dust clouds. Let cloud be spherically symmetric and uniform initially, consisting of a huge number of small dust particles. By statistical fluctuations the cloud will have an angular momentum, of the order of $M\,D^2\,\omega$ where M is the total mass of the cloud, D is the diameter of the cloud and ω is a tiny number with the unit of a rotation frequency. For a solar system the cloud will collapse until virtually all of its mass is concentrated eventually in the sun in its very center. This involves a change of the diameter of the region holding the mass of about 10^4. For conserved angular momentum the frequency ω is growing by a factor of 10^8. In Problem 4.17 you will show that the initial angular momentum can not be coped by a spin of the central star. The competing constraints of the tendency of gravity to lump together the matter in the cloud and the need to conserve angular moment eventually form a solar system with a central very massive star or double star that is surrounded by planets moving around the star at a distance large as compare to the size of the star.

4.6.3 *Self Test*

Problem 4.16. **The CM of the solar system and the position of the sun**

Verify that the center of mass of Sun can lie more than a sun-diameter away from the center of mass of the solar system.

Hint: Relevant parameters are provided in Table A.1.

Problem 4.17. **Angular momentum of the solar system**

The solar system has a total angular momentum of about $L_{SoSy} = 3.3212 \times 10^{45}\,\mathrm{kgm^2s^{-1}}$.

a) Assume that the mass was initially distributed in a ball of a radius of about 40 AU. Estimate the corresponding effective frequency ω.

b) Assume that the mass in concentrated in two point particles that circulate around each other at a distance of about the sun diameter. Compare their rotation speed to the speed of light.

c) Verify that 98% of L_{SoSy} is accounted for by the orbital angular momenta of the planets.

d) How does this imply the disk-like structure of our solar system?

e) Speculate about other effects that contribute to the remaining 2% of the total angular momentum.

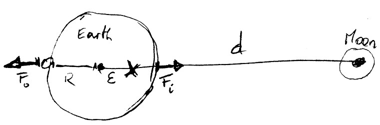


Figure 4.12: Distances adopted for the estimate of the forces inducing tidal forces F_o and F_i: The distance d between Earth and Moon, the Earth radius R, and the distance ϵ between the center of mass of Earth and the joint center of mass of Earth and Moon (indicated by a cross $\times$). Tides emerge at the side facing the Moon (inwards) and opposing the Moon (outwards).

Problem 4.18. **Tidal forces**

Gravitational forces of Moon and centripetal forces due to the rotation with frequency $\Omega = 1/\text{month}$ of Earth around the common center of mass of Earth and Moon (cross in Figure 4.12) give rise to tides. On the outwards facing side the resulting acceleration on a mass element on the Earth surface can be estimated as

$$a_0 = g - (\epsilon + R)\,\Omega^2 + \frac{G\,M_M}{(d+R)^2}$$

where M_M is the Moon mass.

a) Assume that Earth and Moon evolve on circular paths and employ the force balance for a stable motion in order to show that

$$a_0 = g - R\,\Omega^2 \left[1 - \frac{M_M}{M_E + M_M}\,\frac{R}{d}\,\left(1 + \mathcal{O}(R/d)\right)\right]$$

and determine the higher-order correction terms that are indicated here as $(1 + \mathcal{O}(R/d))$.

b) Determine also the change of the acceleration on the side towards the moon. How does it differ from a_i?

c) Determine the relative change of the gravitational acceleration due to the presence of moon, and the difference between a_i and a_o.

d) So far we only discussed the component of the acceleration along a line connecting Earth and Moon at the innermost and outermost points of the Earth surface. What about the other components of the gravitational acceleration:
when considering tides at mid latitudes?
at positions half-way between the two points (i. e. top and bottom sides of Earth in the figure).

e) What is the impact of the Earth rotation? How does it break the symmetry? What does this imply about the relative strength of the two tidal waves every day?

Problem 4.19. **Center of mass and constants of motion**

How do the expressions for the constants of motion discussed in Section 3.4 behave when separating the center of mass motion and the relative motion, $\boldsymbol{q}_i(t) = \boldsymbol{Q}(t) + \boldsymbol{r}_i(t)$.

a) Show that the kinetic energy $T = \sum_i m_i \dot{\boldsymbol{q}}_i^2$ takes the new value

$$T = \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 + \sum_i \frac{m_i}{2}\,\dot{\boldsymbol{r}}_i^2$$

b) Assume that the system is moving in a gravitational field, and that the other forces on the particle arise from pair-wise conservative interactions as discussed Equation (4.6.4). Show that the total energy can be written as

$$E = \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 - M\boldsymbol{g}\cdot\boldsymbol{Q} + \sum_i \frac{m_i}{2}\,\dot{\boldsymbol{r}}_i^2 + \sum_{i<j} \Phi_{ij}\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)$$

c) Show that the total angular momentum is conserved for a systems with the particles interactions given in Equation (4.6.4) and an additional external force

$$\boldsymbol{F}_i = c\,m_i\,\boldsymbol{q}_i$$

acting on each particle i.

4.7 Worked example: the Kepler problem

The first problem tackled in theoretical mechanics was the motion of two point particles with gravitational interaction. It is formulated in terms of three laws. The second law holds for all central forces, the 3rd law is a consequence of mechanical similarity, and the 1st law is based on a solution of the EOM. We first explore the general arguments, and then illustrate their application to the Kepler problem.

4.7.1 Conservation of angular momentum and Kepler's 2nd Law

Angular momentum conservation also has important consequences for the motion of two particles. The center of mass of the two particles takes the form

$$\boldsymbol{Q} = \frac{m_1}{m_1 + m_2}\,(\boldsymbol{Q} + \boldsymbol{r}_1) + \frac{m_2}{m_1 + m_2}\,(\boldsymbol{Q} + \boldsymbol{r}_2) = \boldsymbol{Q} + \frac{m_1\,\boldsymbol{r}_1 + m_2\,\boldsymbol{r}_2}{m_1 + m_2}$$

such that

$$m_1\,\boldsymbol{r}_1 + m_2\,\boldsymbol{r}_2 = \boldsymbol{0} \quad \text{and in particular} \quad \boldsymbol{p} = m_2\,\dot{\boldsymbol{r}}_2 = -m_1\,\dot{\boldsymbol{r}}_1\,.$$

This has important consequences for the evolution of the conserved angular momentum of the relative motion

$$\boldsymbol{L} = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{p}\,.$$

In view of

$$\begin{aligned} \boldsymbol{p} = m_2\,\dot{\boldsymbol{r}}_2 = m_2\,\dot{\boldsymbol{q}}_2 - m_2\,\dot{\boldsymbol{Q}} &= \frac{m_1 m_2}{m_1 + m_2}\,\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{q}_2 - \boldsymbol{q}_1) \\ &= \mu\,\dot{\boldsymbol{R}} \quad \text{with} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad \boldsymbol{R} = \boldsymbol{q}_2 - \boldsymbol{q}_1 \end{aligned}$$

the angular momentum of the relative motion can be expressed in terms of the vector $\boldsymbol{R}$ connecting the two masses

$$\boldsymbol{L} = \boldsymbol{R} \times \mu\dot{\boldsymbol{R}}$$

It takes the then form the angular momentum of a single particle with mass μ, and this also applies for the relation between the acceleration and the force

$$\mu\ddot{\boldsymbol{R}} = m_2\,\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}\,.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} \times \mu\dot{\boldsymbol{R}} = \boldsymbol{R} \times \boldsymbol{F} = \boldsymbol{0}$$

for forces $\boldsymbol{F}$ acting along the line $\boldsymbol{R}$ connecting the two particles.

The conservation of angular momentum has two important consequences:

1. The direction of $\boldsymbol{L}$ is fixed. As a consequence the positions and the velocities of the planet and the sun always lie in a plane that is orthogonal to $\boldsymbol{L}$, and the force $\mu\ddot{\boldsymbol{R}}$ also lies in the plane because the

force is parallel to $\boldsymbol{R}$. Therefore, the motion is constrained to the plane for all times.

2. The absolute value of $\boldsymbol{L}$ is fixed, and this has a geometric interpretation that was first formulated in the context of planetary motion

Theorem 4.2: Kepler's second law

A segment joining the two particles, planet and sun in the Kepler problem, sweeps out equal areas Δa in equal time intervals Δt.

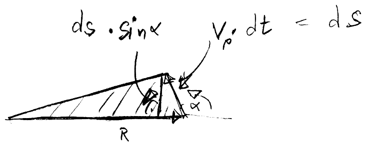


Figure 4.13: Area passed over by the trajectory.

Proof. For the time interval $[t_0, t_1]$ with length $\Delta t = t_1 - t_0$ one has

$$|\boldsymbol{L}|\, \Delta t = \int_{t_0}^{t_1} \mathrm{d}t \, |\boldsymbol{R} \times (m_2 \boldsymbol{v}_2)| = m_2 \int_{t_0}^{t_1} \mathrm{d}t \, |\boldsymbol{R}|\, |\boldsymbol{v}_2| \sin\alpha$$

where α is the angle between $\boldsymbol{R}$ and $\boldsymbol{v}_2$. Further, $\mathrm{d}\boldsymbol{s} = \boldsymbol{v}_2 \, \mathrm{d}t$ is the path length that the trajectory traverses in a time unit $\mathrm{d}t$, such that $\mathrm{d}a = \mathrm{d}t \, |\boldsymbol{R}|\, |\boldsymbol{v}_2| \sin\alpha / 2$ is the area swiped over in $\mathrm{d}t$ (see the sketch in Figure 4.13). Hence,

$$|\boldsymbol{L}|\, \Delta t = \frac{1}{2} \int_0^{\Delta a} \mathrm{d}a = \Delta a \qquad \Rightarrow \qquad \Delta a = \frac{2\, |\boldsymbol{L}|}{m_2}\, \Delta t$$

such that Δa is proportional to Δt. □

4.8 *Mechanical similarity — Kepler's 3rd Law*

Two solutions of a differential equations are called *similar* when they can be transformed into one another by a rescaling of the time-, length-, and mass-scales. We indicate the rescaled quantities by a prime, and denote the scale factors as τ, λ, and α, respectively,

$$t' = \tau t\,, \qquad q'_i = \lambda \boldsymbol{q}_i\,, \qquad m'_i = \alpha m_i$$

We explore the consequences of this idea for the Kepler problem, i. e. for two point particles interacting by a gravitation force $\boldsymbol{F}$ deriving from the following potential

$$\Phi(|\boldsymbol{R}|) = \frac{m_1 m_2 G}{|\boldsymbol{R}|} \qquad \Rightarrow \qquad \boldsymbol{F} = -\nabla\Phi(|\boldsymbol{R}|) = \frac{m_1 m_2 G}{|\boldsymbol{R}|^3}\, \boldsymbol{R}$$

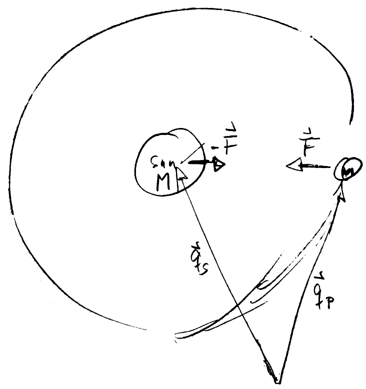


Figure 4.14: Setup of and notations for the motion of a planet around the sun. Here m_S and m_P are the mass of the sun and the planet, respectively, and $\boldsymbol{q}_S$ and $\boldsymbol{q}_P$ are their positions. The relative position is $\boldsymbol{R} = \boldsymbol{q}_P - \boldsymbol{q}_S$.

adapt Figure 4.14

The setup for a planet going around the sun is sketched in Figure 4.14. acting on the planet and pointing towards the sun. We only consider the relative motion and assume that there are no other forces acting on the sun and the planet.

Information about the period and the shape of the trajectory is obtained from the energy for the relative motion

$$E = \frac{\mu}{2}\, \dot{\boldsymbol{R}}^2 + \Phi(|\boldsymbol{R}|)$$

This energy is conserved because

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\, \dot{\boldsymbol{R}}^2 + \Phi(|\boldsymbol{R}|)\right) = \mu\, \dot{\boldsymbol{R}} \cdot \ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}} \cdot \nabla\Phi(|\boldsymbol{R}|) \\ &= \dot{\boldsymbol{R}} \cdot \left(\mu\, \ddot{\boldsymbol{R}} - \boldsymbol{F}\right) = \boldsymbol{0} \end{aligned}$$

Here, we used that $\boldsymbol{F} = \mu\, \ddot{\boldsymbol{R}}$.

In our planetary system the trajectories of the planets are all circular to a good approximation. They are therefore described by the *same* solution of the EOM up to a rescaling of the length scale and the time scale. The former accounts to their different distance to the Sun, and the latter to the different periods of their motion. We observe now that $\mu = m_1\, m_2 / (m_1 + m_2)$ and that the Sun mass m_S is 1000 times larger than the mass of Jupiter, the largest planet. Therefore, for the motion of the planets we have $(m_1 + m_2)/\mu \simeq m_S$ and

$$\frac{E}{\mu} \simeq \frac{\dot{\boldsymbol{R}}^2}{2} + \frac{m_S\, G}{|R|}$$

We expect that different planets follow the same trajectory up to rescaling space and time units, and a different constant value of their energy. We hence explore the consequences of the scaling $\lambda \boldsymbol{R}(t)$ and τt

$$\frac{E}{\mu} \simeq \frac{\lambda^2}{\tau^2}\frac{\dot{\boldsymbol{R}}^2}{2} + \frac{1}{\lambda}\frac{m_S\, G}{|R|}$$
$$\Leftrightarrow \frac{E}{\mu}\frac{\tau^2}{\lambda^2} \simeq \frac{\dot{\boldsymbol{R}}^2}{2} + \frac{\tau^2}{\lambda^3}\frac{m_S\, G}{|R|}$$

where the right-hand side remains invariant iff $\tau^2/\lambda^3 = \text{const}$. This entails

Theorem 4.3: Kepler's third law

The square of the period T of the planets in our planetary system are proportional to the third power of their distance D to the sun.

4.9 *Solving ODEs by coordinate transformations: Kepler's 1st law*

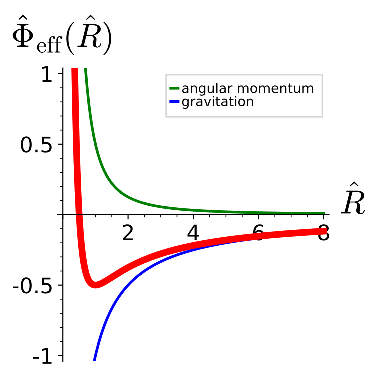


Figure 4.15: Effective potential $\hat{\Phi} = \Phi\, R_0 / m_1\, m_2\, G$ for the Kepler problem as function of the dimensionless distance $\hat{R} = R/R_0$, where $R_0 = L^2/\mu\, m_1\, m_2\, G$.

In polar coordinates $\boldsymbol{R} = (R, \theta)$ the kinetic energy takes the form $\mu \dot{\boldsymbol{R}}^2/2 = \mu \left(\dot{R}^2 + (R\dot{\theta})^2\right)/2$ while the conservation of angular momentum implies $R\dot{\theta} = L/(\mu R)$ with $L = |\boldsymbol{L}|$. Consequently,

$$E = \frac{\mu}{2}\,\dot{R}^2(t) + \frac{L^2}{2\mu\, R^2(t)} - \frac{m_1 m_2 G}{R(t)} \tag{4.9.1}$$

which is equivalent to the motion of a particle of mass μ at position R in the one-dimensional effective potential (Figure 4.15)

$$\Phi_{\text{eff}}(R) = \frac{L^2}{2\mu\, R^2} - \frac{m_1 m_2 G}{R}\,.$$

The first, repulsive contribution to the effective potential arises from angular momentum conservation, and the second, attractive contribution is due to gravity.

There is no elementary way to determine the function $R(t)$. However, based on Equation (4.9.1) one can plot the trajectories in phase space, $\dot{R}(R)$ for different energies. This plot is provided in Figure 4.16. For negative energies there are bounded trajectories that oscillate in the minimum of the potential Φ_{eff}. For zero energy the trajectory reaches till $R = \infty$, and reaches infinity with zero speed. For a positive energy the trajectory reaches till $R = \infty$, and it will go there with speed $\dot{R} = \sqrt{2E/\mu}$.

However, one can determine the shape $R(\theta)$ of the trajectories by observing

$$\dot{R}(\theta) = \dot{\theta}\, \frac{\mathrm{d}R(\theta)}{\mathrm{d}\theta} = \frac{L}{\mu\, R^2}\, R'(\theta)$$

such that

$$E = \frac{L^2}{2\,\mu} \left(\frac{R'^2}{R^4} + \frac{1}{R^2} \right) - \frac{m_1\, m_2\, G}{R}$$

In terms of $w(\theta) = 1/R(\theta)$ this implies

$$\frac{\mu\, E}{L^2} = \frac{1}{2} \left(w'(\theta)\right)^2 + \frac{1}{2}\, w^2(\theta) - \frac{m_1\, m_2\, \mu\, G}{L^2}\, w(\theta) \tag{4.9.2}$$

and differentiating with respect to θ provides

$$0 = w'(\theta) \left[w''(\theta) + w(\theta) - \frac{m_1\, m_2\, \mu\, G}{L^2} \right] .$$

The expression in the square bracket is a second order linear ODE with solution

$$w(\theta) = \frac{\mu\, m_1\, m_2\, G}{L^2} \left[1 + \epsilon\, \cos\left(\theta - \theta_0\right) \right]$$

where ϵ and θ_0 are integration constants that must be determined from the initial conditions. Inserting $w(\theta)$ into Equation (4.9.2) yields

$$\frac{\mu\, E}{L^2} = \frac{\epsilon^2 - 1}{2} \left(\frac{m_1\, m_2\, \mu\, G}{L^2} \right)^2 \quad \Rightarrow \quad \epsilon^2 = 1 + \frac{2\, E\, L^2}{\mu\, \left(m_1\, m_2\, G\right)^2}$$

Hence, ϵ is fully determined by the parameters and the conservation laws of the Kepler problem, while θ_0 determines the orientation of the trajectory in the plane. Commonly, one chooses the coordinate frame where $\theta_0 = 0$.

For the motion of a planet around the sun this entails

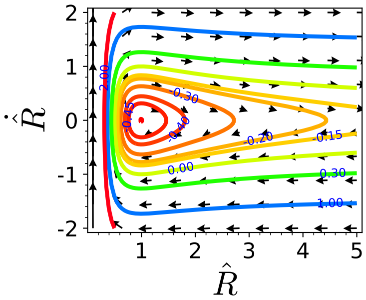


Figure 4.16: The phase-space flow for the EOM of $R(t)$ provided by Equation (4.9.1). The plot adopts dimensionless units with length scale R_0 introduced in Figure 4.15 and a time scale $t_0 = \sqrt{\mu\, R_0^3 / m_1\, m_2\, G}$. Solid lines refer to solutions for different dimensionless energy, with values marked on the contour lines.

Theorem 4.4: Kepler's first law

The trajectories of planets around the sun are described by sections of the cone with a plane,

$$R(\theta) = \frac{R_0}{1 + \epsilon\, \cos(\theta - \theta_0)} \tag{4.9.3}$$

where $R_0 = L^2 / m_1\, m_2\, \mu\, G$ sets the length scale of the trajectory and θ_0 the orientation in the plane. The parameter ϵ sets its shape: for $\epsilon = 0$ the shape amounts to a circle with radius R_0, for $0 < \epsilon < 1$ to an ellipse, for $\epsilon = 1$ to a parabola, and for $\epsilon > 1$ to a hyperbola.

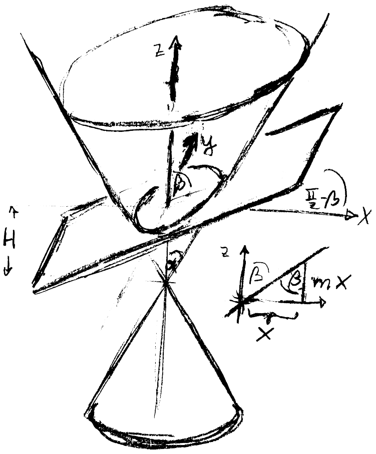


Figure 4.17: Section of a cone double and a plane. The axis are drawn here at the intersection point of the plane with the cone axis, in order to emphasize the the plane is tilted around the y axis. For calculations in the main text the vertex of the cone will be chosen as origin of the coordinate system.

Proof. We consider the section of a cone with opening angle α and its symmetry axis aligned along the z-axis, and a plane, as sketched in Figure 4.17. The origin of the coordinate system is a the vertex of the cone. The plane is tilted with respect to the y-axis such that it forms an angle β with the z-axis, and it intersects the z-axis at height H. The points in the plane have coordinates

$$\boldsymbol{q} = \begin{pmatrix} x \\ y \\ H + m\,H \end{pmatrix} \qquad \text{with} \quad m^{-1} = \tan\beta$$

The point $\boldsymbol{q}$ lies on the cone when $\boldsymbol{q} \cdot \hat{\boldsymbol{z}} = |\boldsymbol{q}| \cos\alpha$, which entails

$$(H + m\,x)^2 = \cos^2\alpha \, \left(x^2 + y^2 + (H + m\,x)^2\right)$$

Henceforth, we adopt dimensionless coordinates $\hat{x} = x/H \tan\alpha$ and $\hat{y} = y/H \tan\alpha$, and we introduce the abbreviation $\epsilon = m \tan\alpha$. We will denote the distance from the origin in the (x, y)-plane as $R = \sqrt{\hat{x}^2 + \hat{y}^2}$, and introduce θ such that $\hat{x} = R \cos\theta$. This entails

$$(1 + \epsilon\,R\,\cos\theta)^2 = R^2$$

with solutions

$$R_\pm = \frac{\epsilon\,\cos\theta}{1 - \epsilon^2\cos^2\theta} \pm \frac{\left[\epsilon^2\cos^2\theta + \left(1 - \epsilon^2\cos^2\theta\right)\right]^{1/2}}{1 - \epsilon^2\cos^2\theta}$$
$$= \frac{\epsilon\,\cos\theta \pm 1}{1 - \epsilon^2\cos^2\theta} = \frac{-1}{\pm 1 + \epsilon\,\cos\theta}$$

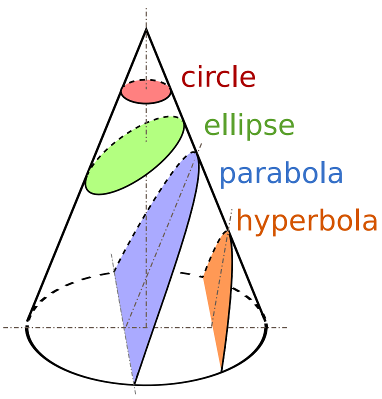


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Figure 4.18: Shape of conic sections for parameters, $\epsilon = 0$ circle, $0 < \epsilon < 1$ ellipse, $\epsilon = 1$ parabola, and $\epsilon > 1$ hyperbola.

Hence, Equation (4.9.3) describes a cone section with length scale $R_0 = H \tan\alpha$ and eccentricity $\epsilon = m \tan\alpha$.

The eccentricity amounts to the ratio of the slope m of the z coordinates of points in the plane as function of x, and the slope $1/\tan\alpha$ of the line obtained as intersection of the double cone and the (x, z) plane. This ratio determines the shape of the conic section (see Figure 4.18).

For $\epsilon = 0$ the shape is a circle $R^2 = 1$.

For $\epsilon = 1$ the shape is a parabola described by $1 + 2\,\epsilon\,\hat{x} = \hat{y}^2$.

For $0 < \epsilon < 1$ the shape is an ellipse described by

$$\frac{1 + \epsilon^2}{1 - \epsilon^2} = \hat{y}^2 + \left(1 - \epsilon^2\right) \left(\hat{x} - \frac{\epsilon}{1 - \epsilon^2}\right)^2$$

For $1 < \epsilon$ the shape is a hyperbola described by

$$\hat{y} = \pm\sqrt{\frac{-1}{\epsilon^2 - 1} + (\epsilon^2 - 1) \left(\hat{x} + \frac{\epsilon}{\epsilon^2 - 1}\right)^2}$$

□

add remark on physical interpretation of cone and plane

4.9.1 *Self Test*

Problem 4.20. **Keeping the Moon at a distance**

Something goes wrong at the farewell party for the settlers of the new Moon colony *Sleeping Beauty 1* such that an extremely annoyed evil fairy switches off gravity for the Moon. Luckily there also is a good fairy at the party. She cannot undo the curse but offers to strip all protons from all water-molecules in a bucket of water that you give to her, and hide them on Moon. The Coulomb attraction between electrons on Earth and protons on Moon can then undo the damage.

a) How much water would you give to her?

b) What will happen to the Earth-Moon system when you are off by 20%, by a factor of two, or even by an order of magnitude?
Hint: The idea is that you discuss the motion for an initial condition where Earth and Moon are at their present position and move with their present velocity, while the gravitational force is changed by a the specified factor.

Problem 4.21. **Mechanical similarity and dimensional analysis**

We discuss here the relation between dimensional analysis, introduced in Section 1.2, and mechanical similarity, adopting the notations introduced in the beginning of Section 4.8.

a) We consider a system with kinetic energy $T = \frac{1}{2} \sum_i m_i \, \dot{\boldsymbol{q}}_i^2$, and consider a potential that admits the following scaling

$$V' = \mu^\alpha \, \lambda^\beta \, V$$

Show that the EOM are then invariant when one rescales time as

$$\tau = \mu^{(1-\alpha)/2} \, \lambda^{(2-\beta)/2}$$

b) Consider now two pendulums, $V = mgz$ with different masses and length of the pendulum arms. Which factors τ, λ, and μ relate their trajectories? How will the periods of the pendulums thus be related to the ratio of the mass and the length of the arms? Which scaling do you expect based on a dimensional analysis?

c) What do you find for the according discussion of the periods of a mass subjected to a harmonic force, $V = k \, |\boldsymbol{q}|^2/2$?

d) Discuss the period of the trajectories in the Kepler problem, $V = mMG/|\boldsymbol{q}|$. In this case the dimensional analysis is tricky because the masses of the sun and of the planet appear in the problem. What does the similarity analysis reveal about the relevance of the mass of the planet for Kepler's third law?

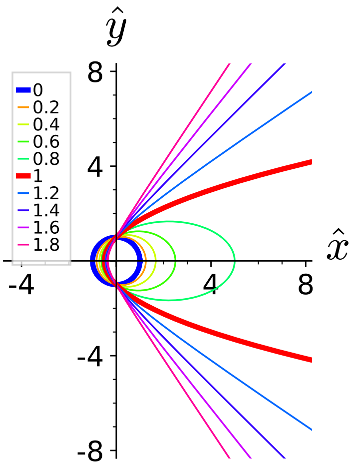

Figure 4.19: Conic sections for different eccentricity ϵ.

Problem 4.22. **Conic Sections**

In the margin we show the shape of conic sections for different eccentricity ϵ.

a) Show that all conic sections intersect the $\hat{y}$ axis at ± 1.

b) Show that the conic sections intersect the $\hat{x}$ axis at $-1/(\epsilon \pm 1)$. Where are these points located for different conic sections?

c) How does Equation (4.9.1) look like after introducing the dimensionless units adopted in Figures 4.15 and 4.16? Write down the solution of the EOM in dimensionless units.

d) Find an alternative choice of the length scale such that all trajectories intersect the x-axis at the potion -1, and prepare the corresponding plot of the trajectory shapes in the (x, y)-plane.

4.10 Problems

4.10.1 Rehearsing Concepts

Problem 4.23. **Maximum distance of flight**

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceeds on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.

What is the optimum choice of θ for the shot put?

c) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?

Problem 4.24. **Phase-space portraits for a scattering problem**

a) Sketch the potential $\Phi(x) = 1 - 1/\cosh x$ for $x \in \mathbb{R}$.

b) Sketch the direction field in the phase space for the EOM $\ddot{x} = -\partial_x \Phi(x)$.

c) Show that $E = \frac{1}{2}\,\dot{x}^2 + \Phi(x)$ is a constant of motion of the EOM.

d) Use energy conservation to determine the shape of the trajectories in phase space, and add a few trajectories to the plot started in b).

Add to the sketch a the phase portrait of the motion in this potential, i. e. the solutions of in the phase space $(x, \dot{x})$.

Problem 4.25. **Another linear ODEs with constant coefficients**

Consider the ODE

$$\ddot{x} = a\,x \qquad \text{with} \quad a \in \mathbb{R}_+$$

a) Sketch the direction field in phase space.

b) Find the solutions of $x(x)$.

c) Add the trajectories the are proceeding through the points $(x(t_0), \dot{x}(t_0) \in \{(1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1), (1,-1)\}$ to the plot started in a).
Hint: Only two cases must be solved explicitly. All other solutions can be inferred from symmetry arguments.

Problem 4.26. **Stokes drag**

The EOM for Stokes friction, Equation (4.3.1) is a linear differential equation. Adopt the strategy for solving linear differential equations, Algorithm 4.3, to find the solution Equation (4.3.3b).

4.10.2 Practicing Concepts

Problem 4.27. **Egyptian water clocks**

In ancient Egypt time was measured by following how water is running out of a container with a constant cross section A. At a water level h in the container, the water will then run out at a speed

$$v(t) = -c\,\sqrt{2g\,h(t)}$$

where the numerical constant c accounts for the viscosity of water and the geometry of the vessel. The Egyptian water clocks this constant takes values of the order of $c \simeq 0.6$.

a) How does the height $h(t)$ of the water in he container evolve after the plug is pulled?

For use as a clock it would be desirable to change the design of the clock such that $h(t)$ would decrease linearly in time. How can the construction of the water clock be amended to reach that aim?

Problem 4.28. **Damped oscillator**

Physical systems are subjected to friction. This can be taken into account by augmenting the EOM of a particle suspended from a spring, Equation (4.5.1), by a friction term

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) - \mu\,\dot{z}(t)$$

a) How does friction affect the motion $z(t)$ of the particle? What is the condition that there are still oscillations, even though with a damping? For which parameters will they disappear, and how do the solutions look like in that case?

b) Sketch the evolution of the trajectories in phase space, for the two settings with and without oscillations.

c) For the borderline case the characteristic polynomial will only have a single root, λ. Verify that the general solution can then be written as

$$z(t) = z_0 + A_1\,\mathrm{e}^{\lambda\,(t-t_0)} + A_2\,t\,\mathrm{e}^{\lambda\,(t-t_0)}$$

d) Determine the solutions for a particle for the following initial conditions:
the particle is at rest and at a distance A from its equilibrium position,
the particle is at the equilibrium position, but it has an initial velocity v_0.
Indicate the form of these trajectories in the phase-space plots.

Problem 4.29. **One-dimensional collisions in the center-of-mass frame**

In Example 3.12 we discussed one-dimensional collisions for settings where the second particle is initially at rest. Now, we consider the situation where both particles are moving from the beginning. Specifically, we consider a setting with two particles of masses m_1 and m_2 with the initial conditions $\big(q_1(t_0), v_1\big)$ and $\big(q_2(t_0), v_2\big)$.

a) Show that the center of mass $Q(t) = \big(m_1\,x_1(t) + m_2\,x_2(t)\big)/M$ with $M = m_1 + m_2$ of the two particles evolves as

$$Q(t) = Q(t_0) + \dot{Q}(t_0)\,(t - t_0) \qquad \text{where} \qquad \dot{Q}(t_0) = a_1\,v_1 + a_2\,v_2$$

and determine the associated real constants a_1 and a_2.

b) We denote the relative coordinates as $x_i = q_i - Q$ and associate it with a momentum $m_i\,\dot{x}_i$. Show that the relative momenta add up to zero before and after the collision,

$$0 = m_1\,x_1 + m_2\,x_2 = m_1\,(q_1 - Q) + m_2\,(q_2 - Q)$$

and that they swap signs upon collision.
Hint: This is a consequence of energy conservation.

c) Determine the time evolution before and after the collision.

d) Verify the consistency of your result with the special case treated in Example 3.12.

Problem 4.30. **Motion in a harmonic central force field**

A particle of mass m and at position $\boldsymbol{r}(t)$ is moving under the influence of a central force field

$$\boldsymbol{F}(\boldsymbol{r}) = -k\,\boldsymbol{r}\,.$$

a) We want to use the force to build a particle trap,[5] i.e. to make sure that the particle trajectories $\boldsymbol{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\boldsymbol{r}(t)| < B$ for all times t. What is the requirement on k to achieve this aim?

[5] Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

b) Determine the energy of the particle and show that the energy is conserved.

c) Demonstrate that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times m\,\dot{\boldsymbol{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field is no longer coincide with the origin of the coordinate system in that case.

d) Let (x_1, x_2) be the coordinates in the plane that is singled out by the angular momentum conservation. Show that $m\ddot{x}_i(t) + k\,x_i(t) = 0$ for $i \in \{1, 2\}$. Determine the solution of these equations. Sketch the trajectories in the phase space $(x_i, \dot{x}_i)$. What determines the shape of the trajectories?

e) Show that the trajectories in the configuration space (x_1, x_2) are ellipses. What determines the shape of these trajectories?

f) Discuss the relation between the amplitude and shape of the trajectory, as determined by the ratio and the geometric mean of the major axes of the ellipse in configuration space, and the period of the trajectory.

4.10.3 *Mathematical Foundation*

Problem 4.31. **Differential equations and functional dependencies**

Determine ODEs whose general solutions are of the form

a) $y(x) = C\,x^2 - x$

b) $y^2(x) = A\,x + B$

Here, A, B, and C are real constants that will be determined by the IC of the ODE.

Problem 4.32. **Separation of variables for a non-autonomous ODE**

We consider the ODE

$$y'(x) = \frac{x}{y}$$

a) How many degrees of freedom does this system have? What is its space? State it as a first order ODE in terms of the phase-space variables.

b) Sketch the direction field in phase space.

c) Find the solution of the ODE for ICs (x_0, y_0) with $y_0 \neq 0$ and

  i. $x_0 < 0$ and $x_0 < y_0 < -x_0$

  ii. $x_0 > 0$ and $x_0 > y_0 > -x_0$

  iii. other ICs with $|x_0| \neq |y_0|$

  iv. $|x_0| = |y_0|$

d) Determine the largest interval of values $x \in \mathbb{R}$ where the solutions $y(x)$ obtained in b) are defined.

e) Is the function $y(x) = |x|$ a solution of the ODE? If in doubt: Where do you see problems for this solution?

Problem 4.33. **Effective potentials and phase-space portraits**

We consider ODEs of the form

$$\ddot{x}(t) = -\frac{\mathrm{d}}{\mathrm{d}x} V_{\text{eff}}(x)$$

Sketch the solutions for trajectories in the following potentials in the phase space $(x, \dot{x})$.

a) $V_{\text{eff}} = x \sin x$	b) $V_{\text{eff}} = x \cos x$
c) $V_{\text{eff}} = x - \sin x$	d) $V_{\text{eff}} = x - \cos x$
e) $V_{\text{eff}} = \mathrm{e}^{x} \sin x$	f) $V_{\text{eff}} = \mathrm{e}^{-x} \sin x$

Problem 4.34. **Central forces conserve angular momentum**

Consider a system of N particles at the positions $\boldsymbol{q}_i$ with masses m_i where each pair (ij) interacts by a force $\boldsymbol{F}_{ij}(|\boldsymbol{d}_{ij}|)$ acting parallel to the displacement vector $\boldsymbol{d}_{ij} = \boldsymbol{q}_j - \boldsymbol{q}_i$ from particle i to j. Proof the following statements:

a) The evolution of the center of mass of the system

$$\boldsymbol{Q} = \frac{1}{M} \sum_{i=0}^{N} m_i \boldsymbol{q}_i \quad \text{with} \quad M = \sum_{i=0}^{N} m_i$$

is force free, i. e. $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$.

b) The total angular momentum can be written as

$$\boldsymbol{L}_{\text{tot}} = M\,\boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \sum_{i<j} \mu_{ij}\, \boldsymbol{d}_{ij} \times \dot{\boldsymbol{d}}_{ij}$$

Determine the factors μ_{ij}.

c) The two contributions to the angular momentum, $M\,\boldsymbol{Q} \times \dot{\boldsymbol{Q}}$ and the sum $\sum_{i<j} \mu_{ij}\, \boldsymbol{d}_{ij} \times \dot{\boldsymbol{d}}_{ij}$ are both conserved.

Problem 4.35. **Impact of translations on conservation laws**

We consider a coordinate transformation where the origin of the coordinate systems is moved to a new time-dependent position $\boldsymbol{x}(t)$,

$$\boldsymbol{q}_i(t) = \boldsymbol{x}(t) + \boldsymbol{r}_i(t)$$

a) Show that the expressions for the kinetic energy are related by

$$T = \sum_i \frac{m_i}{2}\, \dot{\boldsymbol{q}}_i^2 = \frac{M}{2}\, \dot{\boldsymbol{x}}^2 + M\, \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{Q}} + \sum_i \frac{m_i}{2}\, \dot{\boldsymbol{r}}_i^2$$

Here, $M = \sum_i m_i$ and $\boldsymbol{Q} = M^{-1} \sum_i m_i \boldsymbol{q}_i$ are the total mass and the center of mass, respectively.

b) Show that the expressions for the total energy for motion in an external field are related by

$$E = T - M\,\boldsymbol{g} \cdot \boldsymbol{Q} + \sum_{i<j} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) = T - M\,\boldsymbol{g} \cdot \boldsymbol{Q} + \sum_{i<j} \Phi_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) - M$$

c) Show that the angular momentum transforms as follows

$$L = \sum_i m_i\, \boldsymbol{q}_i \times \dot{\boldsymbol{q}}_i = M\,\boldsymbol{x} \times \dot{\boldsymbol{Q}} + M\,(\boldsymbol{x} + \boldsymbol{Q}) \times \dot{\boldsymbol{x}} + \sum_i m_i\, \boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i$$

d) Show that conservation laws are mapped to conservation laws iff we consider a Galilei transformation, i. e. a transformation where $\dot{x}$ =const.

4.10.4 *Transfer and Bonus Problems, Riddles*

Problem 4.36. **Light intensity at single-slit diffraction**

Monochromatic light of wave length λ that is passion through a slit will produce an diffraction pattern on a screen where the intensity follows (cf. Figure 4.20, top panel)

$$I(x) = I_{\max} \left(\frac{\sin x}{x} \right)^2$$

Here the light intensity $I(x)$ is the power per unit area that is observed at a distance x to the side from the direction straight ahead from the light source through the slit to the screen. We are interested in the total power $P(\Delta)$ that falls into a region of width $|x| < \Delta$. Since there is no antiderivative for $I(x)$ we will find approximate solutions by considering Taylor approximations of $I(x)$ that can be integrated without effort.

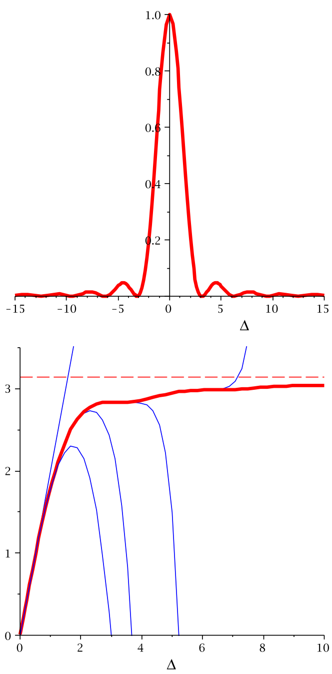

Figure 4.20: The upper panel shows the light intensity $I(x)/I_{\max}$, and the lower panel the fraction of light in the center region of width Δ, i. e. the power $P(\Delta) = \left[\int_{-\Delta}^{\Delta} I(x)\,\mathrm{d}x\right] / I_{\max}$. The red dotted value marks the asymptotic value π and the blue line the approximations obtained by a Taylor approximation up to order 2, 4, 8, 16, and 32. according to the Taylor series evaluated in Problem 4.36.b).

a) Show that $\sin^2 x = (1 - \cos 2x)/2$, and use the Taylor expansion of the cosine-function to show that

$$\frac{\sin^2 x}{x^2} = \frac{1 - \cos 2x}{2x^2} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} \, (2x)^{2n}$$

b) Determine the Taylor approximations for $P(\Delta)$ by integrating the expression found in a).

c) Write a program that is numerically determines $P(\Delta)$ and compares it to Taylor approximations of different order, as shown in the lower panel of Figure 4.20.

Problem 4.37. **Tricky issues in a classical population model**

The Lotka-Volterra model is considered the first model addressing the evolution of populations in theoretical biology. It predicts oscillations of populations, and still today it is cited in the context of data of Lynx and Hare that were collected in Canada in the late 19th century (cf. Figure 4.21).

Let $H(t)$ be the population of prey animals (Hare) and $L(t)$ be the population of its predator (Lynx). When there are no predators the population of prey grows exponentially with a rate a, and this rate is reduced by $-bL(t)$, when prey is consumed by predators. In absence of food the predators die at a rate d, and this rate is reduced by $-cH(t)$, when they find food.

$$\dot{H}(t) = H(t) \;\; [a - b\, L(t)]$$

$$\dot{L}(t) = L(t) \;\; [c\, H(t) - d]$$

a) Let $u(\tau) \propto H(t)$, $v(\tau) \propto L(t)$, and $\tau \propto t$. Find suitable proportionality constants and a dimensionless parameter Π such that

$$\dot{u}(\tau) = \quad u(\tau) \;\; [1 - v(t)]$$

$$\dot{v}(\tau) = \Pi^2 \, v(\tau) \;\; [u(\tau) - 1]$$

b) Show that the EOM for this biological system has fixed points at $(0,0)$ and $(1,1)$. How does the population model behave close to these fixed points?

c) Sketch the evolution of the solutions in the (u,v)-plane, and compare your result with the data reported on the lynx and hare that are shown in Figure 4.21. Can you find the qualitative difference of the data and behavior predicted by the model? **Hint:** Look at the orientation of the flow in phase space. Who would be eating whom?

d) One can infer the form of the trajectories in phase space by observing that

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\dot{v}}{\dot{u}} = \pi^2 \, \frac{v\,(u-1)}{u\,(1-v)} \,.$$

Why does this hold?

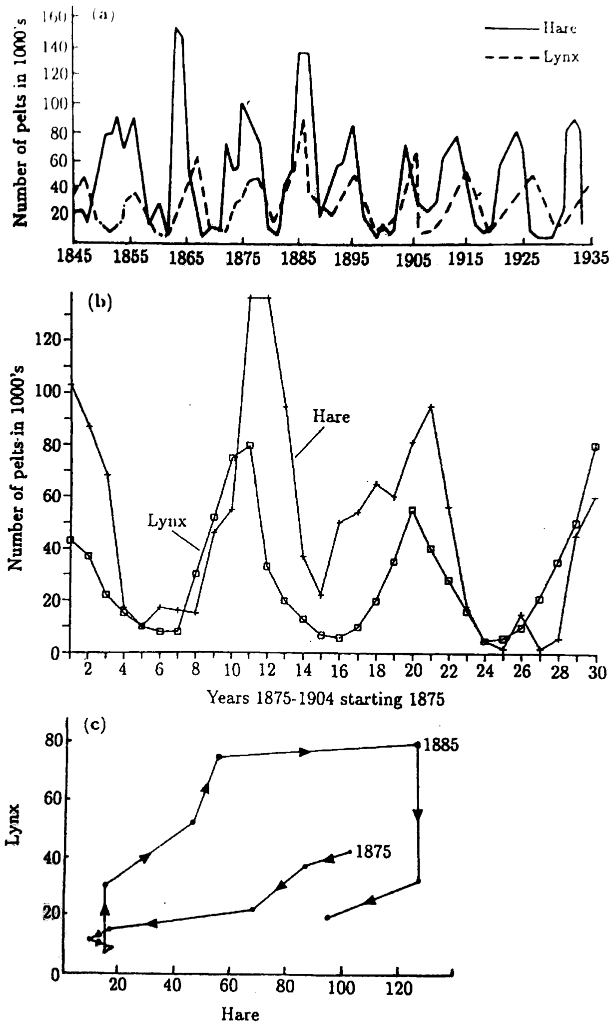

Figure 4.21: (a) Annual oscillations of the skins of hare and lynx offered to the Hudson Bay company. (b) Data with higher time resolution for the 30 years between 1875 and 1904. (c) Presentation of the data presented in (b) as a phase-space plot. [reproduced from Fig. 3.3. of Murray (2002). The book provides a thorough discussion of populations models, their assumptions and artifacts for a range of different populations models.]

e) Find the solution of the ODE by separation of variables and show that the result implies the following constant of motion

$$\Phi(u,v) = \ln\left(v\, u^{\alpha}\right) - v - \alpha\, u\,, \qquad \text{with a suitably chosen } \alpha > 0.$$

Verify this result by also determining the time derivative of $\Phi(u(\tau), v(\tau))$. Here $(u(\tau), v(\tau))$ is a solution of the EOM.

Remark: The presence of a conservation law should be considered an artifact of the model whenever there is no model-immanent (i. e. required by the biological problem in this cases) reason for it to exist.

grav nut cracker

4.11 *Further reading*

An excellent mathematical treatise of the theory of ODEs that is well-accessible for physicists is given by Arnol'd (1992).

fix reference!

A
Physical constants, material constants, and estimates

$$1\,\text{year} \simeq \pi \times 10^7\text{s} \qquad \text{(A.0.1)}$$

A.1 Solar System

The solar system has 1.0014 solar masses, which amounts to 1.991×10^{30} kg.
The Earth-Sun distance is $1\,\text{AU} \simeq 500\,\text{light second} \simeq 1.5 \times 10^{11}$ m.

object	**Sun**	**Mecury**	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**	**Uranus**	**Neptun**
distance	0.005	0.387098	0.723332	1	1.523679	5.2044	9.5826	19.2184	30.11
radius	109	0.3829	0.9499	1	0.533	11.209	9.449	4.007	3.883
mass	333,000	0.055	0.815	1	0.107	317.8	95.159	14.536	17.147
period		0.240846	0.615198	1	2.1354	11.862	29.4571	84.0205	164.8

Table A.1: Properties of Sun and planets of our solar system, provided in multiples of the Earth values. The distance referes to the semi-major axis in AU. For the sun the distance denotes the sun surface, i. e. its radius.

object	**Moon**	**Ceres**	**Pluto**	**Eris**
distance	0.00257	2.769	39.482	67.864
radius	0.2727	0.073	0.1868	0.1825
mass	0.0123	0.00016	0.00218	0.0028
period	0.08085	4.61	247.94	559.07

Table A.2: Properties of the Moon and dwarf planets of our solar system. The properties of the Moon refer to its distrance to and period around Earth. Ceres is the largest object in the meteorite belt between Mars and Jupiter. Eris is a dwarf planet in the Kuiper belt that is larger in mass than Pluto.

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