Lecture Notes by Jürgen Vollmer

Theoretical Mechanics

— Working Copy, Chapter 3 –

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Fakultät für Physik und Geowissenschaften, Universität Leipzig

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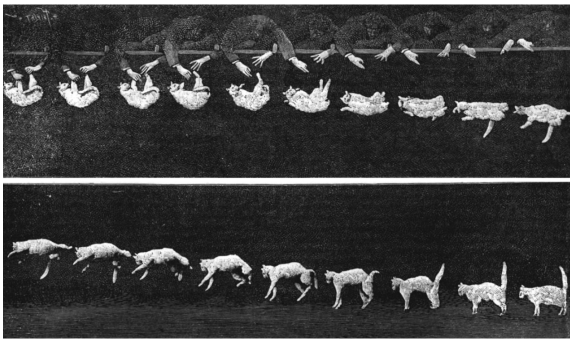
3
Newton's Laws

In Chapter 2 we explored how several forces that act on a body can be subsumed into a net total force and torque. The body stays in rest, say at position $\boldsymbol{q}_0$, when the net force and torque vanish. Now we explore how the forces induce motion and how the position of the body evolves in time, $\boldsymbol{q}(t)$, when it is prepared with an initial condition $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ at the initial time t_0.



Photographs of a Tumbling Cat. Nature 51, 80–81 (1894)

At the end of this chapter we will be able to discuss the likelihood for injuries in different types of accidents, be it men or cat or mice. Why do the cats go away unharmed in most cases when they fall from a balcony, while an old professor should definitely avoid such a fall. As a worked example we will discuss water rockets.

3.1 Motivation and outline: What is causing motion?

Every now and then I make the experience that I sit in a train, reading a book. Then I look out of the window, realize that we are passing a train, feeling happy that we are further approaching my final destination; and then I realize that the train is moving and my train is still in the station. Indeed, the motion of objects in my compartment is exactly identical, no matter whether it is at rest or moves with a constant velocity; be it zero in the station, at 15 m/s in a local commuter train, or 75 m/s in a Japanese high-speed train. However, changes of velocity matter. I forcefully experience the change of speed of the train during an emergency break, and coffee is spilled when it takes too sharp a turn.

Modern physics was born when Galileo and Newton formalized this experience by saying that bodies (e.g. the set of bodies in the compartment of a train) move in a straight line with a constant velocity as long as there is no net force acting on the bodies, and that the change of its velocity is proportional to the applied force.

Outline

In the first part of this chapter we will relate temporal changes of positions and velocities to time derivatives. Subsequently, we can formulate equations of motion that relate these changes to forces. The last part of the chapter deals with strategies to find solutions by making use of conservation laws.

mass	m
position	$\boldsymbol{q}(t)$
velocity	$\dot{\boldsymbol{q}}(t), \boldsymbol{v}(t)$
acceleration	$\ddot{\boldsymbol{q}}(t)$
forces	$\boldsymbol{F}_\alpha(\boldsymbol{q}, t)$

Table 3.1: Notations adopted to describe the motion of a particle. A single dot denotes the time derivative, and double dot the second derivative with respect to time.

3.2 Time derivatives of vectors

In this section we consider the motion of a particle with mass m that is at position $\boldsymbol{q}(t)$ at time t. Its average velocity $\boldsymbol{v}_{\text{av}}(t, \Delta t)$ during the time interval $[t, t+\Delta t]$ is

$$\boldsymbol{v}_{\text{av}}(t, \Delta t) = \frac{\boldsymbol{q}(t+\Delta t) - \boldsymbol{q}(t)}{\Delta t}$$

When the limit $\lim_{\Delta t \to 0} \boldsymbol{v}_{\text{av}}(t, \Delta t)$ exists[1] we can define the velocity of the particle at time t,

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}(t+\Delta t) - \boldsymbol{q}(t)}{\Delta t} \tag{3.2.1}$$

The velocity is then the time derivative of the position, and in an immediate generalization of the time derivative of scalar functions we also write

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t}$$

Finally, we point out that the components of the time derivative of a vector amount to the derivatives of the components.

[1] The discussion of this limit for general functions is a core topic of vector calculus. For our present purpose the intuitive understanding based on the idea that $\boldsymbol{q}(t+\Delta t) \simeq \boldsymbol{q}(t) + \Delta t\, \boldsymbol{v}(t)$ provides the right idea. To provide a hint for the origin of the mathematical subtleties we point out that the approximation works unless there is an *instantaneous* collision with a wall at some point in the time interval $]t, t+\Delta t[$. In physics we try our luck, and fix the problem when we face it. Indeed, upon a close look there are no instantaneous collisions in physics, see Problem 3.17.

Theorem 3.1: Time derivatives of vectors

Let $\boldsymbol{a}(t)$ be a vector with time-dependent components $a_i(t)$ with respect to orthonormal basis $\{\hat{\boldsymbol{e}}_i,\ i = 1 \cdots D\}$ that is fixed in time.
Then $\dot{\boldsymbol{a}}(t) = \sum_i \dot{a}_i(t)\, \hat{\boldsymbol{e}}_i$. The components of $\dot{\boldsymbol{a}}(t)$ amount to the time derivatives of the components of $\boldsymbol{a}(t)$.

Proof. For each time we have $\boldsymbol{a}(t) = \sum_i a_i(t)\, \hat{\boldsymbol{e}}_i$ where it is understood that the sum runs over $i = 1 \cdots D$. We insert this into the definition, Equation (3.2.1), of the the time derivative and use the linearity of scalar products with vectors to obtain

$$\begin{aligned}
\dot{\boldsymbol{a}}(t) &= \lim_{\Delta t \to 0} \frac{\boldsymbol{a}(t+\Delta t) - \boldsymbol{a}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sum_i a_i(t+\Delta t)\, \hat{\boldsymbol{e}}_i - \sum_i a_i(t)\, \hat{\boldsymbol{e}}_i}{\Delta t} \\
&= \lim_{\Delta t \to 0} \sum_i \hat{\boldsymbol{e}}_i \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} = \sum_i \hat{\boldsymbol{e}}_i \lim_{\Delta t \to 0} \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} \\
&= \sum_i \hat{\boldsymbol{e}}_i\, \dot{a}_i(t)
\end{aligned}$$

The subtle step here, from a mathematical point of view, is the swapping of the limit and the sum in the second line of the argument. Courses on vector calculus will spell out the assumptions needed to justify this step (or, more interestingly from a physics perspective, under which conditions it fails). □

The change of the velocity will be denoted as acceleration. Based on an analogous argument as for the velocity, it will be written as a time derivative

Definition 3.1: Acceleration

The time derivative of the velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{q}}(t)$ is denoted as *acceleration*, and written as

$$\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{q}}(t)$$

In the next section it will be related to the action of forces $\boldsymbol{F}(\boldsymbol{q}, t)$ acting on a particle that resides at the position $\boldsymbol{q}$ at time t.

3.2.1 *Self Test*

Problem 3.1. **Derivatives of elementary functions**

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^x = \mathrm{e}^x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \ln x = x^{-1}$$

Use only the three rules for derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)+g(x)\big) = f'(x)+g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) = f'(x)\,g(x)+f(x)\,g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big) = g'(x)\,f'\big(g(x)\big)$$

to work out the following derivatives

a) $\sinh x = \frac{1}{2}\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)$

b) $\cos x = \sin(\pi/2+x)$

c) $x^a = \mathrm{e}^{a\ln x}$ for $a \in \mathbb{R}$
What does this imply for the derivative of $f(x) = x^{-1}$?

d) Use the result from (c) to proof the quotient rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)\,g(x)-f(x)\,g'(x)}{\big(g(x)\big)^2}$$

e) $\tan x = \dfrac{\sin x}{\cos x}$ and $\tanh x = \dfrac{\sinh x}{\cosh x}$

f) Find the derivative of $\ln x$ solely based on $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$.
Hint: Use that $x = \mathrm{e}^{\ln x}$ and take the derivative of both sides.

Problem 3.2. **Integrals of elementary functions**

In a moment we will also perform integrals to determine the work performed on a body when it is moving subject to a force. Practice you skills by evaluating the following integrals.

a) $\int_{-1}^{1}\mathrm{d}x\,(a+x)^2$

b) $\int_{-5}^{5}\mathrm{d}q\,(a+b\,q^3)$

$\int_{0}^{B}\mathrm{d}k\,\tanh^2(kx)$

c) $\int_{0}^{\infty}\mathrm{d}x\,\mathrm{e}^{-x/L}$

d) $\int_{-L}^{L}\mathrm{d}y\,\mathrm{e}^{-y/\xi}$

e) $\int_{0}^{L}\mathrm{d}z\,\dfrac{z}{a+b\,z^2}$

f) $\int_{0}^{\infty}\mathrm{d}x\,x\,\mathrm{e}^{-x^2/(2Dt)}$

g) $\int_{-\sqrt{Dt}}^{\sqrt{Dt}}\mathrm{d}\ell\,\ell\,\mathrm{e}^{-\ell^2/(2Dt)}$

$\int_{-\sqrt{Dt}}^{\sqrt{Dt}}\mathrm{d}z\,x\,\mathrm{e}^{-z\,x^2}$

Except for the integration variable all quantities are considered to be constant.

Hint: Sometimes symmetries can substantially reduce the work needed to evaluate an integral.

3.3 *Newton's axioms and equations of motion (EOM)*

In Section 3.1 we referred to a train compartment to point out that physical observations will be the same — irrespective of the velocity of its motion, as long as it is constant. A setting where we perform an experiment is denoted as reference frame, and reference frames that move with constant velocity are called inertial systems.

Definition 3.2: Reference Frames and Inertial Systems

A *reference frame* $(\boldsymbol{Q}, \{\hat{\boldsymbol{e}}_i(t),\ i = 1 \cdots D\})$ is an agreement about the, in general time dependent, position of the origin $\boldsymbol{Q}(t)$ of the coordinate system and a set of orthonormal basis vectors $\{\hat{\boldsymbol{e}}_i(t),\ \ i \ = \ 1 \cdots D\}$, that are adopted to indicate the positions of particles in a physical model.
The reference frame refers to an *inertial system* when it does not rotate and when it moves with a constant velocity, i. e. if and only if $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$ and $\dot{\hat{\boldsymbol{e}}}_i = \boldsymbol{0}$ for all $i \in \{1 \cdots D\}$.

Remark 3.1. The requirement $\dot{\hat{\boldsymbol{e}}}_i = \boldsymbol{0}$ implies that the orientation of the basis vectors $\hat{\boldsymbol{e}}_i$ does not change, i.e. the reference frame does not rotate.

Remark 3.2. The *rest frame* for a particle is a reference frame where the particle velocity takes the constant velocity $\boldsymbol{0}$.

Remark 3.3. Let $\boldsymbol{q} = (q_1, \dots q_D)$ be the coordinates of a particles, as specified in in the inertial frame $(\boldsymbol{Q}, \{\hat{\boldsymbol{e}}_i\})$, and $\boldsymbol{x} = (x_1, \dots x_D)$ its position given in the inertial frame $(\boldsymbol{X}, \{\hat{\boldsymbol{n}}_i\})$. Then

$$\boldsymbol{q} = \boldsymbol{Q} + \sum_{i=1}^{D} q_i\, \hat{\boldsymbol{e}}_i = \boldsymbol{X} + \sum_{i=1}^{D} x_i\, \hat{\boldsymbol{n}}_i\,.$$

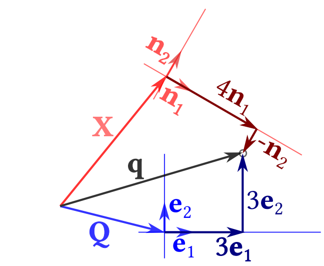


Figure 3.1: Graphical illustration of the description of a position from the perspective of two different reference frames, $\boldsymbol{q} = \boldsymbol{Q} + 3\,\hat{\boldsymbol{e}}_1 + 3\,\hat{\boldsymbol{e}}_2 = \boldsymbol{X} + 4\,\hat{\boldsymbol{n}}_1 - \hat{\boldsymbol{n}}_2$ with the notations of Remark 3.3.

3.3.1 *1st Law*

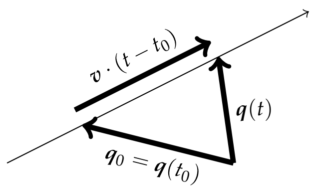
As long as a reference frame moves with a constant velocity, it feels like at rest. Physical measurements can only detect acceleration. This is expressed by

Axiom 3.1: Newton's 1st law

breakable]Newton!1st law | textbf The velocity of a particle moving in an inertial system is constant, unless a (net) force is acting on the particle,

$$\begin{aligned} \forall t \geq t_0 : \boldsymbol{F}(t) = \boldsymbol{0} \quad &\Leftrightarrow \quad \dot{\boldsymbol{q}}(t) = \boldsymbol{v} = \text{const} \\ &\Leftrightarrow \quad \boldsymbol{q}(t) = \boldsymbol{q}_0 + \boldsymbol{v}\,(t - t_0) \end{aligned}$$

as sketched in the margin.



The particle moves then in a straight line with a constant speed. Indeed, when a particle moves with the constant velocity $\boldsymbol{v} = \dot{\boldsymbol{q}}(t)$ in the reference frame $(\boldsymbol{Q}_1, \{\hat{\boldsymbol{e}}_i(t), i = 1 \cdots D\}$ then it is at rest in the alternative reference frame $(\boldsymbol{Q}_2, \{\hat{\boldsymbol{e}}_i(t), i = 1 \cdots D\}$ where $\boldsymbol{Q}_2 = \boldsymbol{Q}_1 + \boldsymbol{v}\,t$. Therefore, in the latter coordinate system the particle is at rest, and it will remain at rest when it is not perturbed by a net external force. After all,

$$\boldsymbol{q} = \boldsymbol{Q}_1 + \boldsymbol{v}\,t = \boldsymbol{Q}_2 + \boldsymbol{0}\,.$$

3.3.2 2nd Law

Newton's second law spells out how the velocity of the particle changes when there is a force.

Axiom 3.2: Newton's 2nd law

The change, $\ddot{\boldsymbol{q}}(t)$, of the velocity of a particle, $\dot{\boldsymbol{q}}(t)$, at position, $\boldsymbol{q}(t)$, is proportional to the sum of the forces $\boldsymbol{F}_\alpha$ acting on the particle, and the proportionality factor is the particle mass m,

$$m\, \ddot{\boldsymbol{q}}(t) = \sum_\alpha \boldsymbol{F}_\alpha(t)\,.$$

Remark 3.4. In general the time dependence of the forces can be decomposed into three contributions

a) An implicit time dependence, $\boldsymbol{F}(\boldsymbol{q}(t))$, when the force depends on the position, $\boldsymbol{q}(t)$ of the particle. For instance, for a Hookian spring with spring constant k one has, $\boldsymbol{F}(\boldsymbol{q}) = -k\, \boldsymbol{q}$.[2]

b) An implicit time dependence, $\boldsymbol{F}(\dot{\boldsymbol{q}}(t))$, when the force depends on the velocity, $\dot{\boldsymbol{q}}(t)$ of the particle.
For instance, the sliding friction for a particle with mass m and friction coefficient γ is, $\boldsymbol{F}(\dot{\boldsymbol{q}}) = -m\,\gamma\, \dot{\boldsymbol{q}}$.

c) An explicit time dependence when the force is changing in time. For instance, when pushing a child sitting on a swing one will only push when the swing is moving in forward direction.

[2] The spring constant k is a positive constant of dimension Newton per meter that characterizes the strength of the spring, and the minus sign makes it explicit that the Hoookian force is a restoring force pushing the particle back towards $\boldsymbol{q} = \boldsymbol{0}$.

Typically, one explicitly sorts out these dependencies and writes

$$m\, \ddot{\boldsymbol{q}}(t) = \sum_\alpha \boldsymbol{F}_\alpha(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)$$

The resulting relation between the acceleration and the force is called equation of motion of the particle.

Definition 3.3: Equation of Motion (EOM)

Newton's second law establishes a relation between the position $\boldsymbol{q}(t)$ of a particle of mass m, its velocity $\dot{\boldsymbol{q}}(t)$, and acceleration $\ddot{\boldsymbol{q}}(t)$,

$$m\, \ddot{\boldsymbol{q}}(t) = \boldsymbol{F}(\dot{\boldsymbol{q}}(t), \boldsymbol{q}(t), t)$$

that is referred to as the *equation of motion* (EOM) of the particle.
The motion of N particles residing at the positions $\boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t) \in \mathbb{R}^D$ and interacting with each other amounts to $N\,D$ coupled equations

$$\begin{aligned} m\, \ddot{\boldsymbol{q}}_1(t) &= \boldsymbol{F}_1\left(\dot{\boldsymbol{q}}_1(t), \dots, \dot{\boldsymbol{q}}_N(t), \boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t), t\right) \\ \vdots\quad & \quad \vdots \\ m\, \ddot{\boldsymbol{q}}_N(t) &= \boldsymbol{F}_N\left(\dot{\boldsymbol{q}}_1(t), \dots, \dot{\boldsymbol{q}}_N(t), \boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t), t\right) \end{aligned}$$

The primary aim of Theoretical Mechanics is to determine the solution of the EOM for given *initial conditions* (cf. Definition 1.6),

$$\Gamma_0 = \Big(\boldsymbol{q}_1(t_0), \dots, \boldsymbol{q}_N(t_0), \dot{\boldsymbol{q}}_1(t_0), \dots, \dot{\boldsymbol{q}}_N(t_0)\Big)$$

for the positions and velocities of the particles at time t_0. Bundles of phase-space trajectories characterize the motion of sets of trajectories, and they can be analyzed to determine how the behavior of a system changes upon varying the parameters of the setup.

Example 3.1: Particle moving in the gravitational field

The gravitational field induces a constant force $m\,\boldsymbol{g}$ on a particle with mass m. Let it have velocity v_0 at time t_0 when it is taking off from the position $\boldsymbol{q}_0$. Then Newton's 2nd law states that $\ddot{\boldsymbol{q}}(t) = \boldsymbol{g}$, and this equation must be solved subject to the initial conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}(t_0) = \boldsymbol{v}$. By working out the derivatives one readily checks that this is given for

$$\boldsymbol{q}(t) = \boldsymbol{q}_0 + \boldsymbol{v}\,(t - t_0) + \frac{1}{2}\,\boldsymbol{g}\,(t - t_0)^2$$

Example 3.2: Particle moving in a circle

Let a particle of mass m move with constant speed in a circle of radius R such that its position can be written as

$$\boldsymbol{q}(t) = \begin{pmatrix} R\,\cos(\omega t) \\ R\,\sin(\omega t) \end{pmatrix}$$

with a constant angular velocity ω. Then its velocity and acceleration take the form

$$\dot{\boldsymbol{q}}(t) = \begin{pmatrix} -\omega\,R\,\sin(\omega t) \\ \omega\,R\,\cos(\omega t) \end{pmatrix}$$

$$\text{and} \quad \ddot{\boldsymbol{q}}(t) = \begin{pmatrix} -\omega^2\,R\,\cos(\omega t) \\ -\omega^2\,R\,\sin(\omega t) \end{pmatrix} = -\omega^2\,\boldsymbol{q}(t)$$

The speed is constant, taking the value $\sqrt{\dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}} = \omega\,R$. The force is antiparallel to $\boldsymbol{q}$ with magnitude $m\,\omega^2\,R$. Moreover, $\dot{\boldsymbol{q}} \cdot \boldsymbol{F} = 0$ at all times. Hence, the force only changes the direction of motion, and not the speed.

3.3.3 *3rd Law*

Newton's third law states that the reference frame does not matter for the description of the evolution of two particles, even when they interact with each other — i.e. when they exert forces on each other. Consider for instance the motion of two particles of the same mass m that reside at the positions $\boldsymbol{q}_1(t)$ and $\boldsymbol{q}_2(t)$. We decide to observe them from a position right in the middle between the two particles $\boldsymbol{Q} = \big(\boldsymbol{q}_1(t) + \boldsymbol{q}_2(t)\big)/2$. In the absence of external forces this is an inertial frame, such that $\ddot{Q} = \mathbf{0}$ according to Newton's first law. However, Newton's second law implies that also

$$\mathbf{0} = 2m\ddot{Q} = m\,\ddot{q}_1 + m\,\ddot{q}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2$$

where $\boldsymbol{F}_1 = m\,\ddot{q}_1$ and $\boldsymbol{F}_2 = m\,\ddot{q}_2$ are the forces acting on particle 1 and 2, respectively. Up to a change of sign the forces are the same, $\boldsymbol{F}_1 = -\boldsymbol{F}_2$. This action-reaction principle is stipulated by

Axiom 3.3: Newton's 3rd law

Forces act in pairs:

actio when a body A is pushing a body B with force $\boldsymbol{F}_{A\to B}$

reactio then B is pushing A with force $\boldsymbol{F}_{B\to A} = -\boldsymbol{F}_{A\to B}$,

and these forces are always balanced, $\boldsymbol{F}_{A\to B} + \boldsymbol{F}_{B\to A} = \mathbf{0}$.

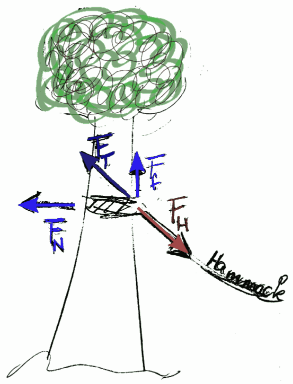


Figure 3.2: Graphical illustrations of forces involved in hanging a hammock on a tree, Example 3.3.

Example 3.3: Fixing a hammock at a tree

When you lie in a hammock that is fixed at a tree, your hammock exerts a force $\boldsymbol{F}_H$ on the tree (actio). The hammock stays where it is because the tree pulls back with exactly the same force $-\boldsymbol{F}_T$, up to a change of sign (reactio), and, in turn, this force can be written as the sum of two components accounting for the normal force F_N of the tree on the rope and a friction force F_f that prevents the rope from sliding down the tree.

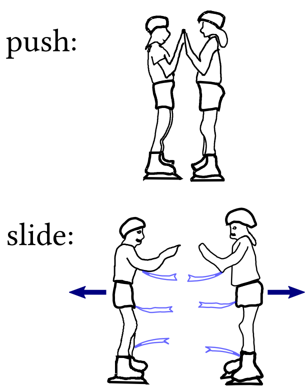


Figure 3.3: Graphical illustrations of motion of the two ice-skaters of Example 3.4.

Example 3.4: Ice skaters

- When two ice skaters of the same mass push each other starting from a position at rest, then they will move in opposite directions with the same speed (unless they brake).
- When they have masses m_1 and m_2 their velocities will be related by $m_1\,\boldsymbol{v}_1 + m_2\,\boldsymbol{v}_2 = \mathbf{0}$ because $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \mathbf{0}$ initially, and $m_1\,\dot{\boldsymbol{v}}_1 + m_2\,\dot{\boldsymbol{v}}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \mathbf{0}$ at any instant of time. As long as they push, the velocities are non-zero and speed increases. When they slide there is no force any longer, and they go at constant speed—except for the impact of friction of the skates on the ice.

Example 3.5: Water Rocket

A water rocket receives its thrust by the repulsive force in response of accelerating and releasing a water jet. Let M the mass of a rocket at a given time, and V_R its speed. To determine the acceleration of the rocket we consider a short time interval Δt where water of mass ΔM is ejected with speed v_f. In the absence of gravitation the momentum balance implies that at any given time the momentum of the rocket $M(t)\,V_R(t)$ must amount to the sum of the water $\Delta M\ (v_R(t) - v_f$ emitted during a short time Δt and the momentum of the rocket $M(t + \Delta t)\,V_R(t + \Delta t)$ after that time,

$$\begin{aligned} M\,V_R &= (M - \Delta M)\,(V_R + \Delta V_R) + \Delta M\,(V_R - v_f) \\ \Leftrightarrow \qquad 0 &= M\,\Delta V_R - \Delta M\,v_f - \Delta M\,\Delta V_R \end{aligned}$$

Now we observe that $\Delta M = a\,\rho, v_f\,\Delta t$ where a is the cross section of the ejected jet, and ρ the mass density of the ejected water:

$$M\,\frac{\Delta V_R}{\Delta t} = a\,\rho, v_f^2 + a\,\rho, v_f\,\frac{\Delta V_R}{\Delta t}\,\Delta t$$

and in the limit of small time increments $\Delta t \to 0$ we obtain the force F_R that is accelerating the rocket

$$F_R = M\,\dot{V}_R = a\,\rho\,v_f^2$$

The rocket trajectory results from interplay of gravity and F_R. One case will be discussed as worked example at the end of this chapter, in Section 3.5. Solving the general case has been suggested as an instructive computer-based example for teaching mechanics (Gale, 1970; Finney, 2000). Instructions about how to build and discuss the rocket in school is available from the NASA and the instructables community.

Michal Richard Trowbridge / wikimedia CC BY-SA 3.0
Figure 3.4: Launching a water rocket, as introduced in Example 3.5.

3.3.4 *Punchline*

Newton's equations are stated nowadays in terms of derivatives, a concept in calculus that has been pioneered by Leibniz.[3] In this language they take the following form for a particle of mass m that is at position $\boldsymbol{q}(t)$ at time t,

[3] Even though these principles of calculus were independently understood by Newton which lead to a very long fight for authorship and fame.

$$\begin{aligned} \dot{\boldsymbol{q}}(t) &= \boldsymbol{v}(t) \\ \dot{\boldsymbol{v}}(t) &= \frac{1}{m}\,\boldsymbol{F}_{\text{tot}}(\boldsymbol{q}(t), \boldsymbol{v}(t), t) \end{aligned}$$

Prior to Newton, physical theories adopted the Aristotelian point of view that $\boldsymbol{v}$ is proportional to the force. Indeed in those days many scientists were regularly inspecting mines, and from the perspective of pushing mine carts is is quite natural to assert that their velocity is proportional to the pushing force. Galileo's achievement is to add the 'tot' of the force side of the equation, pointing out that there also is a friction force acting on the mine cart. Newton's achievement is to add the 'dot' on the left side of the equation, stating that the velocity stays constant when the pushing force and the friction force balance.

Example 3.6: Pushing a mine cart

The motion of the mine cart is one-dimensional along its track such that the position, q, velocity, x, and forces are one-dimensional, i. e. scalar functions. Once the mine cart is moving it experiences a friction force $\boldsymbol{F}_f = -\gamma\, \boldsymbol{v}$, that (to a first approximation) is proportional to its velocity, $\boldsymbol{v}$. Now, let the mine worker push with a constant force F_M such that

$$m\,\ddot{q} = m\,\dot{v} = F_{\text{tot}} = F_M - \gamma\, v\,.$$

The mine cart travels with constant velocity $\dot{v} = 0$, when the attacking forces balance, i. e. for $v_c = F_M/m\,\gamma$.
For a different initial velocity, $v(t_0) = v_0$, one finds an exponential approach to the asymptotic velocity,

$$v(t) = v_c + \left(v_0 - v_c\right)\, \mathrm{e}^{-\gamma\,(t-t_0)}$$

After all, $v(t_0) = v_c + \left(v_0 - v_c\right) = v_0$ and

$$\begin{aligned}\dot{v}(t) &= \left(v_0 - v_c\right)\,(-\gamma)\,\mathrm{e}^{-\gamma\,(t-t_0)}\\ &= \left(-\gamma\,\left(v(t) - v_c\right) = -\gamma\, v(t) + F_M\right)/m\end{aligned}$$

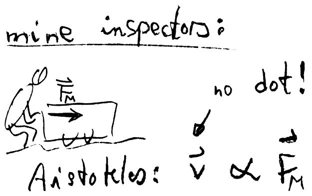


Figure 3.5: The pre-Newtonian understanding of the relation between force and velocity of a body.

clarify: Aristotelian mechanics + criticism

The advantage of the Newtonian approach above earlier modeling attempts is that it makes a quantitative prediction about the asymptotic velocity, and that it also addresses the regime where the velocity is changing, e. g. when the mine cart is taking up speed.

3.3.5 *Self Test*

Problem 3.3. **Terminal velocity for turbulent drag**

Rather than a friction of the type of the mine cart, a golf ball experiences a drag force

$$\boldsymbol{F}_d = -\frac{\rho |u|^2}{2} c_d A\, \hat{\boldsymbol{u}}$$

where A is the cross section of the ball, ρ the density of air, $\boldsymbol{u}$ the velocity of the golf ball, and $c_d \simeq 0.5$ the drag coefficient.

a) The drag coefficient is a dimensionless number that depends on the shape of the object that experiences drag. For the rest the expression for the drag force follows from dimensional analysis. Verify this claim.

b) A slightly more informed derivation of $\boldsymbol{F}_d$ introduces also the diameter D of the golf ball and states that drag arises because the ball has to push air out of its way. When moving it has to push air out of the way at a rate $A\,u$. The air was at rest initially and must move roughly with a velocity u to get out of the way. Subsequently, its kinetic energy is lost. Check out, how this leads to the expression provided for $\boldsymbol{F}_d$.

c) What is the terminal velocity of a golf ball that is falling out of the pocket of a careless hang glider?

d) Use dimensional analysis to estimate the distance after which the ball acquires its terminal velocity, and how long it takes to reach the velocity.

Problem 3.4. **Orbit of the Moon around Earth**

The Moon is circling around Earth due to the gravitational force of modulus

$$F_{ME} = \frac{G M_E M_M}{R_{ME}^2}$$

where $G = \frac{2}{3} \times 10^{-12} \mathrm{m}^3/\mathrm{kg\,s}$ is the gravitational constant, $M_E \simeq 6 \times 10^{24}$ kg and $M_M \simeq \frac{3}{4} \times 10^{-23}$kg are the masses of Moon and Earth, respectively, and $R_{ME} = \frac{7}{4} \times 10^6$m is the distance from Earth to Moon.

a) Calculate the force that Moon is experiencing due to the Earth. Compare it to the gravitational acceleration $g \simeq 10\,\mathrm{m/s}^2$ scaled by $(R_{ME}/R_E)^2$ where $R_E = 2\pi \times 10^6$m is the Earth radius. Why would one scale by this factor?

b) Assume that the Moon trajectory is circular and identify F_{ME} with the centripetal force that keeps the moon on its orbit. What does this tell about the dependence of the period T of the motion on G, R_{ME} and the masses.

c) Evaluate T and compare it to the duration of a month.

Problem 3.5. **Escape velocities**

The escape velocity is the minimum speed of a projectile that would allow it to escape into outer space when friction due to the atmosphere is neglected.

a) Estimate the escape velocity of Earth based on the gravitational force law F_{ME} given in Problem 3.4, the gravitational acceleration $g = 10\,\mathrm{m/s}^2$ on Earth, and the fact that the Earth circumference was set to $2\pi R_E = 4 \times 10^4$ km.

b) Recall the relation between gravity on Earth and Moon given in Problem 1.8, and estimate also the escape velocity from Moon.

c) **After you performed the calculations:**
Compare your estimates to the values provided by Wikipedia.

Problem 3.6. **Pulling a cow**

A child is pulling a toy cow with a force of $F = 5\,\mathrm{N}$. The cow has a mass of $m = 100\,\mathrm{g}$ and the chord has an angle $\theta = \pi/5$ with the horizontal.

[3] For this angle one has $\tan\theta \approx 3/4$.

a) Describe the motion of the cow when there is no friction. In the beginning the cow is at rest.


Children's Museum of Indianapolis, CC BY-SA 3.0

b) What changes when there is friction with a friction coefficient of $\gamma = 0.2$, i.e. a horizontal friction force of magnitude $-\gamma m g$ acting on the cow.

c) Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

3.4 *Constants of motion (CM)*

In the previous section we saw that Newton's laws can be expressed as equations relating the second derivative of the position of a particle to the forces acting on the particle. The forces are determined as part of setting up the physical model. Subsequently, determining the time dependence of the position is a mathematical problem. Often it can be solved by finding constraints on the solution that must hold for all times. Such a constraint is called a

Definition 3.4: Constant of motion

A function $\mathcal{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is a *constant of motion (CM)* iff its time derivative vanishes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0$$

It provides us with an opportunity to take a closer look at the expressions that emerge when taking derivatives of functions with arguments that are vectors. In order to evaluate the time derivative of $\mathcal{C}$ we write $\boldsymbol{q} = (q_1, \ldots\ldots, q_D)$, and apply the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(q_1(t), \cdots, q_D(t), \dot{q}_1(t), \cdots, \dot{q}_D(t), t) \\ &= \sum_{i=1}^{D} \frac{\mathrm{d}q_i}{\mathrm{d}t}\frac{\partial \mathcal{C}}{\partial q_i} + \sum_{i=1}^{D} \frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\frac{\partial \mathcal{C}}{\partial \dot{q}_i} + \frac{\partial \mathcal{C}}{\partial t}\end{aligned} \tag{3.4.1}$$

In this expression the operation ∂ is called '*partial*', and the derivative $\partial\mathcal{C}/\partial q_i$ is denoted as partial derivative of $\mathcal{C}$ with respect to q_i. For the purpose of calculating the partial derivative, we consider $\mathcal{C}$ to be a function of only the single argument q_i. For sake of a more compact notation we also write $\partial_{q_i}\mathcal{C}$ rather than $\partial\mathcal{C}/\partial q_i$. Moreover, when it is not clear from the context which conditions are adopted, they can explicitly be stated as subscript of a vertical bar to the right of the derivative (or even square brackets).

Example 3.7: Partial derivatives

For $f(x,y) = x/\sqrt{x^2+y^2}$ and $R = \sqrt{x^2+y^2}$ we have

$$\partial_x f(x,y)|_y = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{R^3}$$

$$\partial_x f(x,y)|_R = \partial_x \left[\frac{x}{R}\right]_R = \frac{1}{R}$$

A compact notation that allows us to state the expression of Equation (3.4.1) in a more transparent way is achieved as follows: We observe that the expressions in the sums amount to writing out in components a scalar product of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ with vectors that are obtained by the partial derivatives. These vectors are denoted *gradients* with respect to $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, and they will be written as

In the literature one also finds the alternative notations

$$\nabla_q \mathcal{C} = \frac{\partial \mathcal{C}}{\partial \boldsymbol{q}} = \partial_{\boldsymbol{q}} \mathcal{C}$$

$$\nabla_q \mathcal{C} = \begin{pmatrix} \partial_{q_1} \mathcal{C} \\ \vdots \\ \partial_{q_D} \mathcal{C} \end{pmatrix} \qquad \text{and} \qquad \nabla_{\dot{q}} \mathcal{C} = \begin{pmatrix} \partial_{\dot{q}_1} \mathcal{C} \\ \vdots \\ \partial_{\dot{q}_D} \mathcal{C} \end{pmatrix} \tag{3.4.2}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) = \dot{\boldsymbol{q}} \cdot \nabla_q \mathcal{C} + \ddot{\boldsymbol{q}} \cdot \nabla_{\dot{q}} \mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

In terms of the phase-space coordinates $\boldsymbol{\Gamma} = (\boldsymbol{q}, \dot{\boldsymbol{q}})$ one can also adopt the even more compact notation

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) = \dot{\boldsymbol{\Gamma}} \cdot \nabla_\Gamma \mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

or even

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}(\boldsymbol{\Gamma}(t), t) = \dot{\boldsymbol{\Gamma}} \cdot \nabla \mathcal{C}(\boldsymbol{\Gamma}(t), t) + \frac{\partial \mathcal{C}}{\partial t}(\boldsymbol{\Gamma}(t), t)$$

where the index of the nabla operator has been dropped with the understanding that it is clear from the context what the operator refers to.

We make use of these derivatives while introducing some important physical quantities that are constants of the motion in specific settings.

3.4.1 *The kinetic energy*

When no forces are acting on a particle, $\boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$, it moves with constant velocity. All functions that depend only on the velocity will then be constant. In particular this holds for the kinetic energy, T, that will play a very important role in the following.

Theorem 3.2: Conservation of kinetic energy

The *kinetic energy* $T = \frac{m}{2}\, \dot{\boldsymbol{q}}^2$ of a particle is conserved iff no net force acts on the particle, i. e. iff $\boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}T = \frac{m}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\sum_i \dot{q}_i \cdot \dot{q}_i = m\,\sum_i \dot{q}_i \cdot \ddot{q}_i$$
$$= m\dot{\boldsymbol{q}} \cdot \ddot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \cdot (m\,\ddot{\boldsymbol{q}}) = \dot{\boldsymbol{q}} \cdot \boldsymbol{F}_{\text{tot}} = 0$$

In the last two steps we used Newton's 2nd law, and the assumption that $\boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$. □

3.4.2 *Work and total energy*

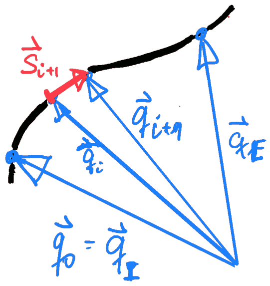


Figure 3.6: Breaking a particle track $\boldsymbol{q}(t)$ into a sequence of discrete points $\boldsymbol{q}_i$ with segments $\boldsymbol{s}_{i+1} = \boldsymbol{q}_{i+1} - \boldsymbol{q}_i$.

From a physics perspective, work is performed when a body is moved in the presence of an external force.

- When the force $\boldsymbol{F}$ is constant along a straight path of displacement $\boldsymbol{s} = \boldsymbol{q}_E - \boldsymbol{q}_I$, from a position $\boldsymbol{q}_I$ to the position $\boldsymbol{q}_E$, then the work W amounts to the scalar product $W = \boldsymbol{F} \cdot \boldsymbol{s}$.

- When the force depends on the position along the path, we parameterize the motion along the path by time, $\boldsymbol{q}(t)$, with $\boldsymbol{q}(t_I) = \boldsymbol{q}_I$ and $\boldsymbol{q}(t_E) = \boldsymbol{q}_E$ and break it into sufficiently small pieces $\boldsymbol{s}_i = \boldsymbol{q}(t_i) - \boldsymbol{q}(t_i - \Delta t)$ where the force $\boldsymbol{F}_i = \boldsymbol{F}(t_i)$ and the velocity of the particle $\dot{q}(t_i)$ may be assumed to be constant, such that $\dot{\boldsymbol{q}}(t_i) = \big(\boldsymbol{q}(t_i) - \boldsymbol{q}(t_{i-1})\big)/\Delta t$. Then

$$W = \sum_i \boldsymbol{F}_i \cdot \boldsymbol{s}_i = \lim_{\Delta t \to 0} \boldsymbol{F}_i \cdot \dot{\boldsymbol{q}}\,\Delta t = \int_{t_0}^{t_1} \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)\,\mathrm{d}t = \int_{\boldsymbol{q}(t)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$$

  The last equality should be understood here as a definition of the final expression that is interpreted here in the spirit of the substitution rule of integration.

Definition 3.5: Work and Line Integrals

The *work*, W, of a particle that performs a path $\boldsymbol{q}$ under the influence of a force $\boldsymbol{F}(t)$ amounts to the result of the *line integral*

$$W = \int_{\boldsymbol{q}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$$

When the path is parameterized by time, then W amounts to the time integral of dissipated power $P(t) = \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)$,

$$W = \int \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)\,\mathrm{d}t = \int P(t)\,\mathrm{d}t$$

Remark 3.5. The scalar product $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$ or $P(t) = \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)$ singles out only the action of the force parallel to the trajectory. The perpendicular components do not perform work. Hence, a force that is always acting perpendicular to the velocity, i. e. perpendicular to the path of the particle, does not perform any work,

$$W = \int \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)\,\mathrm{d}t = \int 0\,\mathrm{d}t = 0$$

It only changes the direction of motion.

Remark 3.6. The result of the integral does not rely on the parameterization of the path by time. For instance mathematicians prefer to use the length ℓ of the path. The speed of the particle is then $\dot{\ell}(t) = |\dot{\boldsymbol{q}}(t)|$ and one finds

$$W = \int \boldsymbol{F}(t) \cdot \boldsymbol{s} = \int \boldsymbol{F}(t(\ell)) \cdot \dot{\boldsymbol{q}}(t(\ell)) \, \frac{\mathrm{d}\ell}{\dot{\ell}} = \int \boldsymbol{F}(\ell) \cdot \frac{\boldsymbol{q}}{\mathrm{d}\ell}(\ell) \, \mathrm{d}\ell$$

where $\mathrm{d}\hat{\boldsymbol{q}}/\mathrm{d}\ell$ is a unit vector pointing in the direction of the trajectory.

The calculation of work simplifies dramatically when the force can be written as gradient of another function, Φ.

Definition 3.6: Potentials and Conservative Forces

A force $\boldsymbol{F}(\boldsymbol{q})$ that can be expressed as the negative gradient of a function $\Phi(\boldsymbol{q})$,

$$\boldsymbol{F}(\boldsymbol{q}) = -\nabla\Phi(\boldsymbol{q}) = -\begin{pmatrix} \partial_{q_1}\Phi(q_1, \dots, q_D) \\ \vdots \\ \partial_{q_D}\Phi(q_1, \dots, q_D) \end{pmatrix}$$

is called a *conservative force* and the function Φ is the *potential* associated to the force.

Remark 3.7. Conservative forces only depend on position, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{q})$. They neither explicitly depend on time nor on the velocity $\boldsymbol{q}$.

Remark 3.8. Conservative forces only depend on position, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{q})$. They neither explicitly depend on time nor on the velocity $\boldsymbol{q}$.

Example 3.8: Conservative forces: (counter-)examples

- Gravitational acceleration $\boldsymbol{g}$ is is constant in space. Hence, gravity is a conservative force.
- Friction of a cube sliding over a table is proportional to the particle speed v. Therefore, friction is *not* a conservative force.
- Setting the rope into motion for rope skipping requires an oscillatory force. Due to its time time dependence such a force is not conservative.

Rope skipping on the poster of the movie "Doubletime", wikimedia, CC BY 2.0

Theorem 3.3: Work for conservative forces

For conservative forces, $\boldsymbol{F} = -\nabla\Phi(\boldsymbol{q})$, the work for a path $\boldsymbol{q}(t)$ from $\boldsymbol{q}_0$ to $\boldsymbol{q}_1$ amounts to the difference of the potential evaluated at the initial and at the final point of the path

$$W = \int_{\boldsymbol{q}(t)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q} = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1)$$

Proof.

$$
\begin{aligned}
W &= \int_{t_0}^{t_1} \boldsymbol{F} \cdot \dot{\boldsymbol{q}} \, \mathrm{d}t = - \int_{t_0}^{t_1} \nabla\Phi \cdot \, \dot{\boldsymbol{q}} \, \mathrm{d}t \\
&= - \int_{t_0}^{t_1} \sum_i \frac{\partial \Phi}{\partial q_i} \frac{\partial q_i}{\partial t} \, \mathrm{d}t = - \int_{t_0}^{t_1} \frac{\mathrm{d}\,\Phi}{\mathrm{d}t} \, \mathrm{d}t \\
&= - \big(\Phi(\boldsymbol{q}(t_1)) - \Phi(\boldsymbol{q}(t_0)) \big) = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1) \qquad \square
\end{aligned}
$$

Remark 3.9. The work performed along a closed path vanishes for conservative forces. After all, in that case $\boldsymbol{q}_1 = \boldsymbol{q}_0$ such that $W = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1) = 0$.

Remark 3.10. The potential in itself is not an observable.[4] One can only observe the work, which is the potential difference between two positions, and the force, which is the negative gradient of the potential. Therefore, the potential is only defined up to adding a constant.

[4] An *observable* is a quantity that can be measured by direct observation.

Example 3.9: Gravitational Potential

For a particle of mass m gravity on the Earth surface gives rise to a force of magnitude $\boldsymbol{F}(x,y,z) = -m\,g\,\hat{\boldsymbol{z}}$ that can be derived from the potential $\boldsymbol{\Phi}(x,y,z) = m\,g\,z$,

$$
-\nabla\Phi_1(x,y,z) = \begin{pmatrix} -\partial_x \Phi(x,y,z) \\ -\partial_y \Phi(x,y,z) \\ -\partial_z \Phi(x,y,z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -m\,g \end{pmatrix} = \boldsymbol{F}(x,y,z)
$$

Far away, at a position $\boldsymbol{q} = (q_1, q_2, q_3)$ from the center of Earth, gravity induces a force $\boldsymbol{F}(\boldsymbol{q}) = -G\,M_E\,m\,\boldsymbol{q}/|\boldsymbol{q}|^3$ on a body of mass m. This force can be obtained as

$$
\begin{aligned}
-\nabla\phi_2(\boldsymbol{q}) &= \nabla \frac{G\,M_E\,m}{\sqrt{q_1^2 + q_2^2 + q_3^2}} = G\,M_E\,m \begin{pmatrix} \partial_{q_1} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \\ \partial_{q_2} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \\ \partial_{q_3} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \end{pmatrix} \\
&= G\,M_E\,m \begin{pmatrix} \frac{-q_1}{[q_1^2+q_2^2+q_3^2]^{3/2}} \\ \frac{-q_2}{[q_1^2+q_2^2+q_3^2]^{3/2}} \\ \frac{-q_3}{[q_1^2+q_2^2+q_3^2]^{3/2}} \end{pmatrix} = \frac{-G\,M_E\,m}{[q_1^2 + q_2^2 + q_3^2]^{3/2}} \, \boldsymbol{q} = \boldsymbol{F}(\boldsymbol{q})
\end{aligned}
$$

Remark 3.11. According to Theorem 3.3 differences of the value of the potential between two positions amount to the work performed in the potential. Different approaches to calculate the value of this scalar observable must yield identical results. Therefore, the functional dependence of the potential must not depend on the choice of the coordinate system. This invariance requires that the potential can always be expressed in terms of scalar products. For the potentials in Example 3.9 this is achieved by writing

$$
\begin{aligned}
\Phi_1(\boldsymbol{q}) &= m\,\boldsymbol{g} \cdot \boldsymbol{q} \quad \text{with} \quad \boldsymbol{g} = (0, 0, -g) \\
\Phi_2(\boldsymbol{q}) &= -G\,M_E\,m/\sqrt{\boldsymbol{q} \cdot \boldsymbol{q}}
\end{aligned}
$$

Remark 3.12. One can make use of the properties of scalar products to reduce the computational work to determine the force for a given potential by working out the component i of the gradient where i is can be any index of the vector. For conciseness we also write then ∂_i for the partial derivative with respect to component q_i of the argument $\boldsymbol{q}$ of $\Phi(\boldsymbol{q})$.

For the potentials in Example 3.9 this works as follows

$$-\partial_i \Phi_1(\boldsymbol{q}) = -m\,\partial_i \sum_j g_j q_j = -m \sum_j g_j \delta_{ij} = -m g_i$$
$$-\partial_i \Phi_2(\boldsymbol{q}) = G\,M_E\,m\,\partial_i \Big[\sum_j q_j^2\Big]^{-1/2} = \frac{-G\,M_E\,m\,q_i}{\big[\sum_j q_j^2\big]^{3/2}}$$

In particular in the second case the advantage is evident.

Example 3.10: Falling men and cat

When a cat, that has a mass of $m = 3\,\text{kg}$, falls from a balcony in the fourth floor, i. e. from a height $H \simeq 4 \times 3\,\text{m} = 12\,\text{m}$, the initial potential energy

$$V_{\text{cat}} = mgH = 3\,\text{kg} \times 10\,\text{m/s}^2 \times 12\,\text{m} = 360\,\text{kg}\,\text{m}^2/\text{s}^2$$

will be transformed into kinetic energy and then dissipated when the cat hits the ground.

To get an idea about this energy we compare it to the energy dissipated when a man of mass $M = 80\,\text{kg}$, falls out of his bed that has a height of $h = 50\,\text{cm}$,

$$V_{\text{man}} = Mgh = 80\,\text{kg} \times 10\,\text{m/s}^2 \times 0.5\,\text{m} = 400\,\text{kg}\,\text{m}^2/\text{s}^2$$

From the point of view of the dissipated energy the fall of the cat is not as bad as it looks at first sight.

Conservative forces are called conservative forces because motion in such a potential conserves the sum of the potential energy and the kinetic energy.

Theorem 3.4: Conservation of the total energy

The *total energy* $E = T + \Phi$ of a particle is conserved if it moves in a conservative force field $\boldsymbol{F} = -\nabla\Phi$.

Proof.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}\Phi}{\mathrm{d}t} = m\,\dot{\boldsymbol{q}} \cdot \ddot{\boldsymbol{q}} + \nabla\Phi \cdot \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \cdot \underbrace{(m\ddot{\boldsymbol{q}} - \boldsymbol{F})}_{=\,\boldsymbol{0}} = \boldsymbol{0}$$

In the third equality we used that the force is conservative, and in the final step, we used Newton's second law which states that $m\ddot{\boldsymbol{q}} = \boldsymbol{F}$. □

Example 3.11: Accidents at work and on the street

A paramedic emergency ambulance receives two calls from an accident site:
i. a craftsmen fell from a roof of height H
ii. a teenager hit a tree with his motorcycle with a speed v
For which height does the energy of the craftsman approximately match the one of the motor cyclist when he drove

in the city, $v_C = 50\,\mathrm{km/h}$,
outside the city, $v_L = 100\,\mathrm{km/h}$,
on a German autobahn with $v_A = 150\,\mathrm{km/h}$
or was really speeding with $v_S = 200\,\mathrm{km/h}$.

We assume that they both have comparable mass.
Energy conservation entails that we have to compare the potential energy V_{worker} of the craftsman on the roof and the kinetic energy of the teenager on the motorcycle T_{teenager},

$$mgH = V_{\text{worker}} = T_{\text{teenager}} = \frac{m}{2}\,v^2 \quad \Leftrightarrow \quad H = \frac{v^2}{2g}$$

Hence we find

v	50 km/h	100 km/h	150 km/h	200 km/h
H	12 m	50 m	110 m	200 m
floor	4	16	36	64

Most likely, the teenager will encounter more severe injuries, unless the craftsman is working on a really high building.

3.4.3 *Momentum*

Theorem 3.5: Conservation of momentum

The *momentum* $\boldsymbol{P} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\boldsymbol{q}_i(t)$ is conserved if no net force $\boldsymbol{F}_{\text{tot}}$ acts on the system.

Proof. The time derivative of the total momentum is $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P} = \sum_{i=1}^{N} m_i \ddot{\boldsymbol{q}}_i(t)$ where $m_i \ddot{\boldsymbol{q}}_i(t)$ amounts to the force on particle i. This force amounts to the sum of an external force $\boldsymbol{F}_i$ on particle i and the forces $\boldsymbol{f}_{ji}$ exerted by other particles j on i. The net force amounts to the sum of the external forces, $\boldsymbol{0} = \boldsymbol{F}_{\text{tot}} = \sum_i \boldsymbol{F}_i$. Newton's third law requires that $\boldsymbol{f}_{ji} = -\boldsymbol{f}_{ij}$, and we will set $\boldsymbol{f}_{ii} = \boldsymbol{0}$ to simplify notations of indices in the sums. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P} = \sum_{i=1}^{N}\left(\boldsymbol{F}_i + \sum_{j=1}^{N}\boldsymbol{f}_{ji}\right) = \sum_{i=1}^{N}\boldsymbol{F}_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{f}_{ij}$$

$$= \boldsymbol{F}_{\text{tot}} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\boldsymbol{f}_{ij} + \boldsymbol{f}_{ji}) = \boldsymbol{0}$$

□

Figure 3.7: Newton's cradle. When the excited ball to the right is released it will come down, hit the rightmost ball that is hanging down at rest. The momentum is transferred to the leftmost ball, and that is moving up (almost) as far to the left as the initial ball was excited to the right. Its motion reverses, and by the same sequence of events the motion proceeds from left to right.

Example 3.12: One-dimensional collisions

We consider two steel balls that can freely move along a line. They have masses m_1 and m_2 and reside at positions x_1 and x_2, respectively. Initially ball two is at rest in the origin, and ball one is approaching from the right with a constant speed v_1. What is the speed of the balls after the collision? Before and after the collision the particles feel no forces such that their velocity is constant. We assume that the collision is elastic such that energy is preserved. Hence,

	before collision	=	after collision
momentum:	$m_1\, v_1$	=	$m_1\, v_1' + m_2\, v_2'$
energy:	$\frac{m_1}{2}\, v_1^2$	=	$\frac{m_1}{2}\,(v_1')^2 + \frac{m_2}{2}\,(v_2')^2$

where the prime indicates the post-collision velocities. These velocities can best be determined by writing the momentum and energy balance in the form

$$m_1\,(v_1 - v_1') = m_2\, v_2' \quad \text{and} \quad m_1\,(v_1^2 - v_1'^2) = m_2\, v_2'^2$$

and dividing the second by the first equation. This provides

$$v_1 + v_1' = v_2'$$

Together with the momentum balance it provides

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}\, v_1 \qquad \text{and} \qquad v_2' = \frac{2\, m_1}{m_1 + m_2}\, v_1$$

In particular, when the two particles have the same mass one obtains that $v_1' = 0$ and $v_2' = v_1$ which is beautifully exemplified by the dynamics of Newton's cradle.

3.4.4 *Angular Momentum*

In the immediate vicinity of the collisions the balls in Newton's cradle perform a motion along a horizontal line, as discussed in Example 3.12. However, during the excursions to the left and right they follow a circular track where the chains act as arms and their suspension as fulcrum of the circular motion. In such settings it is often desirable to also consider the evolution of the angular momentum.

Theorem 3.6: Conservation of angular momentum

The *angular momentum* $\boldsymbol{L} = \sum_{i=1}^{N} m_i \boldsymbol{q}_i(t) \times \dot{\boldsymbol{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\boldsymbol{q}_i(t)$ is conserved if no external forces act on the system and if the interaction forces between pairs of particles act parallel to the line connecting the particles.

Proof.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} &= \sum_{i=1}^{N} m_i \Big(\dot{\boldsymbol{q}}_i(t) \times \dot{\boldsymbol{q}}_i(t) + \boldsymbol{q}_i(t) \times \ddot{\boldsymbol{q}}_i(t) \Big) \\ &= \sum_{i<j} \Big(\boldsymbol{q}_i(t) \times \boldsymbol{f}_{ij} + \boldsymbol{q}_j(t) \times \boldsymbol{f}_{ji} \Big) \\ &= \sum_{i<j} \Big(\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t) \Big) \times \boldsymbol{f}_{ij} = \boldsymbol{0} \end{aligned}$$

where we used that $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$ due to Newton's third law, and that $(\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t))$ is parallel to $\boldsymbol{f}_{ij}$ by assumption on the particle interactions. □

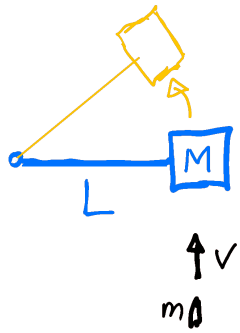


Figure 3.8: Notations adopted in the measurement of the speed v of a bullet of mass m that is hitting a rotor of mass M attached to an arm of length L; see **??** 3.13.

Example 3.13: Determine the speed of a bullet.

In a CSI lab one tests the speed of a bullet by shooting it into a rotor where a mass $M = 1\,\mathrm{kg}$ can move horizontally with minimal fraction on an arm with length $L = 1\,\mathrm{m}$. For a bullet of a mass $m = 8\,\mathrm{g}$ we find a rotation frequency $f = 0.16\,\mathrm{Hz}$. What is the muzzle velocity v of the gun? During the collision the bullet gets stuck in the rotor mass. Before and after the collision the angular momentum thus is

$$\begin{aligned} & m\,L\,v = (m+M)\,L^2\,\omega = (m+M)\,L^2\,2\pi\,f \\ \Leftrightarrow \qquad & v = \frac{m+M}{m}\,2\pi\,f\,L = \frac{1008}{8} \times 2\pi \times 0.16\,\mathrm{m/s} \simeq 125\,\mathrm{m/s} \end{aligned}$$

3.4.5 *Self Test*

Problem 3.7. **Derivatives of common composite expressions**

Evaluate the following derivatives.

a) $\frac{\mathrm{d}}{\mathrm{d}x}(a+x)^b$

b) $\frac{\partial}{\partial x}(x+b\,y)^2$

c) $\frac{\mathrm{d}}{\mathrm{d}x}(x+y(x))^2$

d) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta(t)$

e) $\frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta(t)\ \cos\theta(t))$

f) $\frac{\mathrm{d}}{\mathrm{d}t}\sin(2\theta(t))$

g) $\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{a+b\,z^2}$

h) $\frac{\partial}{\partial x_3}\left[\sum_{j=1}^{6} x_j^2\right]^{-1/2}$

i) $\frac{\partial}{\partial y_1}\ln(\boldsymbol{x}\cdot\boldsymbol{y})$

In these expressions a and b are real constants, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are 6-dimensional vectors.

Problem 3.8. **Running mothers**

Demonstrate that

$$I = \dot{x}_1(t)\,\dot{x}_2(t) + \omega^2\,x_1(t)\,x_2(t)$$

is a constant of motion of a two-dimensional harmonic oscillator with equation of motion

$$\ddot{\mathbf{x}}(t) = -\omega^2\,\mathbf{x}(t) \qquad \text{with} \quad \omega \in \mathbb{R} \quad \text{and} \quad \mathbf{x}(t) = \big(x_1(t), x_2(t)\big) \in \mathbb{R}^2$$

Problem 3.9. **Anvil shooting**

Rex Hammock from USA/ wikimedia CC BY-SA 2.0

Anvil shooting is a tradition in some US communities to celebrate St. Clement's Day, honoring Pope Clement I, the patron saint of blacksmiths and metalworkers. Typical anvils have a mass of about 150 kg and they are shot up to a height of 60 m. Which energy must the gun powder release to the anvil for such a feat?

Problem 3.10. **Running mothers**

In the Clara Zetkin Park one regularly encounters blessings[5] of dozens of mothers jogging in the park while pushing baby carriages. Troops of kangaroo mothers rather carry their youngs in pouches.

[5] Look up "terms of venery" if you ever run out of collective nouns.

a) Estimate the energy consumption spend in pushing the carriages as opposed to carrying the newborn.
The carriages suffer from friction. Let the friction coefficient be $\gamma = 0.3$.
When carrying the baby the kangaroo must lift it up in every jump and the associated potential energy is dissipated.

b) How does the running speed matter in this discussion?

c) How does the mass of the babies/youngs make a difference?

Problem 3.11. **The sledgehammer experiment**

In his magnificent book "Thinking Physics" Lewis Carroll Epstein (2009) sets out a class room experiment that he used to perform in his physics class: He placed an anvil on his chest and asked a student from the audience to hit the anvil with a sledge hammer as hard as he could manage. What will happen?

Epstein changed the way of presentation of this experiment when a very nervous student missed the anvil and hit his hand. Have a look into the book for the full story.

Problem 3.12. **The rotating chair experiment**

deerstop, wikimedia, CC0

The spin increases when an ice dancer pulls inwards arms and legs. This is illustrated in the picture of Yuko Kawaguti in the margin, and the physical principle has beautifully been demonstrated in a wikimedia movie by Oliver Zajkov from the Physics Institute at the University of Skopje.

a) Assume that a less careful experimenter starts his motion with a spin of 1 Hz, holding 5 kg barbells with stretched-out arms 1 m away from the rotation axis. Estimate his spin rate when he pulls in his arms till the barbells reach a distance of 20 cm from the rotation axis.

b) Which trajectory will they take when the careless experimenter gets dizzy and looses hold on the barbells?

3.5 *Worked example: Flight of an Earth-bound rocket*

In order to illustrate the applications of Newton's laws we discuss now the flight of a rocket. We will deal with the case a) where the rocket is moving in vertical direction, b) where the fuel is ejected with a constant speed v_f (or zero when it is exhausted), and c) where the rocket does not reach heights with a noticeable change of the gravitational acceleration. At the end of this section we discuss the impact of relaxing these assumptions, and point to the literature for a further discussion.

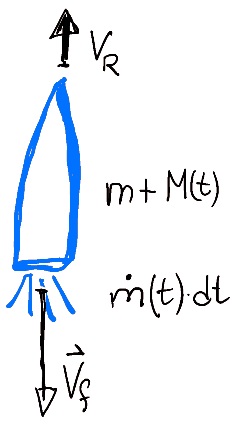


Figure 3.9: Notations adopted for the discussion of the flight of a rocket in Section 3.5.

Let V_R be the speed of the rocket. It is positive when the rocket goes up, and negative when it falls down. On the way down, its mass will be m. Initially, it has a mass $m + M_0$, where M_0 is mass of the fuel (cf. Figure 3.9). As long as the rocket is firing, Newton's third law implies that

$$F_R = \big(m + M(t)\big)\, \dot V_R = a\,\rho\, v_f^2 - \big(m + M(t)\big)\, g$$

The first force on the right-hand side of this equation accounts for the recoil from ejection of the fuel (cf. Example 3.5) and the latter to gravitational acceleration. We also observed in Example 3.5 that the mass $M(t)$ of the remaining fuel at time t obeys the differential equation $\dot M = -a\,\rho\, v_f$ such that[6]

$$M(t) = M_0 - a\,\rho\, v_f\, t\,.$$

At some time T all fuel is consumed, and we have

$$0 = M_0 - a\,\rho\, v_f\, T \quad\Rightarrow\quad T = \frac{M_0}{a\,\rho\, v_f}\,.$$

Moreover, for the rocket acceleration we find

$$\dot V_R(t) = \frac{F_R}{m + M(t)} = -g + \frac{v_f/T}{\mu - t/T} \quad \text{with} \quad \mu = \frac{m + M_0}{M_0}$$

The rocket speed is obtained by integrating the acceleration from the initial time, where the rocket is at rest, till time t. The integral takes a simpler form when one adopts the dimensionless integration variable, $\tau = t/\mu T$,

$$V_R(t) = \mu T \int_0^{t/\mu T} \mathrm{d}\tau\, V_R(t) = -g\,t - v_f \int_0^{t/\mu T} \mathrm{d}\tau\, \frac{1}{1-\tau}$$

$$= -g\,t - v_f\, \ln\left(1 - \frac{t}{\mu T}\right) \tag{3.5.1a}$$

[6] One easily checks that this expression is correct for the initial mass, $M(0) = M_0$ and its derivative agrees with $\dot M(t)$. The same applies also for the expressions for the speed and height of the rocket discussed below. Problem 4.2 gives clues how the solutions are determined systematically. In Chapter 4 we discuss systematic approaches to find the solution.

Thus, at time T the rocket has acquired the speed

$$V_R(T) = -g\,T + v_f\,\ln\frac{\mu-1}{\mu}\,. \tag{3.5.1b}$$

The rocket height $z(t)$ is obtained by observing that $\dot{z}(t) = V_R(t)$, which in turn is given by Equation (3.5.1a). The solution where the rocket starts at height zero is the given by

$$\begin{aligned} z(t) &= \mu T \int_0^{t/\mu T} \mathrm{d}\tau\, V_R(t) = -\frac{g\,t^2}{2} - \mu T\,v_f \int_0^{t/\mu T} \mathrm{d}\tau\, \ln(1-\tau) \\ &= -\frac{g\,t^2}{2} - \mu T\,v_f\, \big[(\tau-1)\,\left(-1+\ln(1-\tau)\right)\big]_0^{t/\mu T} \\ &= -\frac{g\,t^2}{2} + v_f\,t + v_f\,T\left(\mu - \frac{t}{T}\right)\ln\left(1-\frac{t}{\mu T}\right). \end{aligned} \tag{3.5.2a}$$

At time T this simplifies to

$$z(T) = -\frac{g\,T^2}{2} + v_f\,T\left[1 + (\mu-1)\,\ln\frac{\mu-1}{\mu}\right] \tag{3.5.2b}$$

Starting from that position the rocket will perform a ballistic flight with initial velocity $V_R(T)$ that will add to its height another height increment of $V_R^2(T)/2g$. The additional height increment ΔH before the rocket reaches the crest of its height is found by energy conservation and Equation (3.5.1b)

$$\begin{aligned} m\,g\,\Delta H &= \frac{m}{2}\,V_R^2(T) \\ \Rightarrow\quad \Delta H &= \frac{V_R^2(T)}{2\,g} = \left[\frac{g\,T^2}{2} - T\,v_f\,\ln\frac{\mu-1}{\mu} + \frac{v_f^2}{2\,g}\left(\ln\frac{\mu-1}{\mu}\right)^2\right] \end{aligned} \tag{3.5.3}$$

Combining Equations (3.5.2b) and (3.5.3) yields the total height, H, reached by the rocket,

$$\begin{aligned} H = z(T) + \Delta H &= \left[-\frac{g\,T^2}{2} + v_f\,T + (\mu-1)\,v_f\,T\,\ln\frac{\mu-1}{\mu}\right] \\ &\quad + \left[\frac{g\,T^2}{2} + v_f\,T\,\ln\frac{\mu-1}{\mu} + \frac{v_f^2}{2\,g}\left(\ln\frac{\mu}{1+\mu}\right)^2\right] \\ &= v_f\,T\left[1 + \mu\,\ln\frac{\mu-1}{\mu}\right] + \frac{v_f^2}{2\,g}\left(\ln\frac{\mu-1}{\mu}\right)^2 \end{aligned} \tag{3.5.4}$$

For $m > 0$ we have $\mu > 1$, and the expression in the square bracket is always negative, as one can see based on the inequality $\ln x \le x - 1$ shown Figure 3.10,

$$1 + \mu\,\ln(1+\mu^{-1}) \le 1 + \mu\left(1 + \mu^{-1} - 1\right) = 0$$

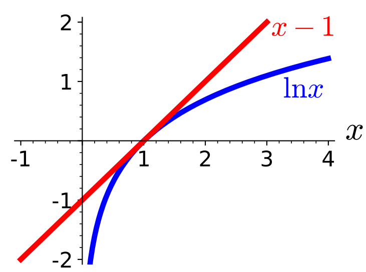


Figure 3.10: The function $x-1$ (red) is always larger (or equal) than $\ln x$ (blue).

The best strategy to achieve a large height is to go for a small T in order to suppress the first term in Equation (3.5.4) and large v_f to achieve large values of the second term.

When energy efficiency is a concern, e. g. when the rocket is used for a measurement of the atmosphere at height H, one might

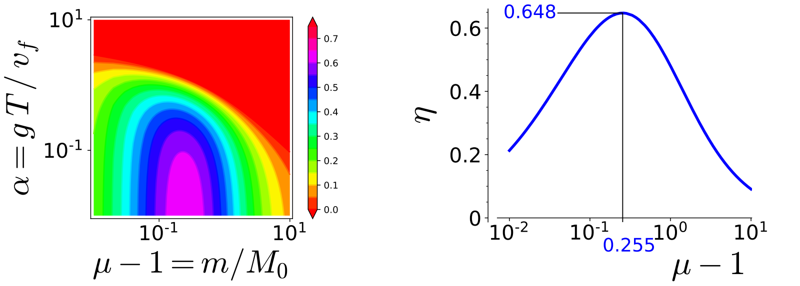


Figure 3.11: (left) Contour line for the efficiency, Equation (3.5.5), as function of the mass ratio $m/M_0 = \mu - 1$ and the dimensionless inverse rocket acceleration $\alpha = gT/v_f$. The maximum is taken for $\alpha = 0$. (right) Plot of the μ dependence of the efficiency for $\alpha = 0$. The maximum efficiency of $\eta_{\text{opt}} \simeq 0.648$ is obtained for $\mu_c \simeq 1.255$.

be interested to reach the height H with minimum energy cost. This means one is interested to minimize the ratio of the potential energy of the rocket at height H and the the energy $M_0 v_f^2/2$ burned to deliver the freight,

$$\eta = \frac{m\,g\,H}{M_0 v_f^2/2} = \frac{2\,m\,g\,T}{M_0\,v_f}\left[1 + \mu\,\ln\frac{\mu-1}{\mu}\right] + \frac{m}{M_0}\left(\ln\frac{\mu}{1+\mu}\right)^2$$
$$= \frac{2\,g\,T}{v_f}\,(\mu-1)\left[1 + \mu\,\ln\frac{\mu-1}{\mu}\right] + (\mu-1)\left(\ln\frac{\mu-1}{\mu}\right)^2 \tag{3.5.5}$$

The efficiency is a function of μ and of the dimensionless number $\alpha = g\,T/v_f$. The contour lines of $\eta(\mu,\alpha)$ are plotted in the left panel of Figure 3.11.

Definition 3.7: Contour lines and isosurfaces

The *contour lines* of a two variable function $f(x,y)$ are those lines in the (x,y)-plane, where $f(x,y)$ takes some constant value. More generally these lines are also called *isolines*, the two-dimensional surfaces where a three-variable function $g(x,y,z)$ in the (x,y,z)-space takes constant values are called isosurfaces, and the $N-1$-dimensional hypersurfaces of $\mathbb{R}^N$ where the function $h(\boldsymbol{q})$ with $\boldsymbol{q} \in \mathbb{R}^N$ takes a constant values will also be denoted as isosurfaces.

Example 3.14: Isosurfaces of the 3D Gaussian distribution

The 3D Gaussian distribution

$$P(x,y,z) = \frac{1}{(2\pi D\,t)^{3/2}}\,\exp\left(-\frac{x^2+y^2+z^2}{2Dt}\right)$$

describes the distribution of dye molecules at time t when a tiny droplet of dye is added without motion in a large container of water (Brownian motion). At any given instant of time the surfaces where the concentration take constant values C amount to

$$C = (2\pi D\,t)^{-3/2}\,\exp\left(-(x^2+y^2+z^2)/(2Dt)\right)$$
$$\Leftrightarrow\quad x^2+y^2+z^2 = -2Dt\,\ln\left(C\,(2\pi D\,t)^{3/2}\right) = R^2$$

where R^2 is an abbreviation of the (positive) constant on the right-hand side of the equation. Hence, the isosurface I_R for a given R amounts to a sphere of radius R,

$$I_R = \{(x,y,y) \in \mathbb{R}\ :\ x^2+y^2+z^2 = R^2\}\,.$$

The contour lines of the efficiency reveal that the maximum efficiency is obtained for $\alpha = 0$, which can expected since the expression in square brackets in Equation (3.5.5) is negative. From a physics perspective it means that high efficiencies require a large fuel expulsion speed v_f. The maximum efficiency amounts to the maximum of $\eta(\mu,\alpha=0) = \mu\ \ln^2[\mu/(1+\mu)]$, which amounts to the root μ_c of the equation $2/(1+\mu)+\ln[\mu/(1+\mu)]$. Numerically it is found to be $\mu_c \simeq 0.255$. Hence, the maximum efficiency is obtained when the mass of the fuel M_0 is roughly four times larger than the mass of the empty rocket. The maximum efficiency amounts then to

$$\eta_{\max} = \frac{4\,\mu_c}{(1+\mu_c)^2} \simeq 0.648\,.$$

Irrespective of the rocket design one can not transform more than 2/3 of the energy of the fuel into potential energy of the rocket. The remaining energy is dissipated in the kinetic energy of the exhaust.

Further discussion of the trajectories of rockets can be found in Finney (2000); Gale (1970); Seifert *et al.* (1947). A discussion of water rockets that addresses the change of speed v_f of the ejected water was given in Kagan *et al.* (1995); Gommes (2010).

3.6 *Problems*

3.6.1 *Practicing Concepts*

Problem 3.13. **Car on an air-cushion**

We consider a car of mass $m = 20\,\mathrm{g}$ moving – to a very good approximation without friction – on an air-cushion track. There is a string attached to the car that moves over a roll and hangs vertically down on the side opposite to the car.

a) Sketch the setup and the relevant parameters.

b) Which acceleration is acting on the car when the string is vertically pulled down with a force of $F = 2\,\mathrm{N}$. Determine the velocity $v(t)$ and its position $x(t)$.

c) Determine the force acting on a 200 g chocolate bar, in order to get a feeling for the size of the force that was considered in (b).

d) Now we fix the chocolate bar at the other side of the string. The velocity of the car can then be obtained based on energy conservation

$$E = E_{\text{kin}} + E_{\text{pot}} = \frac{m+M}{2}\, v^2 + Mgh = \text{konst},$$

where M is the mass of the chocolate bar. Is the acceleration the same of different as in the cases (b) and (c)? Provide an argument for your conclusion.

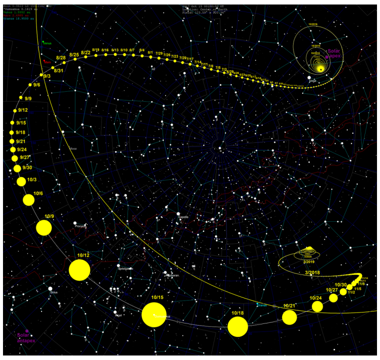


Figure 3.12: 'Oumuamua trajectory as seen by an observer on Earth.

Problem 3.14. **'Oumuamua**

On 19 October 2017 astronomers at the Haleakala Observatory in Hawaii discovered 'Oumuamua, the first interstellar object observed in our solar system. It approached the solar system with a speed of about $v_I = 26\,\text{km/s}$ and reached a maximum speed of $v_P = 87.71\,\text{km/s}$ at its perihelion, i. e. upon closest approach to the sun on 9 September 2017.

a) Show that at the perihelion the speed and 'Oumuamua's smallest distance to the sun, D, obey the relation

$$\frac{v_P^2 - v_I^2}{2} = \frac{M_S\, G}{D}$$

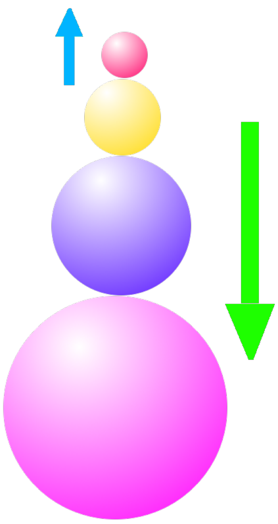
while for the Earth we always have

$$\frac{4\pi^2 R}{T^2} \simeq \frac{M_S\, G}{R^2}$$

Here, M_S is the mass of Sun, R is the Earth-Sun distance, and $T = 1\,\text{year}$ is the period of Earth around Sun.

b) Show that this entails that $\dfrac{D}{R} = \dfrac{2\, v_E^2}{v_P^2 - v_I^2}$, where $v_E = 2\pi R/T$ is the speed of Earth around sun.

c) Use the relation obtained in (b) to determine D in astronomical units, and compare your estimate with the observed value $D = 0.25534(7)\,\text{AU}$.



Problem 3.15. **Galilean cannon**

In the margin we show a sketch of a Galilean cannon. Assume that the mass mass ratio of neighboring balls with always two, and that they perform elastic collisions.

a) Initially they are stacked exactly vertically such that their distance is negligible. Let the distance between the ground the lowermost ball be 1 m. How will the distance of the balls evolve prior to the collision of the lowermost ball with the ground?

b) After the collision with the ground the balls will move up again. Determine the maximum height that is reached by each of the balls.

Problem 3.16. **Motion in a harmonic central force field**

A particle of mass m and at position $\boldsymbol{r}(t)$ is moving under the influence of a central force field

$$\boldsymbol{F}(\boldsymbol{r}) = -k\,\boldsymbol{r}\,.$$

a) We want to use the force to build a particle trap,[7] i. e. to make sure that the particle trajectories $\boldsymbol{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\boldsymbol{r}(t)| < B$ for all times t. What is the requirement on the sign of the constant k to achieve this aim?

[7] Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

b) Determine the energy of the particle and show that its energy is conserved.

c) Demonstrate that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times m\,\dot{\boldsymbol{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field is no longer coincide with the origin of the coordinate system in that case.

Problem 3.17. **Collision with an elastic bumper**

Consider two balls of radius R with masses m_1 and m_2 that are moving along a line. Their positions will be denoted as x_1 and x_2 in such a way that they touch when $x_1 = x_2$ and they do not feel each other when $x_1 < x_2$. When they run into each other, the balls can slightly be deformed such that the distance between their centers takes the value $2R - d$, and they experience a harmonic repulsive forces $\pm k\,d$. We will say then that $d = x_2 - x_1 < 0$.

a) Newton's equations for the collision of the two balls take the form

$$m_1\,\ddot{x}_1(t) = -k\,d(t) \qquad\qquad m_2\,\ddot{x}_2(t) = k\,d(t)$$

Show that this implies

$$\ddot{d} = -\omega^2\,d$$

for some positive constant ω. How does ω depend on the spring constant k and on the masses m_1 and m_2?

b) Let $d(t) = -d_M\,\sin\big(\omega\,(t - t_0)\big)$ describe the deformation of the balls for a collision at $t = t_0$, and contact in the time interval $t_0 \le t \le t_R$. Verify that it is a solution of the equation of motion. At which time t_R will the particles release (i.e. there is no overlap any longer)? What is the maximum potential energy stored in the harmonic potential?

c) We consider initial conditions where particle 1 arrives with a constant velocity v_0 from the left, and particle 2 is at rest. What is the total kinetic energy in this situation? Assume that at most a fraction α of the kinetic energy is transferred to potential energy. What is the relation between v_0 and the maximum deformation d_M?

d) The velocity of the two particles at times $t_0 \leq t \leq t_R$ can now be obtained by solving the integrals

$$m_i\, \dot{x}_i(t) = m_i\, x_i(t_0) + (-1)^i \int_{t_0}^{t} \mathrm{d}t'\, k\, d(t')\,, \quad \text{with } i \in \{1,2\}$$

Why does this hold? Which values does $x_i(t_0)$ take? Solve the integral and show that

$$\dot{x}_1 = v_0 \left[1 + \sqrt{\alpha\beta}\left(\cos(\omega\,(t-t_0)) - 1\right)\right]$$
$$\dot{x}_2 = v_0\, \frac{m_1}{m_2}\, \sqrt{\alpha\beta}\left(\cos(\omega\,(t-t_0)) - 1\right)$$

How does β depend on the masses?

e) Verify that at release we have

$$\dot{x}_1 = v_0\, (1 - 2\sqrt{\alpha\beta})$$
$$\dot{x}_2 = v_0\, \frac{2\, m_1}{m_2}\, \sqrt{\alpha\beta})$$

Verify that these expressions comply to momentum conservation. Verify that the expressions obey energy conservation iff $\alpha = \beta = m_2/(m_1 + m_2)$.

f) What does this imply for particles of identical masses, $m_1 = m_2$? How does your result fit to the motion observed in Newton's cradle? What does it tell about the assumption of instantaneous collisions of balls that is frequently adopted in theoretical physics?

"Free Metal Jacket" movie poster (wikimedia fair use)

Problem 3.18. **Inelastic collisions, ballistics, and cinema heroes**

Let us take a look at how cinema heroes shoot.

a) The title of Stanley Kubrick's movie Full Metal Jacket refers to full metal jacket bullets, i. e. projectiles as they were used in the M16 assault rifle used in the Vietnam war. Its bullets have a mass of 10 g and they set a 1 kg wooden block revolving at a 1 m arm into a 8 Hz motion. What is the velocity of the bullets?

The bullets of a 9 mm Luger pistol have a mass of 8 g and they are fired with a muzzle velocity of $350\,\mathrm{m\,s^{-1}}$. What is the resulting angular speed $\dot{\theta}$ of the wooden block?

b) Alternatively one can preform this measurement by shooting the bullet into a swing where a wooden block of mass M is attached to ropes of length ℓ. Initially it is at rest. Consider angular momentum conservation to determine its velocity immediately after impact. What does this tell about the kinetic energy immediately

of Planet Terror.
icense)

after the impact, and what about the maximum height of reached by the swing in its subsequent motion?

Let L be 2 m. Which mass is required to let the swing go up to the height of its spindle?

What does this tell about the recoil of the pistol and the rifle? What do you think now about the shooting scenes that you might recall from Rambo movies or grindhouse movies like Planet Terror.

3.6.2 *Mathematical Foundation*

add: problems for line integrals, in particular parameterization with length

add: calculation of the length of a path

add: problems for contour lines

Problem 3.19. **Solving integrals by partial integration**

Evaluate the following integrals by partial integration

$$\int \mathrm{d}x\, f(x)\, g'(x) = f(x)\, g(x) - \int \mathrm{d}x\, f'(x)\, g(x)$$

a) $\displaystyle\int_a^b \mathrm{d}x\, x\, \mathrm{e}^{kx}$

b) $\displaystyle\int_a^b \mathrm{d}x\, x^2\, \mathrm{e}^{kx}$

c) $\displaystyle\int_0^\infty \mathrm{d}x\, x^3\, \mathrm{e}^{-x^2}$

d) $\displaystyle\int_a^b \mathrm{d}x\, x^n\, \mathrm{e}^{kx},\ n \in \mathbb{N}$

The integral d) can only be given as a sum over $j = 0, \dots, n$.

Problem 3.20. **Substitution with trigonometric and hyperbolic functions**

Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{f(x_I)}^{f(x_F)} \mathrm{d}f\, g(f) = \int_{x_I}^{x_F} \mathrm{d}x\, \frac{\mathrm{d}f(x)}{\mathrm{d}x}\, g(f(x))$$

with a function $f(x)$ that is bijective on the integration interval $[x_I, x_F]$. A graphical illustration of the rule is given in Figure 3.13.

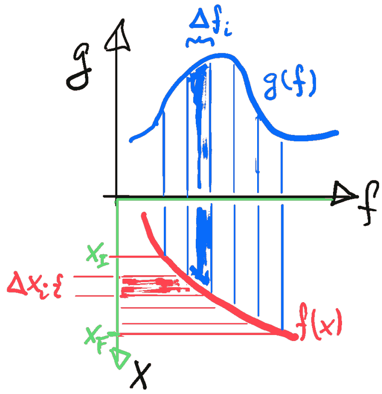


Figure 3.13: Illustration of the substitute rule for integrals that may be represented in terms of a Riemann sum ():

$$\int_{f(x_I)}^{f(x_F)} \mathrm{d}f\, g(f) \simeq \sum_i \Delta f_i\, g(f_i) \simeq \sum_i \Delta x_i\, \frac{\Delta f_i}{\Delta x_i}\, g(f(x_i)) \simeq \int_{x_I}^{x_F} \mathrm{d}x\, \frac{\mathrm{d}f(x)}{\mathrm{d}x}\, g(f(x))$$

provide reference

a) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}}$ by substituting $x = \sin\theta$

b) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1+x^2}}$ by substituting $x = \sinh z$

c) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{1+x^2}$ by substituting $x = \tan\theta$

d) $\displaystyle\int_a^b \mathrm{d}x\, \frac{1}{1-x^2}$ by substituting $x = \tanh z$

Problem 3.21. **Gradients and contour lines**

a) Contour lines in the (x, y)-plane are lines $y(x)$ or $x(y)$ where a functions $f(x, y)$ takes a constant value (cf. Definition 3.7). Sketch the contour lines of the functions

$$f_1(x,y) = (x^2 + y^2)^{-1} \qquad \text{and} \qquad f_2(x,y) = -x^2\, y^2$$

b) Determine the gradients $\nabla f_1(x, y)$ and $\nabla f_2(x, y)$.
Hint: The gradient $\nabla f(x, y)$ of a function $f(x, y)$ is the vector $(\partial_x f(x, y), \partial_y f(x, y))$ that contains the two partial derivatives of the (scalar) function $f(x, y)$ (cf. Equation (3.4.2)).

c) Indicate the direction and magnitude of the gradient by appropriate arrows in the sketch showing the contour lines. In which direction is the gradient pointing?

3.6.3 *Transfer and Bonus Problems, Riddles*

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Problem 3.22. **Moeschenbroeks double-cone experiment**

In the margin we show Moeschenbroeks double-cone experiment. The setup involves three angles:

1. The opening angle α between the two rails.
2. The angle ϕ of the rail surface with the horizontal.
3. The opening angle θ of the cone.

When it is released from the depicted position the cone might move to the right, to the left, and it could stay where it is. How does the selected direction of motion depend on the choice of the three angles?

Problem 3.23. **Coulomb potential and external electric forces**

We consider the Hydrogen atom to be a classical system as suggested by the Bohr-Sommerfeld model. Let the proton be at the center of the coordinate system and the electron at the position $\boldsymbol{r}$. The interaction between the proton and the electron is described by the Coulomb potential $\alpha/|\boldsymbol{r}|$. In addition to this interaction there is a constant electric force acting, that is described by the potential $\boldsymbol{F} \cdot \boldsymbol{r}$. Altogether the motion of the electron is therefore described by the potential

$$U = -\frac{\alpha}{|\boldsymbol{r}|} - \boldsymbol{F} \cdot \boldsymbol{r}$$

a) Sketch the system and the relevant parameters.

b) Which force is acting on the particle? How do its equation of motion look like?

c) Verify that the energy is conserved.

d) Show that also the following quantity is a constant of motion,

$$I = \boldsymbol{F} \cdot (\dot{\boldsymbol{r}} \times \boldsymbol{L}) - \alpha \, \frac{\boldsymbol{F} \cdot \boldsymbol{r}}{|\boldsymbol{r}|} + \frac{1}{2} \, (\boldsymbol{F} \times \boldsymbol{r})^2$$

Here $\boldsymbol{L}$ is the angular momentum of the particle with respect to the origin of the coordinate system.

3.7 *Further reading*

Sommerfeld's (1994) classical discussion of Newton's axioms dates back to the 1940s, but still is a one of the most superb expositions of the topic.

A comprehensive discussion of the flight of water bottle rockets has been given in Finney (2000), and it has been augmented by a discussion of subtle corrections involving the thermodynamic expansion of air in Gommes (2010).

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