Lecture Notes by Jürgen Vollmer

Theoretical Mechanics

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Fakultät für Physik und Geowissenschaften, Universität Leipzig

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Die Philosophie steht in diesem großen Buch geschrieben, dem Universum, das unserem Blick ständig offen liegt. Aber das Buch ist nicht zu verstehen, wenn man nicht zuvor die Sprache erlernt und sich mit den Buchstaben vetraut gemacht hat, in denen es geschrieben ist. Es ist in der Sprache der Mathematik geschrieben, und deren Buchstaben sind Kreise, Dreiecke und andere geometrische Figuren, ohne die es dem Menschen unmöglich ist, ein einziges Wort davon zu verstehen; ohne diese irrt man in einem dunklen Labyrinth herum.

Galileo Galilei, Il Saggiatore, 1623

Die Mathematik ist das Instrument, welches die Vermittlung bewirkt zwischen Theorie und Praxis, zwischen Denken und Beobachten: sie baut die verbindende Brücke und gestaltet sie immer tragfähiger. Daher kommt es, daß unsere ganze gegenwärtige Kultur, soweit sie auf der geistigen Durchdringung und Dienstbarmachung der Natur beruht, ihre Grundlage in der Mathematik findet.

David Hilbert, Ansprache "'Naturerkennen und Logik"' am 8.9.1930 während des Kongresses der Vereinigung deutscher Naturwissenschafter und Mediziner

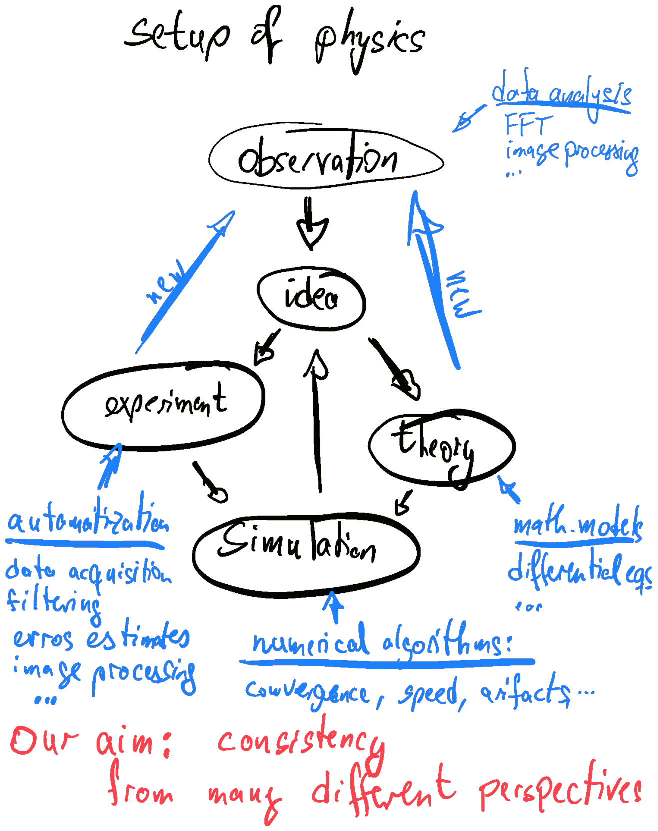
Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit.

Albert Einstein Festvortrag "'Geometrie und Erfahrung"' am 27.1.1921 vor der Preußischen Akademie der Wissenschaften

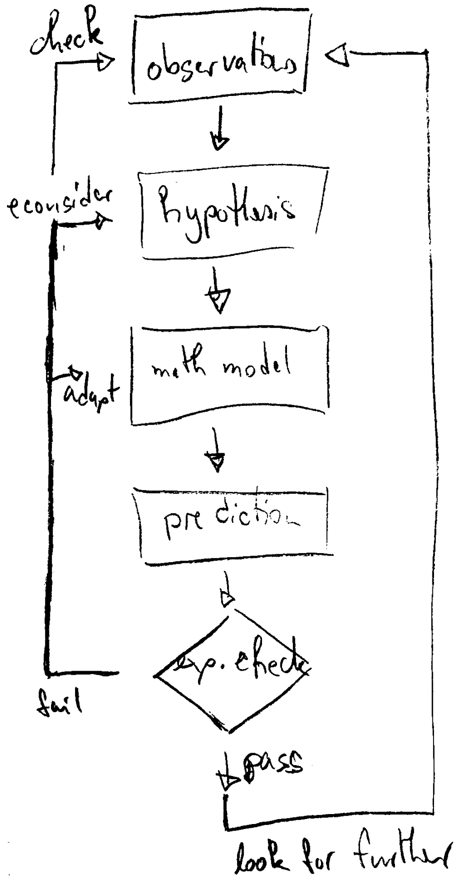
Preface

Die ganzen Zahlen hat der liebe Gott geschaffen,
alles andere ist Menschenwerk.
Leopold Kronecker

Almost 400 years ago Galilei Galileo expressed the credo of modern sciences: The language of mathematics is the appropriate instrument to decode the secrets of the universe. Arguably the fruits of this enterprise are more visible today than they have ever been in the past. Mathematical models are the cornerstone of modern science and engineering. They provide the tools for optimizing engines, and the technology for data and communication sciences. No car will run, no plane will fly, no cell phone ring without the technical equipment and the software to make it run. Moreover, again and again the challenges of physics models inspired the development of new mathematics. Physics and mathematics take complementary perspectives: Mathematicians strive for a logically stringent representation of the structure of theories and models. Physicists adopt mathematics as a tool to speak about and better understand nature:



The present Lecture Notes are developed to accompany courses on "Theoretical Mechanics" for physics freshmen in the international physics program and for students in the teacher education program of the Universität Leipzig. The course addresses mechanics problems to introduce the students to concepts and strategies aiming at a quantitative description of observations.



To meet that aim the lectures strive to meet several purposes:

a) They introduce the concept of a mathematical model, its predictions, and how they relate to observations.

b) They present strategies adopted to develop a model, to explore its predictions, to falsify models, and to refine them based on comparison to observations.

c) They introduce mathematical concepts used in this enterprise: dimensional analysis, non-dimensionalization, complex numbers, vector calculus, and ordinary differential equations.

d) They provide an introduction to Newtonian and Lagrangian Mechanics.

Our approach to mathematical concepts is strongly biased to developing skills to apply mathematical tools in a modeling context, rather than striving for mathematical rigor. For the latter we point out potential pitfalls based on physical examples, and refer students to maths classes.

The material is organized in chapters that address subsequent mathematical and physical topics. Each chapter is introduced by a physics illustration problem. Then, we develop and discuss relevant new concepts. Subsequently, we provide a worked examples. One of them will be the solution of the problem sketched in the introduction. At the end of each section we provide problems:

add reading plan

a. quickies to test conceptual understanding and highlight the new concepts. At times they involve a small twist or highlight pitfalls.

b. exercises to gain practice in employing the concepts.

At the end of chapter further problems are given:

c. more elaborate exercises where the new concepts are used to discuss non-trivial problems.

d. exercises that provide complementary insight based on Python and Sage programs

e. teasers with challenging problems. Typically these exercises require a non-trivial combination of different concepts that have been introduced in earlier chapters.

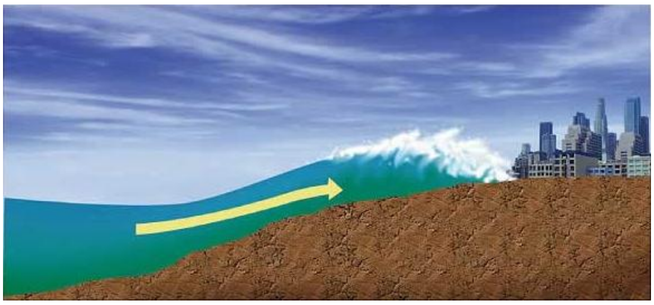
At the end of the chapter we recommend additional literature and provide an outlook for further reading.

I am grateful to Robin Barta, Lennart Buchwald, Fabian Giese, Kolya Lettl, Menna Noufal, Annemarie Wenzel, and Maurice Zeuner for feedback on the notes.

I am eager to receive further feedback. It is crucial for the development of this project to learn about typos, inconsistencies, confusing or incomplete explanations, and suggestions for additional material (contents as well as links to papers, books and internet resources) that should be added in forthcoming revisions. Everybody who is willing to provide feedback will be invited to a coffee in Café Corso.

Contents

1
Basic Principles



-Ilhador-/wikimedia, Public domain

At the end of this chapter we will be able to estimate the speed of a Tsunami wave.

1.1 *Basic notions of mechanics*

Definition 1.1: System

A mechanical *system* is comprised of particles labeled by an index $i \in \mathbb{I}$, that have masses m_i, reside at the positions $\boldsymbol{x}_i$, and move with velocities $\boldsymbol{v}_i$.

Remark 1.1. We say that the system has N particles when $\mathbb{I} = \{1, \ldots, N\}$.

Remark 1.2. The arrows indicate here that $\boldsymbol{x}_i$ describes a position in space. For a D-dimensional space one needs D numbers[1] to specify the position, and $\boldsymbol{x}_i$ may be thought of as a vector in $\mathbb{R}^D$. We say that $\boldsymbol{x}_i$ is a D-vector.

[1] Strictly speaking we do not only need numbers, but must also indicate the adopted units.

Remark 1.3. In order to emphasize the close connection between positions and velocities, the latter will also be denoted as $\dot{\boldsymbol{x}}$.

Example 1.1: A piece of chalk

We wish to follow the trajectory of a piece of chalk through the lecture hall. In order to follow its position and orientation in space, we *decide* to model it as a set of two masses that are localized at the tip and at the tail of the chalk. The positions of these two masses $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ will both be vectors in $\mathbb{R}^3$. For instance we can indicate the shortest distance to three walls that meet in one corner of the lecture hall. In this model we have $N = 2$ and $D = 3$.

Definition 1.2: Degrees of Freedom (DOF)

A system with N particles whose positions are described by D-vectors has $D\,N$ *degrees of freedom (DOF)*.

Remark 1.4. Note that according to this definition the number of DOF is a property of the model. For instance, the model for the piece of chalk has $D\,N = 6$ DOF. However, the length of the piece of chalk does not change. Therefore, one can find an alternative description that will only evolve 5 DOF. (We will come back to this in due time.)

Definition 1.3: State Vector

The position of all particles can be written in a single *state vector*, $\boldsymbol{q}$, that specifies the positions of all particles. Its components are called coordinates.

Remark 1.5. For a system with N particles whose positions are specified by D-dimensional vectors, $\boldsymbol{x}_i = (x_{i,1}, \ldots, x_{i,D})$, the vector $\boldsymbol{q}$ takes the form $\boldsymbol{q} = (x_{1,1}, \ldots, x_{1,D}, x_{2,1}, \ldots, x_{2,D}, \ldots, x_{N,1}, \ldots, x_{N,D})$, which comprises the coordinates $x_{1,1}, \ldots, x_{N,D})$. For conciseness we will also write $\boldsymbol{q} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$. The vector $\boldsymbol{q}$ has DOF number of entries, and hence $\boldsymbol{q} \in \mathbb{R}^{DN}$.

Remark 1.6. The velocity associated to $\boldsymbol{q}$ will be denoted as $\dot{\boldsymbol{q}} = (\dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N)$.

Definition 1.4: Phase Vector

The position and velocities of all particles form the *phase vector*, $\boldsymbol{\Gamma} = (\boldsymbol{q}, \dot{\boldsymbol{q}})$.

Definition 1.5: Trajectory

The *trajectory* of a system is described by specifying the time dependent functions

$$\begin{aligned} & \boldsymbol{x}_i(t), \boldsymbol{v}_i(t), \quad i = 1, \ldots, N \\ \text{or} \quad & \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t) \\ \text{or} \quad & \boldsymbol{\Gamma}(t) \end{aligned}$$

Definition 1.6: Initial Conditions (IC)

For $t \in [t_0, \infty)$ the trajectory is uniquely determined by its *initial conditions (IC)* for the positions $\boldsymbol{x}_i(t_0)$ and velocities $\boldsymbol{v}_i(t_0)$, i.e. the point $\boldsymbol{\Gamma}(t_0)$ in phase space.

Remark 1.7. This definition expresses that the future evolution of a system is *uniquely* determined by its ICs. Such a system is called deterministic. Mechanics addresses the evolution of deterministic systems. At some point in your studies you might encounter stochastic dynamics where different rules apply.

Example 1.2: Throwing a javelin

The ICs for the flight of a javelin specify where it is released, $\boldsymbol{x}_0$, when it is thrown, the velocity $\boldsymbol{v}_0$ at that point of time, and the orientation of the javelin. In a good trial the initial orientation of the javelin is parallel to its initial velocity $\boldsymbol{v}_0$, as shown in Figure 1.1

based on Atalanta, creativecommons, CC BY-SA 3.0
Figure 1.1: Initial conditions for throwing a javelin, cf. Example 1.2.

Remark 1.8. In repeated experiments the ICs will be (slightly) different, and one observes different trajectories.

1. A seasoned soccer player will hit the goal in repeated kicks. However, even a professional may miss occasionally.
2. A bicycle involves a lot of mechanical pieces that work together to provide a predictable riding experience.
3. A lottery machine involves a smaller set of pieces than a bike, but it is constructed such that unnoticeably small differences of initial conditions give rise to noticeably different outcomes. The outcome of the lottery can not be predicted, in spite of best efforts to select identical initial conditions.

Definition 1.7: Constant of Motion

A function of the positions x_i and velocities v_i is called a *constant of motion*, when it does not evolve in time.

Remark 1.9. For a given initial conditions a constant of motion takes the same value for the full trajectory. However, it may take different values for different trajectories, i.e. different choices of initial conditions.

Example 1.3: Length of a piece of chalk

During the flight the positions x_1 and x_2 of the piece of chalk will change. However, the length L of the piece of chalk will not, and at any given time it can be determined from x_1 and x_2. Hence, L is a constant of motion that takes the same value for all trajectories of the piece of chalk.

Example 1.4: Energy conservation for the piece of chalk

We will see that the sum of the potential and the kinetic energy is conserved during the flight of the piece of chalk. This sum, the total energy E, is a constant of motion. The potential energy depends on the position and the kinetic energy is a function of the velocity. Trajectories that start at the same position with different speed will therefore have different total energy. Hence, E is a constant of motion that can take different value for different trajectories of the piece of chalk.

Definition 1.8: Parameter

In addition to the ICs the trajectories will depend on *parameters* of the system. Their values are fixed for a given system.

Example 1.5: A piece of chalk

For the piece of chalk the trajectory will depend on whether the hall is the Theory Lecture Hall in Leipzig, a briefing room in a ship during a heavy storm, or the experimental hall of the ISS space station. To the very least one must specify how the gravitational acceleration acts on the piece of chalk, and how the room moves in space.

Remark 1.10. The set of parameters that appear in a model depends on the *choices* that one makes upon setting up the experiment. For instance

Beckham's banana kicks can only be understood when one accounts for the impact of air friction on the soccer ball.

Air friction will not impact the trajectory of a small piece of talk that I through into the dust bin.

By adopting a clever choice of the parameterization the trajectory of the piece of chalk can be described in a setting with 5 DOF. The length of the piece of chalk will appear as a parameter in that description.

Definition 1.9: Physical Quantities

Positions, velocities and parameters are *physical quantities* that are characterized by at least one number and a unit.

in 3. length or duration?

Example 1.6: Physical Quantities

1. The mass, M, of a soccer ball can be fully characterized by a number and the unit kilogram (kg), e.g. $M \approx 0.4\,\mathrm{kg}$.
2. The length, L, of a piece of chalk can be fully characterized by a number and the unit meter (m), e.g. $L \approx 7 \times 10^{-2}\,\mathrm{m}$.
3. The duration, T, of a year can be characterized by a number and the unit second, e.g. $T \approx \pi \times 10^{7}\,\mathrm{s}$.
4. The speed, v, of a car can be fully characterized by a number and the unit, e.g. $v \approx 42\,\mathrm{km\,h^{-1}}$.
5. A position in a D-dimensional space can fully be characterized by D numbers and the unit meter.
6. The velocity of a piece of chalk flying through the lecture hall can be characterized by three numbers and the unit m/s. However, one is missing information in that case about its rotation.

Remark 1.11. Analyzing the units of the parameters of a system provides a fast way to explore and write down functional dependencies. When doing so, the units of a physical quantity Q are denoted by $[Q]$. For instance for the length L of the piece of chalk, we have $[L] = \mathrm{m}$. For a dimensionless quantity d we write $[d] = 1$.

Example 1.7: Changing units

Suppose we wish to change units from km/h to m/s. A transparent way to do this for the speed of the car in the example above is by multiplications with one

$$v = 72 \frac{\mathrm{km}}{\mathrm{h}} \frac{1\,\mathrm{h}}{3.6 \times 10^{3}\,\mathrm{s}} \frac{1 \times 10^{3}\,\mathrm{m}}{1\,\mathrm{km}} = \frac{72}{3.6} \mathrm{m\,s^{-1}} = 20\,\mathrm{m\,s^{-1}}$$

Definition 1.10: Dynamics

The characterization of all possible trajectories for all admissible ICs is called *dynamics* of a system.

1.1.1 *Self Test*

Problem 1.1. **The degrees of freedom of a frisbee**

a) How would you describe the position of a frisbee in space?

b) How many degrees of freedom does your parameterization involve?

c) Are there constants of motion in your description?

d) Specify at least three parameters required for the description.

Problem 1.2. **Useful numbers and unit conversions**

a) Verify that
   - one nano-century amounts to π seconds,
   - a colloquium talk at our Physics Department must not run take longer than a micro-century,
   - a generous thumb-width amounts to one atto-parsec.

b) The Physics Handbook of Nordling and Österman (2006) defines a beard-second, i. e. the length an average beard grows in one second, as 10 nm. In contrast, Google Calculator uses a value of only 5 nm. I prefer the one where the synodic period of the moon amounts to a beard-inch. Which one will that be?

c) In the furlong–firkin–fortnight (FFF) unit system one furlong per fortnight amounts to the speed of a tardy snail (1 centimeter per minute to a very good approximation), and one micro-fortnight was used as a delay for user input by some old-fashioned computers (it is equal to 1.2096 s). Use this information to determine the length of one furlong.

1.2 *Dimensional analysis*

Mathematics does not know units. Experimental physicists hate large sets of parameters because the sampling of high-dimensional parameter space is tiresome. A remedy to both issues is offered by the Buckingham-Pi-Theorem. We state it here in a form accessible with our present level of mathematical refinement. The discussion of a more advanced formulation may appear as a homework problem later on on this course.

Theorem 1.1: Buckingham-Pi-Theorem

A dynamics with n parameters, where the positions $\boldsymbol{q}$ and the parameters involve the three units meter, seconds and kilogram, can be rewritten in terms of a *dimensionless dynamics* with $n-3$ parameters, where the positions $\boldsymbol{\xi}$, velocities $\boldsymbol{\zeta}$, and parameters π_j with $j \in \{1, \dots, n-1\}$ are given solely by numbers.

Example 1.8: Non-dimensionalization for a pendulum

Let $\boldsymbol{x}$ denote the position of a pendulum of mass M that is attached to a chord of length L and swinging in a gravitational field $\boldsymbol{g}$ of strength g (see Figure 1.2).
The units of these quantities are $[\boldsymbol{x}] = \text{m}$, $[M] = \text{kg}$, $[L] = \text{m}$, and $[g] = \text{m/s}^2$, respectively. The position $\boldsymbol{x}$ describes the position of the system. Its evolution will depend (potentially) on the three parameters, M, L, and g, plus the direction of $\boldsymbol{g}$.
In this problem we choose L as length scale and $\sqrt{L/g}$ as time scale. Then the dimensionless positions will be $\boldsymbol{\xi} = \boldsymbol{x}/L$, the dimensionless velocities will be $\boldsymbol{\zeta} = \dot{\boldsymbol{x}}/\sqrt{g\,L}$.
There is no way to turn M into a dimensionless parameter. Therefore, the evolution of (ξ, ζ) can not depend on M. The only dimensionless parameter that remains in the model is the direction of $\boldsymbol{g}$.

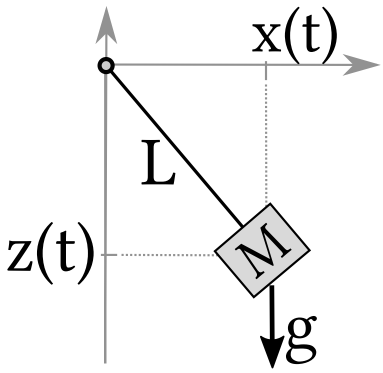


Figure 1.2: Pendulum discussed in Example 1.8

Example 1.9: Non-dimensionalization for the flight of a piece of chalk

Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ denote the position of the tip and the tail of a model for a piece of chalk, where tip and tail are associated to masses m_1 and m_2. The piece of chalk has a length L. It performs a free flight in a gravitational field with acceleration $\boldsymbol{g}$ of strength g.
The units of these quantities are $[\boldsymbol{x}_i] = \text{m}$, $[m_i] = \text{kg}$, $L = \text{m}$, and $[g] = \text{m/s}^2$, respectively. There are four parameters, $n = 4$, plus the direction of $\boldsymbol{g}$.
In this problem we choose L as length scale and $\sqrt{L/g}$ as time scale. Then the dimensionless positions will be $\boldsymbol{\xi}_i = \boldsymbol{x}_i/L$, the dimensionless velocities will be $\boldsymbol{\zeta} = \dot{\boldsymbol{x}}_i/\sqrt{g\,L}$.
The two masses m_1 and m_2 give rise to the dimensionless parameter $\pi_1 = m_1/m_2$, and in three dimensions the direction of $\boldsymbol{g}$ must be characterized by another two dimensionless parameters.

Proof of the Buckingham-Pi-Theorem. We first look for combinations of the parameters with the following units

$$\text{m} = \left[p_1^{\alpha_1}\right] \left[p_2^{\alpha_2}\right] \dots \left[p_n^{\alpha_n}\right]$$

$$\text{s} = \left[p_1^{\beta_1}\right] \left[p_2^{\beta_2}\right] \dots \left[p_n^{\beta_n}\right]$$

$$\text{kg} = \left[p_1^{\gamma_1}\right] \left[p_2^{\gamma_2}\right] \dots \left[p_n^{\gamma_n}\right]$$

Each of these equations involves constraints on the exponents in order to match the exponents of the three units that can be expressed as a system of linear equations. The solvability conditions for such systems imply that they conditions can always be met by an appropriately chosen set of three parameters. Without loss of generality we denote them as p_1, p_2 and p_3, and we have

$$\begin{aligned} \mathrm{m} &= \left[p_1^{\alpha_1}\right] \left[p_2^{\alpha_2}\right] \left[p_3^{\alpha_n}\right] \\ \mathrm{s} &= \left[p_1^{\beta_1}\right] \left[p_2^{\beta_2}\right] \left[p_3^{\beta_n}\right] \\ \mathrm{kg} &= \left[p_1^{\gamma_1}\right] \left[p_2^{\gamma_2}\right] \left[p_3^{\gamma_n}\right] \end{aligned} \tag{1.2.1}$$

Thus we use the parameters $p_1, \dots, p_3$ to remove the units from our description. In its dimensionless form it will involve the positions and velocities

$$\begin{aligned} \xi &= q\, p_1^{-\alpha_1}\, p_2^{-\alpha_2}\, p_3^{-\alpha_n} \\ \zeta &= \dot{q}\, p_1^{\beta_1-\alpha_1}\, p_2^{\beta_2-\alpha_2}\, p_3^{\beta_n-\alpha_n} \end{aligned}$$

Similarly, the dimensionless form of the parameters p_i of the dynamics are obtained by multiplying the original parameters with appropriate powers of the expressions (1.2.1) of the units. For p_1 to p_3 this gives rise to one. Additional parameters will turn into dimensionless groups of parameters that provide π_1 to π_{n-3}. □

expand and provide more examples

1.2.1 Self Test

Problem 1.3. **Oscillation period of a particle attached to a spring**

In a gravitational field with acceleration $g_{\text{Moon}} = 1.6\,\mathrm{m/s^2}$ a particle of mass $M = 100\,\mathrm{g}$ is hanging at a spring with spring constant $k = 1.6\,\mathrm{kg/s^2}$. It oscillates with period T when it is slightly pulled downwards and released. We describe the oscillation by the distance $x(t)$ from its rest position.

a) Determine the dimensionless distance $\xi(t)$, and the associated dimensionless velocity $\zeta(t)$.

b) Provide an order-of-estimate guess of the oscillation period T.

Problem 1.4. **Earth orbit around the sun**

a) Light travels with a speed of $c \approx 3 \times 10^8\,\mathrm{m\,s^{-1}}$, and it takes 500 s to travel from Sun to Earth. What is the Earth-Sun distance D, i. e. one Astronomical Unit (AU) in meters?

b) The period of the trajectory of the Earth around the Sun depends on D, on the mass $M = 2 \times 10^{30}\,\mathrm{kg}$ of the sun, and on the gravitational constant $G = 6.7 \times 10^{-11}\,\mathrm{m^3/kg\,s^2}$. Estimate, based on this information, how long it takes for the Earth to travel once around the sun.

c) Express your estimate in terms of years. The estimate of (b) is of order one, but still off by a considerable factor. Do you recognize the numerical value of this factor?

d) Upon discussing the trajectory $\boldsymbol{x}(t)$ of planets around the sun later on in this course, we will introduce dimensionless positions of the planets $\boldsymbol{\xi}(t) = \boldsymbol{x}(t)/L = (x_1(t)/L, x_2(t)/L, x_3(t)/L)$. How would you define the associated dimensionless velocities?

1.3 *Order-of-magnitude guesses*

Many physical quantities take a value close to one when they are expressed in their "natural" dimensionless units. When the choice is unique, then clearly it is also natural. Otherwise, the appropriate choice is a matter of experience.

We will come back to this when we employ non-dimensionalization in the forthcoming discussion. We demonstrate this based on a discussion of

Example 1.10: The period of a pendulum

We consider a pendulum of mass M attached at a stiff bar of negligible mass. With this bar it is fixed to a pivot at a distance L from the mass such that it can swing in a gravitational field inducing an acceleration g. In this example we make use of the fact that the bar has fixed length L, and describe the position of the mass by the angel $\theta(t)$ (see Figure 1.3).

As discussed in Example 1.8 the dimensionless time unit for this problem is $\sqrt{L/g}$. Hence we estimate that the period T of the pendulum is of the order of $T \simeq \sqrt{L/g}$. Explicit calculations to be performed later on will reveal that this estimate is off by a factor 2π when the amplitude is small, $|\theta(t)| \ll 1$. For large oscillation amplitudes θ_0 the period will increase further, tending to infinity when θ_0 approaches π. Hence, we conclude that

$$T = f(\theta_0)\sqrt{L/g} \qquad \text{with} \;\; f(\theta_0) \simeq 2\pi \;\; \text{for} \;\; \theta_0 \ll 1\,.$$

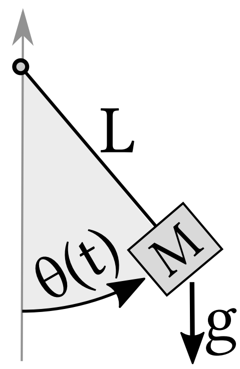


Figure 1.3: Pendulum discussed in Example 1.10

Example 1.11: The speed of Tsunami waves

A Tsunami wave is a water wave that is generated by an earth quake or an underwater land slide. Typical wave lengths are of an order of magnitude $\lambda = 100\,\text{km}$. They travel through the ocean that has an average depth of about $D = 4\,\text{km}$, much smaller than λ. Therefore, we expect that the wave speed v_{Tsunami} is predominantly set by the ocean depth and the gravitational acceleration $g \approx 10\,\text{m/s}^2$, i.e.

$$v_{\text{Tsunami}} \approx \sqrt{gD} = 2 \times 10^2\,\text{m/s} \approx 700\,\text{km/h}$$

This estimate suggests that the 2004 Indian Ocean Tsunami traversed the distance from Indonesia to the East African coast, $L \approx 10\,000\,\text{km}$, in about

$$\frac{L}{v_{\text{Tsunami}}} \approx \frac{1 \times 10^4\,\text{km}}{700\,\text{km/h}} = \frac{100}{7}\text{h} \approx 15\,\text{h}$$

This is very close to the value of 16 h reported in Wikipedia.

Example 1.12: The period of Tsunami waves

In spite of their speed and devastating power, Tsunamis are very hard to detect on the open sea because their period T is very long. It can be estimated as the time that the wave needs to run once through its wavelength[2]

$$T \approx \frac{\lambda}{v_{\text{Tsunami}}} = \frac{\lambda}{\sqrt{gD}} = \frac{100\,\text{km}}{700\,\text{km/h}} = \frac{1}{7\,\text{h}} \approx 10\,\text{min}$$

Here, our estimate is too small by about a factor of three.

[2] Observe that this physical argument goes beyond the blind use of dimensional analysis. The equation for T involves the length scales λ and D in a non-trivial combination that is set by a physical argument.

We conclude that estimates based on dimensional analysis provide valuable insight in time scales of physical processes, even in situations where a detailed mathematical treatment is very delicate.

1.3.1 *Self Test*

Problem 1.5. **Printing the output of Phantom cameras**

With a set of three phantom cameras one can simultaneously follow the motion of 100 particles in a violent 3d turbulent flow. Data analysis of the images provides particle positions with a resolution of 25,000 frames per second. You follow the evolution for 20 minute, print it double paged with 8 coordinates per line and 70 lines per page. A bookbinder makes 12 cm thick books from every 1000 pages. You put these books into bookshelves with seven boards in each shelf. How many meters of bookshelves will you need to store your data on paper?

1.4 *Problems*

Problem 1.6. **Dimensional Analysis of Flight Trajectories**

a) How does the initial velocity v_0 impact the distance W of a thrown object (stone, ball, or shot) or a jump?

b) How does the initial velocity v_0 depend on
the force F acting by the responsible muscle
the accelerated mass M, and
the distance L of the path where the acceleration is performed?

c) Estimate the maximum distance
of throwing a stone of mass $m = 200\,\text{g}$,
of a standing jump for a human and a grass hopper.

d) Make an explicit analysis of standing jumps by exploring how their distance scales with the ratio of characteristic sizes (i.e., body length) of the jumper.

Problem 1.7. **Water waves**

The speed of waves on the ocean depends only on their wave length L and the gravitational acceleration $g \simeq 10\,\text{m/s}^2$.

a) How does the speed of the waves depend on L and g?

b) Unless it is surfing, the speed of a yacht is limited by its hull speed, i.e. the speed of a wave with wave length identical to the length of the yacht. Estimate the top speed of a 30 ft yacht.

c) Close to the beach the water depth H become a more important parameter than the wave length. How does the speed of the crest and the trough of the wave differ? What does this imply about the form of the wave?

Problem 1.8. **Golf on Moon and Earth**

In the end of the Apollo14 mission, on February 6, 1971, astronauts Alan Shepard and Ed Mitchell modified discarded equipment to perform sport on Moon: Mitchell threw a scoop handle as if it were a javelin (see the Apollo14 Lunar surface journal of the NASA). Shepard attached a golf club head to a handle of a sample tool, and hit two golf balls that still reside on Moon. Rumours tell that the golf balls went "miles and miles and miles".

snapshot retrieved from a NASA movie

a) According to Newton's laws of gravity the gravitational acceleration amounts to $G\,M/R^2$, where G is a constant, M the mass of the planet or moon, and R its radius. The Earth radius is four times larger than the one of Moon. Estimate the gravitational acceleration g_M on Moon.
Hint: The acceleration on the Earth surface is $g = 10\,\text{m}\,\text{s}^{-1}$.

b) In contrast to what you have found in a) the gravitational acceleration on Moon is about one sixth of the value on Earth. Use this difference to estimate the difference of the average density of the Moon and of Earth.

c) On Earth a long-distance golf shot can go a few hundred meters. By which factor does this distance increase on Moon?

d) Assume that the shot on Earth can go for 500 m, when one neglects friction due to the Earth atomosphere. Estimate the release velocity of the shot and its time of flight.

e) How long will the golf ball go on Moon, and how long will it fly?

1.4.1 Proofs

1.5 Further reading

The first chapter of Großmann (2012) provides a clear and concise introduction to basic calculus with an emphasis on applications to physics problems.

The introductory chapters of Morin (2014, 2007) provide an excellent introductions to problem solving strategies in physics and dimensional analysis.

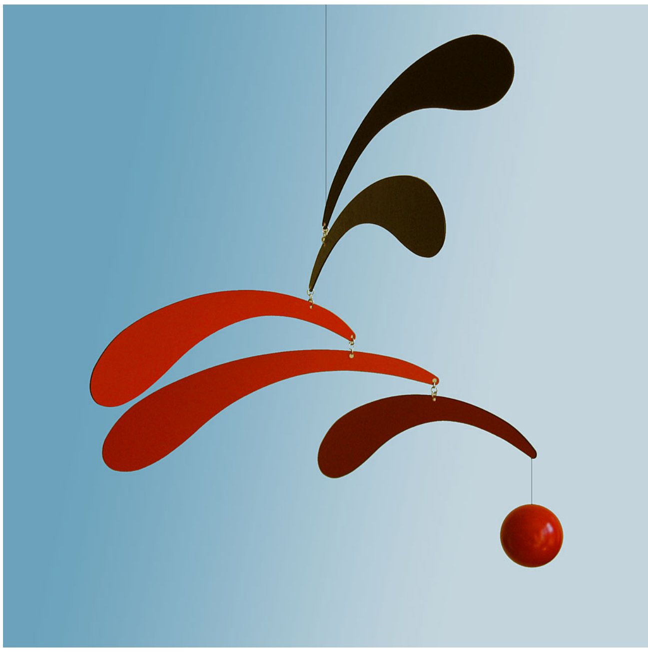
Mahajan (2010) and Zee (2020) provided full-grown text books addressing the art of narrowing down solutions of a broad range of physics and mathematics problems.

Harte (1988) is a classic text that introduces and illustrates modeling strategies for problems derived from environmental science.

2

Balancing Forces and Torques

In Chapter 1 we observed that positions and velocities of particles are specified by indicating their unit, magnitude and directions. Hence, they are vectors. In the present chapter we learn how vectors are defined in mathematics, and how they are used and handled in physics. In order to provide a formal definition we introduce a number of mathematical concepts, like groups, that will be revisited in forthcoming chapters. As first important application we deal with balancing forces and torques.



Mobile (sculpture) in the style of Alexander Calder
Andrew Dunn / wikimedia CC BY-SA 2.0

At the end of this chapter we will be able to determine how a mobile hangs from the ceiling.

2.1 Motivation and outline: forces are vectors

In mechanics we use vectors to describe forces, displacements and velocities. A displacement describes the relative position of two points in space, and the velocity can be thought of as a distance divided by the time needed to go from the initial to the final point. (A mathematically more thorough definition will be given in Chapter 3.) For forces it is of paramount importance to indicate in which direction they are acting. Similarly, in contrast to speed, a velocity can not be specified in terms of a number with a unit, e.g. 5 m/s. By its very definition one also has to specify the direction of motion. Finally, also a displacement involves a length specification and a direction.

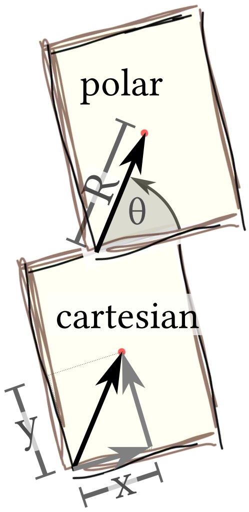


Figure 2.1: The displacement of the red point from the bottom left corner to the the middle of the page can either be specified by the direction θ and the distance R (polar coordinates, top), or by the distances x and y along the sides of the paper (Cartesian coordinates, bottom).

add more explanation

Example 2.1: Displacement of a red dot from the lower left corner to the middle of a paper

This displacement is illustrated in Figure 2.1. It can either be specified in terms of the distance R of the point from the corner and the angle θ of the line connecting the points and the lower edge of the paper (i.e. the direction of the point). Alternatively, it can be given in terms of two distances (x, y) that refer to the length x of a displacement along the edge of the paper and a displacement y in the direction vertical to the edge towards the paper. This can be viewed as result of two subsequent displacements indicated by gray arrows.

In three dimensions, one has to adopt a third direction out of the plane used for the paper, and hence three numbers, to specify a displacements—or indeed any other vector.

	displacement $\boldsymbol{x} = (x_1, x_2, x_3)$	velocity $\boldsymbol{v} = (v_1, v_2, v_3)$	force $\boldsymbol{F} = (f_1, f_2, f_3)$
unit	$[\boldsymbol{x}] = \mathrm{m}$	$[\boldsymbol{v}] = \mathrm{m\,s^{-1}}$	$[\boldsymbol{F}] = \mathrm{kg\ m/s^2}$
magnitude	$\lvert\boldsymbol{x}\rvert = \sqrt{x_1^2 + x_2^2 + x_3^2}$	$\lvert\boldsymbol{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$	$\lvert\boldsymbol{F}\rvert = \sqrt{f_1^2 + f_2^2 + f_3^2}$
direction	$\hat{\boldsymbol{x}} = \boldsymbol{x}/\lvert\boldsymbol{x}\rvert$	$\hat{\boldsymbol{v}} = \boldsymbol{v}/\lvert\boldsymbol{v}\rvert$	$\hat{\boldsymbol{F}} = \boldsymbol{F}/\lvert\boldsymbol{F}\rvert$

A basic introduction of mechanics can be given based on this heuristic account of vectors. However, for the thorough exposition that serve as a foundation of theoretical physics a more profound mathematical understanding of vectors is crucial. Hence, a large part of this chapter will be devoted to mathematical concepts.

Outline

In the first part of this chapter we introduce the mathematical notions of sets and groups that are needed to provide a mathematically sound definition of a vector space. Sets are the most fundamental structure of mathematics. It denotes a collection of elements, e.g., numbers like the digits of our number system $\{1, 2, \ldots, 9\}$ or the set of students in my class. Mathematical structures refer to sets where the elements obey certain additional properties, like in groups and vector spaces. They are expressed in terms of *operations* that take one or several elements of the set, and return a result that may or may not be part of the given set. When an operation f takes an element of a set A and returns another element of A we write $f : A \to A$. When an operation $\circ$ takes two elements of a set A and returns a single element of A we write[1] $\circ : A \times A \to A$. Equipped with the mathematical tool of vectors we will explore the physical concepts of forces and torques, and how they are balanced in systems at rest.

[1] Here $A \times A$ is the set, (a_1, a_2), of all pairs of elements $a_1, a_2 \in A$. Further details will be given in Definition 2.3 below.

2.2 *Sets*

In mathematics and physics we often wish to make statements about a collection of objects, numbers, or other distinct entities.

Definition 2.1: Set

A *set* is a gathering of well-defined, distinct objects of our perception or thoughts.
An object a that is part of a set A is an *element* of A;
we write $a \in A$.
If a set M has a finite number n of elements we say that its *cardinality* is n. We write $|M| = n$.

Remark 2.1. Notations and additional properties:

a) When a set M has a finite number of elements, e.g., $+1$ and -1, one can specify the elements by explicitly stating the elements, $M = \{+1, -1\}$. In which order they are states does not play a role, and it also does not make a difference when elements are provided several times. In other words the set M of cardinality two can be specified by any of the following statements

$$M = \{-1, +1\} = \{+1, -1\} = \{-1, 1, 1, 1, \} = \{-1, 1, +1, -1\}$$

b) If e is not an element of a set M, we write $e \notin M$. For instance $-1 \in M$ and $2 \notin M$.

c) There is exactly one set with no elements, i. e. with cardinality zero. It is denoted as empty set, $\emptyset$.

Example 2.2: Sets

- Set of capitals of German states:

  A_C = {Berlin, Bremen, Hamburg, Stuttgart, Mainz, Wiesbaden, München, Magdeburg, Saarbrücken, Potsdam, Kiel, Hannover, Dresden, Schwerin, Düsseldorf, Erfurt}

- Set of small letters in German:

  A_L = {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z, ä, ö, ü, ß}

- Set of month with 28 days:[2]

  A_M = {January, February, March, April, May, June, July, August, September, October, November, December}

The cardinalities of these sets are
$$|A_C| = 16,\ |A_L = 30|, \text{ and } |A_M| = 12.$$

[2] Most of them have even more days.

Example 2.3: Sets of sets

A set can be an element of a set. For instance the set

$$M = \{1, 3, \{1, 2\}\}$$

has three elements 1, 3 and $\{1, 2\}$ such that $|M| = 3$, and

$$1 \in M\,, \qquad \{1, 2\} \in M\,, \qquad 2 \notin M \qquad \{1\} \notin M\,.$$

Often it is bulky to list all elements of a set. In obvious cases we use ellipses such as A_L = {a, b, c, . . . , z, ä, ö, ü, ß} for the set given in Example 2.2. Alternatively, one can provide a set M by specifying the properties $A(x)$ of its elements x in the following form

$$\underbrace{M}_{\text{The set } M} \underbrace{=}_{\text{contains}} \underbrace{\{}_{\text{all elements}} \underbrace{x}_{x,} \underbrace{:}_{\text{with :}} \underbrace{A(x)}_{\text{properties} \ldots} \}.$$

where the properties specify one of several properties of the elements. The properties are separated by commas, and must all be true for all elements of the set.

Example 2.4: Set definition by property

The set of digits $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ can also be defined as follows $D = \{1, \ldots, 9\} = \{x : 0 < x \leq 9, x \in \mathbb{Z}\}$. In the latter definition $\mathbb{Z}$ denotes the set of all integer numbers.

In order to specify the properties in a compact form we use logical junctors as short hand notation. In the present course we adopt the notations not $\neg$, and $\wedge$, or $\vee$, implies $\Rightarrow$, and is equivalent $\Leftrightarrow$ for the relations indicated in 2.1.

The definition of the digits in Example 2.4 entails that all elements of D are also numbers in $\mathbb{Z}$: we say that D is a subset of $\mathbb{Z}$.

Definition 2.2: Subset and Superset

The set M_1 is a *subset* of M_2, if all elements of M_1 are also contained in M_2. We write[3] $M_1 \subseteq M_2$. We denote M_2 then as *superset* of M_1, writing $M_2 \supseteq M_1$.

[3] Some authors use $\subset$ instead of $\subseteq$, and $\subsetneq$ to denote proper subsets.

Table 2.1: List of the results of different junctors acting on two statements A and B. Here 0 and 1 indicate that a statement is wrong or right, respectively. In the rightmost column we state the contents of the expression in the left column in words. The final three lines provide examples of more complicated expressions.

A	0	0	1	1	
B	0	1	0	1	
$\neg A$	1	1	0	0	not A
$\neg B$	1	0	1	0	not B
$A \vee B$	0	1	1	1	A or B
$A \wedge B$	0	0	0	1	A and B
$A \Rightarrow B$	1	1	0	1	A implies B
$A \Leftrightarrow B$	1	0	0	1	A is equivalent to B
$A \vee \neg B$	1	0	1	1	A or not B
$\neg A \wedge B$	0	1	0	0	not A or B
$A \wedge \neg B$	0	0	1	0	A and not B

The set M_1 is a *proper subset* of M_2 when at least one of the elements of M_2 is not contained in M_1. In this case $|M_1| < |M_2|$, and we write $M_1 \subset M_2$ or $M_2 \supset M_1$.

Example 2.5: Subsets

- The set of month with names that end with "ber" is a subset of the set A_M of Example 2.2

$$\{\text{September, October, November, December}\} \subseteq A_M$$

- For the set M of Example 2.3 one has

$$\{1\} \subseteq M\,, \quad \{1,3\} \subseteq M\,, \quad \{1,2\} \nsubseteq M\,, \quad \{2,\{1,2\}\} \nsubseteq M\,.$$

  Note that $\{1, 2\}$ is an element of M. However, it is not a subset. The last two sets are no subsets because $2 \notin M$.

Two sets are the same when they are subsets of each other.

Theorem 2.1: Equivalence of Sets

Two sets A and B are *equal* or *equivalent*, iff

$$(A \subseteq B) \;\wedge\; (B \subseteq A)\,.$$

Remark 2.2 (iff). In mathematics "iff" indicates that something holds "if and only if". Observe its use in the following two statements: A number is an even number if it is the product of two even numbers. A number is an even number iff it is the product of an even number and another number.

Remark 2.3 (precedence of operations in logical expressions.). In logical expressions we first evaluate $\in$, $\notin$ and other set operations that are used to build logical expressions. Then we evaluate the junctor $\neg$ that is acting on a a single logical expression. Finally the other junctors $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are evaluated. Hence, the brackets are not required in Theorem 2.1.

Proof of Theorem 2.1.
$A \subseteq B$ implies that $a \in A \Rightarrow a \in B$.
$B \subseteq A$ implies $b \in B \Rightarrow b \in A$.
If $A \subseteq B$ and $B \subseteq A$, then we also have $a \in A \Leftrightarrow a \in B$. □

The description of sets by properties of its members, Example 2.4, suggests that one will often be interested in operations on sets. For instance the odd and even numbers are subsets of the natural numbers. Together they form this set, and one is left with the even numbers when removing the odd numbers from the natural numbers. Hence, we define the following operations on sets.

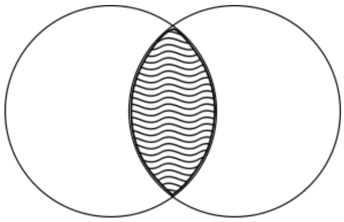


Figure 2.2: Intersection of two sets.

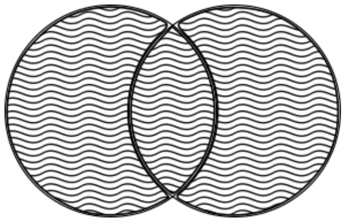


Figure 2.3: Union of two sets.

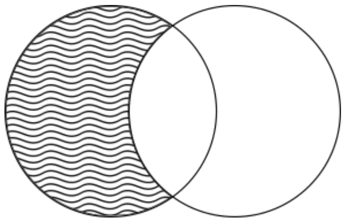


Figure 2.4: Difference of two sets.

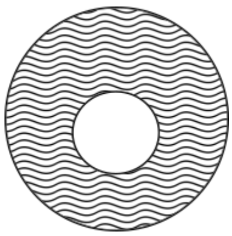


Figure 2.5: Complement of a set.

Definition 2.3: Set Operations

For two sets M_1 and M_2 we define the following operations:

- *Intersection*: $M_1 \bigcap M_2 = \{m \,:\, m \in M_1 \wedge m \in M_2\}$,
- *Union*: $M_1 \bigcup M_2 = \{m \,:\, m \in M_1 \vee m \in M_2\}$,
- *Difference*: $M_1 \backslash M_2 = \{m \,:\, m \in M_1 \wedge m \notin M_2\}$,
- The *complement* of a set M in a *universe* U is defined for subsets $M \subseteq U$ as $M^C = \{m \in U \,:\, m \notin M\} = U \backslash M$.
- The *Cartesian product of two sets* M_1 *and* M_2 is defined as the set of ordered pairs (a, b) of elements $a \in M_1$ and $b \in M_2$: $M_1 \times M_2 = \{(a, b) \,:\, a \in M_1, b \in M_2\}$.

A graphical illustration of the operations is provided in Figures 2.2 to 2.5.

Example 2.6: Set operations: participants in my class

Consider the set of participants P in my class. The sets of female F and male M participants of the class are proper subsets of P with an empty intersection $F \bigcap M$. The set of non-female participants is $P \backslash F$. The set of heterosexual couples in the class is a subset of the Cartesian product $F \times M$. Furthermore, the union $F \bigcup M$ is a proper subset of P, when there is a participant who is neither female nor male.

Definition 2.4: Logical quantors

A logical statements S about elements a of a set A may hold

- for all elements of a set — we write: $\forall a \in A \,:\, S$
- for some elements of a set — we write: $\exists a \in A \,:\, \mathsf{S}$
- for exactly one elements of a set — we write: $\exists! a \in A \,:\, \mathsf{S}$

name	symbol	description
natural numbers	$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
natural numbers with 0	$\mathbb{N}_0$	$\mathbb{N} \bigcup \{0\}$
negative numbers	$-\mathbb{N}$	$\{-n \,:\, n \in \mathbb{N}\}$
even numbers	$2\mathbb{N}$	$\{2n \,:\, n \in \mathbb{N}\}$
odd numbers	$2\,\mathbb{N} - 1$	$\{2n - 1 \,:\, n \in \mathbb{N}\}$
integer numbers	$\mathbb{Z}$	$(-\mathbb{N}) \bigcup \mathbb{N}_0$
rational numbers	$\mathbb{Q}$	$\left\{\frac{p}{q} \,:\, p \in \mathbb{Z}, q \in \mathbb{N}\right\}$
real numbers	$\mathbb{R}$	see below
complex numbers	$\mathbb{C}$	$\mathbb{R} + \mathrm{i}\mathbb{R}$, where $\mathrm{i} = \sqrt{-1}$

Table 2.2: Summary of important sets of numbers.

Example 2.7: Logical quantors and properties of set elements

Let $|m|$ denote the number of days in a month $a \in A_M$ (cf. Example 2.2). Then the following statements are true: There is exactly one month that has exactly 28 days:

$$\exists! a \in A_M \,:\, |a| = 28$$

Some months have exactly 30 days:

$$\exists a \in A_M \,:\, |a| = 30$$

All month have at least 28 days:

$$\forall a \in A_M \,:\, |a| \geq 28$$

2.2.1 *Sets of Numbers*

Many sets of numbers that are of interest in physics have infinitely many elements. We construct them in Table 2.2 based on the natural numbers

$\mathbb{N}$: check ISO norm

$\mathbb{N}$: remark on Neumann construction?

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

or the natural numbers with zero

$$\mathbb{N}_0 = \mathbb{N} \bigcup \{0\} \,.$$

Remark 2.4. Some authors adopt the convention that zero is included in the natural numbers $\mathbb{N}$. When this matters you have to check which convention is adopted.

There are many more sets of numbers. For instance, in mathematics the set of constructable numbers is relevant for certain proofs in geometry, and in physics and computer graphics quaternions are handy when it comes to problems involving three-dimensional rotations. In any case one needs intervals of numbers.

Definition 2.5: Interval of Real Numbers $\mathbb{R}$

An *interval* is a continuous subset of a set of numbers. We distinguish *open, closed, and half-open subsets*.

- closed interval: $[a, b] = \{x : x \geq a, x \leq b\}$,
- open interval: $(a, b) =]a, b[= \{x : x > a, x < b\}$,
- right open interval: $[a, b) = [a, b[= \{x : x \geq a, x < b\}$,
- left open interval: $(a, b] =]a, b] = \{x : x > a, x \leq b\}$.

Subsets of $\mathbb{R}$ will be denoted as real intervals.

add limits, closure, and $\mathbb{R}$ as closure of $\mathbb{Q}$.

2.2.2 *Self Test*

Problem 2.1. **Relations between sets**

Let A, B, C, and D be pairwise distinct elements. Select one of the symbols

$$\in, \quad \notin, \quad \ni, \quad \not\ni, \quad \subset, \quad \not\subset, \quad \supset, \quad \not\supset, \quad =$$

and avoid $\notin$, $\not\ni$, $\not\subset$, $\not\supset$ wherever possible.

a) $\{A, B\} \ \square \ \{A, B, C\}$,

b) $\{A\} \ \square \ B$,

c) $\{\emptyset\} \ \square \ \emptyset$,

d) $\{\{A\}\} \ \square \ \{\{A\}, \{B\}\}$,

e) $A \ \square \ \{A, B, C\}$,

f) $\{A, C, D\} \cap \{A, B\} \ \square \ \{A, B, C, D\}$,

g) $\{A, C, D\} \setminus \{A, B\} \ \square \ \{A, B, C\}$,

h) $\{A, C, D\} \cup \{A, B\} \ \square \ A$.

Problem 2.2. **Intervals**

a) Provide $[1; 17] \cap]0; 5[$ as a single interval.

b) Provide $[-1, 4] \setminus [1, 2[$ as union of two intervals.

Problem 2.3. **Sets of numbers**

Which of the following statements are true?

a) $\{6 \cdot z | z \in \mathbb{Z}\} \subset \{2 \cdot z | z \in \mathbb{Z}\}$.

b) $\{2 \cdot z | z \in \mathbb{Z}\} \cap \{3 \cdot z | z \in \mathbb{Z}\} = \{6 \cdot z | z \in \mathbb{Z}\}$.

c) Let $T(a)$ be the set of numbers that divide a. Then

$$\forall a, b \in \mathbb{N} : \quad T(a) \cup T(b) = T(a \cdot b)$$

Example: $T(2) = \{1, 2\}$, $T(3) = \{1, 3\}$, and $T(6) = \{1, 2, 3, 6\}$.

2.3 Groups

A group G refers to a set of operations $t \in G$ that are changing some data or objects. Elementary examples refer to reflections in space, turning some sides of a Rubik's cube, or translations in space, as illustrated in Figure 2.1. The subsequent action of two group elements t_1 and t_2 of G is another (typically more complicated) transformation $t_3 \in G$. Analogous to the concatenation of functions, we write $t_3 = t_2 \circ t_1$, and we say t_3 is t_2 after t_1. The set of transformations forms a group iff it obeys the following rules.

Definition 2.6: Group

A set $(G, \circ)$ is called a *group* with operation $\circ : G \times G \to G$ when the following rules apply

a) The set is *closed*: $\forall g_1, g_2 \in G : g_1 \circ g_2 \in G.$

b) The set has a *neutral element*: $\exists e \in G\ \forall g \in G : e \circ g = g.$

c) Each element has an *inverse element*: $\forall g \in G\ \exists i \in G : g \circ i = e.$

d) The operation $\circ$ is *associative*: $\forall g_1, g_2, g_3 \in G : (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3).$

Wüstholz rather suggests: $\forall e \in G : e \circ f = f \Rightarrow \forall g \in G\ \exists h \in G : h \circ g = e$

Definition 2.7: Commutative Group

A group $(G, \circ)$ is called a *commutative group* when

e) the group operation is *commutative*: $\forall g_1, g_2 \in G : g_1 \circ g_2 = g_2 \circ g_1.$

Remark 2.5. Commutative groups are also denoted as *Abelian groups*. ◫

When the group has a finite number of elements the result of the group operation can explicitly be specified by a group table. We demonstrate this by the smallest groups. The empty set can not be a group because it has no neutral element. Therefore the smallest groups have a single element and two elements. Both of these groups are commutative.

Example 2.8: Smallest groups

$(\{n\}, \odot)$ comprises only the neutral element.

$\odot$	n
n	n

The smallest non-trivial group has two elements $(\{0, 1\}, \oplus)$ with

$\oplus$	0	1
0	0	1
1	1	0

It describes the turning of a piece of paper:
Not turning, 0, does not change anything (neutral element).
Turning, 1, shows the other side, and turning twice is equivalent to not turning at all (1 is its own inverse).

Remark 2.6. The group properties imply that all elements of the group must appear exactly once in each row and each column of the group table. As a consequence the smallest non-commutative group is the dihedral group of order 6 with six elements that is discussed in Problem 2.7.

Figure 2.6: Rotation of a book by multiples of $\pi/2$ around three orthogonal axes.

watch out: there is a problem with inverse & neutral elts!

Example 2.9: Non-commutative groups: rotations

The rotation of an object in space is a group. In particular this holds for the 90°-rotations of an object around a vertical and a horizontal axis. Figure 2.6 illustrates that these rotations do not commute.

Example 2.10: Non-commutative groups: edit text fields

We consider the text fields of a fixed length n in an electronic form. Then the operations
"Put the letter L into position $\sqcup$ of the field"
with $L \in \{\_, a, \ldots, z, A, \ldots, Z\}$
and $\sqcup \in \{1, \ldots, n\}$ form a group.
Also in this case one can easily check that the order of the operations is relevant. In the left and right column the same operations are preformed for a text field of length $n = 4$:

$$\begin{array}{rr}
|_|_|_|_| & |_|_|_|_| \\
\to |M|_|_|_| & \to |P|_|_|_| \\
\to |M|a|_|_| & \to |P|h|_|_| \\
\to |M|a|t|_| & \to |P|h|y|_| \\
\to |M|a|t|h| & \to |P|h|y|s| \\
\to |M|a|t|s| & \to |P|h|y|h| \\
\to |M|a|y|s| & \to |P|h|t|h| \\
\to |M|h|y|s| & \to |P|a|t|h| \\
\to |P|h|y|s| & \to |M|a|t|h|
\end{array}$$

Remark 2.7. Notations and additional properties:

a) Depending of the context the inverse element is denoted as g^{-1} or as $-g$. This depends on whether the operation is considered a multiplication or rather an addition. In accordance with this choice the neutral element is denoted as 1 or 0.

b) The second property of groups, b) $\exists e \in G \; \forall g \in G : e \circ g = g$, implies that also $g \circ e = g$. The proof is provided as Problem 2.8.

c) When a group is not commutative then one must distinguish the left and right inverse. The condition $g \circ i = e$ does not imply $i \circ g = e$. However, there always is another element $j \in G$ such that $j \circ g = e$. An example is provided in Problem 2.7.

2.3.1 *Self Test*

Problem 2.4. **Checking group axioms**

Which of the following sets are groups?

a) $(\mathbb{N}, +)$
b) $(\mathbb{Z}, +)$
c) $(\mathbb{Z}, \cdot)$
d) $(\{+1, -1\}, \cdot)$
e) $(\{0\}, +)$
f) $(\{1, \ldots, 12\}, \oplus)$

where $\oplus$ in f) revers to adding as we do it on a clock, e.g. $10 \oplus 4 = 2$.

Problem 2.5. **The group with three elements**

Let $\mathcal{G}$ be a group with three elements $\{n, l, r\}$, where n is the neutral element.

a) Show that there only is a single choice for the result of the group operations $a \circ b$ with $a, b \in \mathcal{G}$. Provide the group table.

b) Verify that the group describes the rotations of an equilateral triangle that interchange the positions of the angles.

c) Show that there is a bijective map $\mathrm{m} : \{n, l, r\} \to \{0, 1, 2\}$ with the following property:

$$\forall a, b \in \mathcal{G} \; : \; a \circ b = \big(\mathrm{m}(a) + \mathrm{m}(b)\big) \mathrm{mod} 3 \, .$$

We say that the group $\mathcal{G}$ is isomorphic to the natural numbers with addition addition modulo 3.[4]

[4] The natural number modulo n amount to n classes that represent the remainder of the numbers after division by n. For instance, for the natural numbers modulo two the 0 represents even numbers, and the 1 odd numbers. Similarly, for the natural numbers modulo three the 0 represents numbers that are divisible by three, and for the sum of 2 and 2 modulo 3 one obtains $(2+2) \mathrm{mod} 3 = 4 \mathrm{mod} 3 = 1$.

Problem 2.6. **Symmetry group of rectangles**

A polygon has a symmetry with an associated symmetry operation a when a only interchanges the vertices of the polygon. It does not alter the position. To get a grip on this concept we consider the symmetry operations of a rectangle.

a) Sketch how reflections with respect to a symmetry axis interchange the vertices of a rectangle. What happens when the reflections are repeatedly applied?

b) Show that the symmetry operations form a group with four elements. Provide a geometric interpretation for all group elements.

c) Provide the group table.

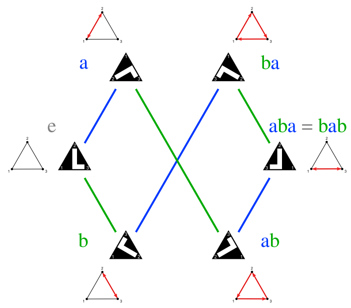


Figure 2.7: Reflections of equilateral triangle with respect to the three symmetry axes form a group with six elements; see Problem 2.7.

Problem 2.7. **Dihedral group of order** 6

Figure 2.7 illustrates the effect of reflections of a triangle with respect to its three symmetry axis. All group elements can be generated by repeated action of two reflections, e.g. those denoted as a and b in the figure.

a) Verify that the group properties, Definition 2.6, together with the three additional requirements

$$a \circ a = b \circ b = e \quad \text{and} \quad a \circ b \circ a = b \circ a \circ b$$

imply that the group has exactly six elements,

$$\mathcal{G} = \{e, a, b, a \circ b, b \circ a, a \circ b \circ a\} .$$

b) Work out the group table.

c) Verify by inspection that e is the neutral element for operation from the right *and* from the left.

d) Verify that the group is *not* commutative, and provide an example of a group element where the left inverse and the right inverse differ.

e) The group can also be represented in terms of a reflection and the rotations described in Problem 2.5. How would the graphical representation, analogous to Figure 2.7, look like in that case.

Problem 2.8. **Uniqueness of the neutral element**

Proof that the group axioms, Definition 2.6, imply that $e \circ g = g$ implies that also $g \circ e = g$.

2.4 *Fields*

Besides being of importance to characterize the action of symmetry operations like reflections or rotations, groups are also important for us because they admit further characterization of sets of numbers.

The natural numbers are not a group. For the addition they are lacking the neutral elements, and for adding and multiplications they are lacking inverse elements.

In contrast the group $(\mathbb{Z}, +)$ is a commutative group with infinitely many elements.

Example 2.11: The group $(\mathbb{Z}, +)$

The numbers $\mathbb{Z}$ with operation $+$ form a group. This is demonstrated here by checking the group axioms.

a) Addition of any two numbers provides a number:

$\forall x, y \in \mathbb{Z} \,:\, (x+y) \in \mathbb{Z}.$

b) The neutral element of the addition is 0:

$\exists\, 0 \in \mathbb{Z}\; \forall z \in \mathbb{Z} \,:\, z+0 = z = 0+z.$

c) For every element $z \in \mathbb{Z}$ there is an inverse $(-z) \in \mathbb{Z}$:

$\forall z \in \mathbb{Z}\; \exists (-z) \in \mathbb{Z} \,:\, z + (-z) = 0 = (-z) + z.$

d) The addition of numbers is associative:

$\forall z_1, z_2, z_3 \in \mathbb{Z} \,:\, z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$

However, the numbers $\mathbb{Z}$ still lack inverse elements of the multiplication. The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are commutative groups for addition and multiplication (with the special rule that multiplication with 0 has no inverse element), and their elements also obey distributivity. Such sets are called number fields.

Definition 2.8: Field

A set $(\mathbb{F}, +, \cdot)$ is called a *field* with neutral elements 0 and 1 for addition $+$ and multiplication $\cdot$, respectively, when its elements comply with the following rules

a) $(\mathbb{F}, +)$ is a commutative group,

b) $(\mathbb{F}\backslash\{0\}, \cdot)$ is a commutative group,

c) Addition and Multiplication are distributive:

$$\forall a, b, c \in \mathbb{F} : a \cdot (b + c) = a \cdot b + a \cdot c$$

Remark 2.8. For the multiplication of field elements one commonly suppresses the $\cdot$ for the multiplication, writing e.g. $a\,b$ rather than $a \cdot b$.

Example 2.12: The smallest field has two elements

The smallest field $(\{0,1\}, \oplus, \odot)$ comprises only the neutral elements 0 of the group $(\{0,1\}, \oplus)$ with two elements, and 1 of the group $(\{1\}, \odot)$ with one element.

Example 2.13: Complex numbers are a field

a) The sum of two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ amounts to

$$z_1 + z_2 = (x_1 + \mathrm{i}y_1) + (x_2 + \mathrm{i}y_2) = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$$

Hence, the group properties for $+$ follow from the properties of the real numbers x_1, x_2 and y_1, y_2, respectively.
b) They also entail distributivity of complex numbers.
c) The product of the complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and

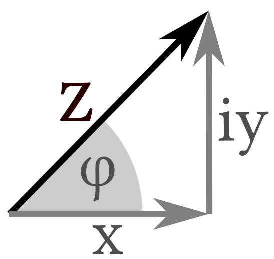


Figure 2.8: Complex numbers z can be represented as $z = x + \mathrm{i}y$ in a plane where (x, y) are the Cartesian coordinates of z. Alternatively, one can adopt a representation in terms of polar coordinates $z = R\,\mathrm{e}^{\mathrm{i}\varphi}$ where $R = \sqrt{x^2 + y^2}$ and φ is the angle with respect to the x-axis.

$z_2 = x_2 + \mathrm{i}y_2$ amounts to

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + \mathrm{i}y_1) \cdot (x_2 + \mathrm{i}y_2) \\ &= (x_1\, x_2 + \mathrm{i}y_1\, x_2 + \mathrm{i}y_1\, x_2 + \mathrm{i}^2\, y_1\, y_2) \\ &= (x_1\, x_2 - y_1\, y_2) + \mathrm{i}\,(y_1\, x_2 + x_1\, y_2) \end{aligned}$$

Checking the group axioms based on this representation of the complex numbers is tedious. One better adopts a representation in terms of polar coordinates, $z_1 = R_1\,\mathrm{e}^{\mathrm{i}\varphi_1}$ and $z_2 = R_2\,\mathrm{e}^{\mathrm{i}\varphi_2}$ (see Figure 2.8) where (cf Problem 2.10)

$$z_1 \cdot z_2 = R_1\,\mathrm{e}^{\mathrm{i}\varphi_1} \cdot R_2\,\mathrm{e}^{\mathrm{i}\varphi_2} = \big(R_1\, R_2\big)\,\mathrm{e}^{\mathrm{i}\,(\varphi_1 + \varphi_2)}$$

Here, the group properties follow from those of multiplying R_1 and R_2, and adding φ_1 and φ_2.

Remark 2.9 (complex conjugation). Each complex numbers z has a complex conjugate, denoted as z^* or $\bar{z}$, that is defined as

$$\forall z = x + \mathrm{i}y = R\,\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C}: \quad \bar{z} = x - \mathrm{i}y = R\,\mathrm{e}^{-\mathrm{i}\varphi} \tag{2.4.1}$$

Complex conjugation provides an effective way to calculate the absolute value $|z| = R$ of complex numbers

$$\begin{aligned} & z\,\bar{z} = (x + \mathrm{i}y)\,(x - \mathrm{i}y) = x^2 - \mathrm{i}^2\,y^2 = x^2 + y^2 = R^2 \\ \text{and} \quad & z\,\bar{z} = R\,\mathrm{e}^{\mathrm{i}\varphi}\, R\,\mathrm{e}^{-\mathrm{i}\varphi} = R^2\,\mathrm{e}^0 = R^2 \\ \Rightarrow \quad & |z| = \sqrt{z\,\bar{z}} = \sqrt{\bar{z}\,z} \end{aligned} \tag{2.4.2}$$

Remark 2.10. In physics complex numbers are commonly applied to describe rotations in a plane: Multiplication by $\mathrm{e}^{\mathrm{i}\theta}$ rotates a complex number z by an angle θ around the origin:

$$\forall z = R\,\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C}: \quad z \cdot \mathrm{e}^{\mathrm{i}\theta} = R\,\mathrm{e}^{\mathrm{i}(\varphi+\theta)} \tag{2.4.3}$$

2.4.1 *Self Test*

Problem 2.9. **Checking field axioms**

Which of the following sets are fields?

a) $(\mathbb{Z}, +, \cdot)$

b) $(\{1, 2, \ldots, 12\}, +\text{mod}12, \cdot\text{mod}12)$
like on a clock: $11 \oplus 2 = 13\text{mod}12 = 1$ and $4 \odot 5 = 20\text{mod}12 = 8$.

c) $(\{0, 1, 2\}, +\text{mod}3, \cdot\text{mod}3)$
for instance $2 \odot 2 = 2 + 2 = 4\text{mod}3 = 1$ and $2 \oplus 1 = 3\text{mod}3 = 0$.

Problem 2.10. **Euler's equation and trigonometric relations**

Euler's equation $e^{ix} = \cos x + i \sin x$ relates complex values exponential functions and trigonometric functions.

a) Sketch the position of $R\, e^{ix}$ in the complex plane, and indicate how Euler's equation is related to the Theorem of Pythagoras.

b) Complex valued exponential functions obey the same rules as their real-valued cousins. In particular, for $R = 1$ one has $e^{i\,(x+y)} = e^{ix}\, e^{iy}$. Compare the real and complex parts of the expressions on both sides of this relation. What does this imply about $\sin(2x)$ and $\cos(2x)$?

2.5 *Vector spaces*

With the notions introduced in the preceding sections we can give now the formal definition of a vector space

Definition 2.9: Vector Space

A *vector space* $(\mathsf{V}, \mathbb{F}, \oplus, \odot)$ is a set of *vectors* $\boldsymbol{v} \in \mathsf{V}$ over a field $(\mathbb{F}, +, \cdot)$ with binary operations $\oplus : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ and $\odot : \mathbb{F} \times \mathsf{V} \to \mathsf{V}$ complying with the following rules

a) $(\mathsf{V}, \oplus)$ is a commutative group

b) associativity: $\forall a, b \in \mathbb{F}\ \forall \boldsymbol{v} \in \mathsf{V} : a \odot (b \odot \boldsymbol{v}) = (a \cdot b) \odot \boldsymbol{v}$

c) distributivity 1:
$$\forall a, b \in \mathbb{F}\ \forall \boldsymbol{v} \in \mathsf{V} : (a + b) \odot \boldsymbol{v} = (a \odot \boldsymbol{v}) \oplus (b \odot \boldsymbol{v})$$

d) distributivity 2:
$$\forall a \in \mathbb{F}\ \forall \boldsymbol{v}, \boldsymbol{w} \in \mathsf{V} : a \odot (\boldsymbol{v} \oplus \boldsymbol{w}) = (a \odot \boldsymbol{v}) \oplus (a \odot \boldsymbol{w})$$

Remark 2.11. It is common to use $+$ and $\cdot$ instead of $\oplus$ and $\odot$, respectively, with the understanding that it is clear from the context in the equation whether the symbols refer to operations involving vectors, only numbers, or a number and a vector.

Moreover, as for the multiplication of numbers, one commonly drops the $\odot$ for the multiplication, writing e.g. $a\,\boldsymbol{v}$ rather than $a \cdot \boldsymbol{v}$.

Example 2.14: Vector spaces: displacements in the plane

For displacements we define the operation $\oplus$ as concatenation of displacements, and $\odot$ as increasing the length of the displacement by a given factor without touching the direction.

a) The neutral element amounts to staying, one can always shift back, move between any two points in a plane, and commutativity follows form the properties of parallelo-

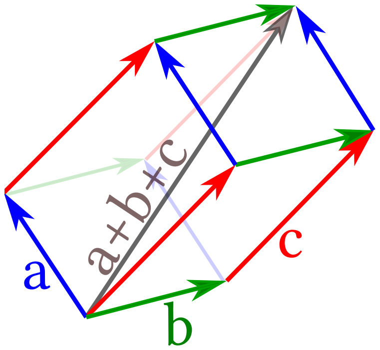


Figure 2.9: The arrows indicate displacements by three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, as discussed in Example 2.14. Their commutativity and associativity follow from the properties of parallelograms. This holds in the plane, and also when the vectors span a three-dimensional volume.

grams, see Figure 2.9.
b,c) The vectors select the direction. Scalar multiplication only changes the length of the vectors, and the length is a real number.
d) Is implied by the Intercept Theorem.

Example 2.15: Vector spaces: $\mathbb{R}^D$

For every $D \in \mathbb{N}$ the D-fold Cartesian product $\mathbb{R}^D$ of the real numbers is a vector space over $\mathbb{R}$ when defining the operation $+$ and $\cdot$ as

$$\forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^D : \boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_D \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_D \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_D + b_D \end{pmatrix}$$

$$\forall s \in \mathbb{R}\ \forall \boldsymbol{a} \in \mathbb{R}^D : s \cdot \boldsymbol{a} = s \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_D \end{pmatrix} = \begin{pmatrix} s\, a_1 \\ s\, a_2 \\ \vdots \\ s\, a_D \end{pmatrix}$$

In a more compact manner this is also written as,

$$\forall \boldsymbol{a} = (a_i), \boldsymbol{b} = (b_i), s \in \mathbb{R} : \boldsymbol{a} + \boldsymbol{b} = (a_i + b_i) \ \wedge \ s\, \boldsymbol{a} = (s\, a_i)$$

Checking the properties of a vector space is given as Problem 2.11a).

Definition 2.10: $N \times M$ Matrix: $\mathbb{M}^{N \times M}(\mathbb{F})$

For $N, M \in \mathbb{N}$ we define $N \times M$ *matrices* $A, B \in \mathbb{M}^{N \times M}(\mathbb{F})$ over the field $\mathbb{F}$ as arrays, $A = (a_{ij})$, $B = (b_{ij})$, with components $a_{ij}, b_{ij}, \in \mathbb{F}$.
The indices $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$ label the rows and columns of the array, respectively.
The sum of matrices and the product with a scalar are defined component-wise as

$$\forall A, B \in \mathbb{M}^{N \times M},\ c \in \mathbb{F} : A + B = (a_{ij} + b_{ij}) \ \wedge \ c \cdot A = (c\, a_{ij})$$

Example 2.16: 2×3 matrices: summation and multiplication with a scalar

To be specific we provide here the sum of two 2×3 matrices and the multiplication by a factor of π. Let

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 12 & 13 \\ 14 & 15 \\ 16 & 17 \end{pmatrix} .$$

Then

$$A + B = \begin{pmatrix} 2+12 & 3+13 \\ 4+14 & 5+15 \\ 6+16 & 7+17 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 18 & 20 \\ 22 & 24 \end{pmatrix}$$

$$\pi\, A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 2\,\pi & 3\,\pi \\ 4\,\pi & 5\,\pi \\ 6\,\pi & 7\,\pi \end{pmatrix}$$

Example 2.17: Vector spaces: $M \times N$ matrices

The $N \times M$ matrices over a field $\mathbb{F}$, $(\mathbb{M}^{N\times M}, \mathbb{F}, +, \cdot)$ form a vector space. The proof is given as Problem 2.11b).

Definition 2.11: Matrix multiplication

For matrices one defines a product as follows

$$\odot : \mathbb{M}^{N\times L} \times \mathbb{M}^{L\times M} \to \mathbb{M}^{N\times M}$$

$$\forall A \in \mathbb{M}^{N\times L}, B \in \times\mathbb{M}^{L\times M} : A \odot B = C = (c_{ij}) = \left(\sum_{k=1}^{L} a_{ik} b_{kj}\right)$$

Remark 2.12. Also for matrix multiplication one commonly suppresses the $\odot$ operator, writing $A\,B$ rather than $A \odot B$.

Remark 2.13. For square matrices $\mathbb{M}^{M\times M}$ the operation $+$ and $\odot$ define a sum and a product that take two elements of $\mathbb{M}^{M\times M}$ and return an element of $\mathbb{M}^{M\times M}$. Nevertheless, $(\mathbb{M}^{M\times M}, +, \odot)$ is *not* a field: In general, $\odot$ is not commutative and matrices do not necessarily have an inverse.

Remark 2.14. Square matrices can be used to represent reflections and rotations. In Problem 2.12 we provide an example of eight matrices that form a symmetry group.

Example 2.18: Vector spaces: Polynomials of degree 2

For a field $\mathbb{F}$ the polynomials P_2 of degree two in the variable x are defined as

$$\mathsf{P}_2 = \{\boldsymbol{p} = [p_0 + p_1\,x + p_2\,x^2] : p_0, p_1, p_2 \in \mathbb{F}\}$$

This set is a vector space with respect to the summation

$$\begin{aligned} \boldsymbol{p} + \boldsymbol{q} &= [p_0 + p_1\,x + p_2\,x^2] + [q_0 + q_1\,x + q_2\,x^2] \\ &= \Big[(p_0 + q_0) + (p_1 + q_1)\,x + (p_2 + q_2)\,x^2\Big] \end{aligned}$$

and the multiplication with a scalar $s \in \mathbb{F}$

$$s \cdot \boldsymbol{p} = s \cdot (p_0 + p_1\,x + p_2\,x^2) = \Big[(s\,p_0) + (s\,p_1)\,x + (s\,p_2)\,x^2)\Big]$$

Proof. Each element $\boldsymbol{p} = [p_0 + p_1\,x + p_2\,x^2]$ of this vector space is uniquely described by the three-tuple $(p_0, p_1, p_2) \in \mathbb{F}^3$ with rules for addition and scalar multiplication analogous to those discussed for $\mathbb{R}^3$ in Example 2.15. Hence, the proof for $\mathbb{R}^3$ also applies here. □

clarify

In physics we heavily make use of the correspondence evoked by the proof in Example 2.18. The relative position of two objects with respect to each other is commonly described in terms of (the sum of several) vectors. In order to gain further information about the positions, we will then recast the *geometric* problem about the positions into an *algebraic* problem stated in terms of linear equations. The latter can then be solved by straightforward analytical calculations. Vice versa, abstract findings about the solutions of sets of equations will be recast in terms of geometry in order to visualize the abstract results. The change of perspective has become a major avenue to drive theoretical physics throughout the 20th century. For mechanical problems it forms the core of the mathematical formulation of problems in robotics and computer vision. Quantum mechanics is entirely build on the principles of vector spaces and their generalization to Hilbert spaces. General relativity and quantum field theory take Noether's theorem as their common starting point, which is build upon concepts from group theory and the requirement that physical predictions must not change when taking different choices how to mathematically describe the system. An important concern of these notes is to serve as a training ground to practice the changing of mathematical perspective for the purpose of solving physics problem. As a first physical application we discuss now force balances. Then we resume the discussion of vector spaces, taking a closer look into the calculation of coordinates and distances.

2.5.1 Self Test

Problem 2.11. **Checking vector-space properties**

a) Verify that $\mathbb{R}^D$ with the operations defined in Example 2.15 is a vector space.

b) Verify that $N \times M$ matrices, as defined in Definition 2.10, form a vector space.

Problem 2.12. **Geometric interpretation of matrices** We explore the set of the eight matrices

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, \quad \text{with } a, b, c, d \in \{\pm 1\} \right\}$$

a) Let the action $\circ$ denotes matrix multiplication. Verify that $(M, \odot)$ is a group with respect to matrix multiplication, as defined in Definition 2.11. We denote its neutral element as $\mathbb{I}$.

b) Show that the group has five non-trivial elements $s_1, \dots s_5$ that are self inverse:

$$s_i \neq \mathbb{I} \quad \wedge \quad s_i \circ s_i = \mathbb{I} \quad \text{for} \quad i \in \{1, \dots 6\} .$$

c) Show that the other two elements d and r obey $d \circ r = r \circ d = \mathbb{I}$, that $r = d \circ d \circ d$, and that $d = r \circ r \circ r$.

d) Show that the set of points $P = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$ is mapped to P by the action of an element of the group:

$$\forall m \in M \ \wedge \ p \in P : \quad p \circ m \in P$$

Hint: The action of the matrix on the vector defined as follows

$$(v_1, v_2) \circ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} v_1\, m_{11} + v_2\, m_{21} \\ v_1\, m_{12} + v_2\, m_{22} \end{pmatrix}$$

e) What is the geometric interpretation of the group M? Illustrate the action of the group elements in terms of transformations of a suitably chosen geometric object.

Problem 2.13. **Polynomials of degree N**

For a field $\mathbb{F}$ the polynomials P_N of degree N in the variable x are defined as

$$\mathsf{P}_N = \left\{ \boldsymbol{p} = \left[\sum_{i=0}^{N} p_i\, x^i \right] : p_0, \dots, p_N \in \mathbb{F} \right\}$$

a) State the rules of addition and multiplication with a scalar $s \in \mathbb{F}$ in analogy to the special case of $N = 2$ discussed in Example 2.18.

b) Verify that the polynomials of degree N are a vector space.

Tug of War, Nikolay Bogdanov-Belsky, 1939
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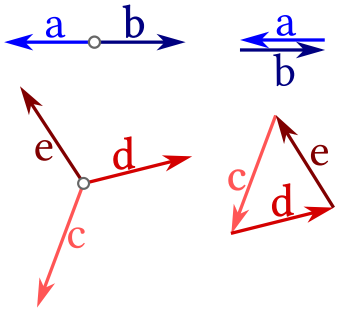


Figure 2.10: The left diagrams show two and three forces acting on a ring. To the right it is demonstrated that they add to zero.

explain center of mass

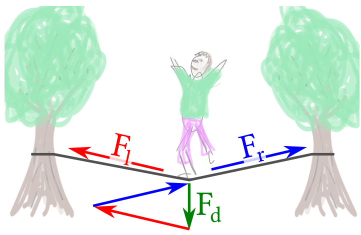


Figure 2.11: For a person balancing on a slackline, the gravitational force $\boldsymbol{F}_d$ (d for down) is balanced by forces $\boldsymbol{F}_l$ and $\boldsymbol{F}_r$ along the line that pull towards the left and right, respectively. See Example 2.19 for further discussion.

2.6 *Physics application: balancing forces*

It is an experience from tug of war that nothing moves as long as forces are balanced. In this example one can add a ring to the rope. The pulling forces act in opposing directions on the ring, as illustrated in the upper left diagram in Figure 2.10. The lower left diagram shows the case, where three parties are pulling on the ring. In any case the total force on the ring amounts to the sum of the acting forces, forces are vectors, and all sums of vectors obey the same rules. As far as graphical illustrations are concerned the sum of forces looks therefore the same as the sum of displacements in Figure 2.1. For the ring the sums of the forces are illustrated in the right panels of Figure 2.10. The ring does not move when they add to zero.

Axiom 2.1: Force balance

Let N forces $\boldsymbol{F}_1, \dots, \boldsymbol{F}_N$ act on a body. The body does not move as long as the forces add to zero, i.e. iff $\boldsymbol{0} = \sum_{i=1}^{N} \boldsymbol{F}_i$.

Remark 2.15. Strictly speaking the body might turn, but its center of mass will not move. We come back to this point in Section 2.9.

Example 2.19: Balancing on a slackline

A person balances on a slackline that is fixed to trees at its opposing sides. At the point where she is standing there are three forces acting:
her weight $\boldsymbol{F}_d = M\boldsymbol{g}$ pushing downwards, and
forces along the slackline towards the left $\boldsymbol{F}_l$ and right $\boldsymbol{F}_r$.
She can stay at rest as long as

$$\boldsymbol{0} = \boldsymbol{F}_d + \boldsymbol{F}_l + \boldsymbol{F}_r$$

The forces $\boldsymbol{F}_l$ and $\boldsymbol{F}_r$ are counterbalanced by the trees. These forces become huge when the slackline runs almost horizontally. Every now a then a careless slackliner roots out a tree or fells a pillar.

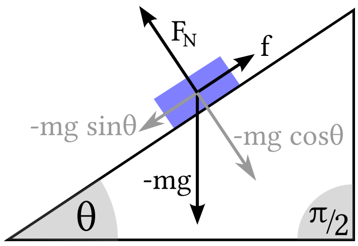
Example 2.20: Measuring the static friction coefficient

In a rough approximation static friction between two surfaces arises due to interlocking or surface irregularities. One must lift a block by a little amount to unlock the surfaces. In line with this argument dimensional analysis suggest that static friction should be proportional to the normal force between the surfaces. It is independent of the contact area, and depends on the material of the surfaces. This is indeed what is observed experimentally: The static friction force, $\boldsymbol{f}$ in Figure 2.12, can take values up to a maximum value of γ

times the normal force, $\boldsymbol{F}_N$, where γ typically takes values slightly less than one. By splitting the gravitational force, $m\boldsymbol{g}$ acting on a block on a plane into its components parallel and normal to the surface (gray arrows in Figure 2.12), one finds that in the presence of a force balance $m\boldsymbol{g} + \boldsymbol{f} + \boldsymbol{F}_N = \boldsymbol{0}$ one has

$$\left.\begin{aligned} F_N &= mg\cos\theta \\ f &= mg\sin\theta \\ f &< \gamma F_N \end{aligned}\right\} \quad \begin{aligned} &\Rightarrow \sin\theta < \gamma\cos\theta \\ &\quad \Rightarrow \theta < \theta_c = \arctan\gamma \end{aligned}$$

When θ exceeds θ_c the block starts to slide. Hence, one can infer γ from measurements of θ_c.



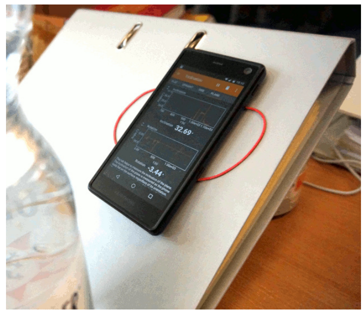
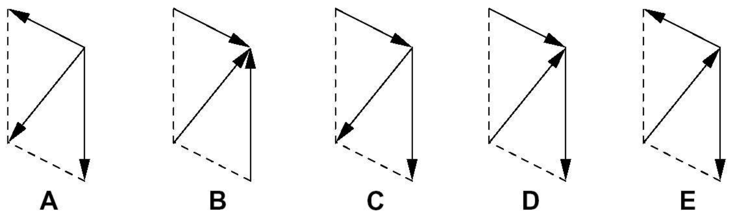


Figure 2.12: (top) As long as θ is smaller than the angle of friction the blue block does not slide. (bottom) Placing my cell phone on two rubber bands on a folder provides a maximum angle of about 33°, i. e. $\mu \simeq 0.5$. Using PhyPhox and a cell phone one can easily measure θ_c and μ for other combinations of materials.

2.6.1 *Self Test*

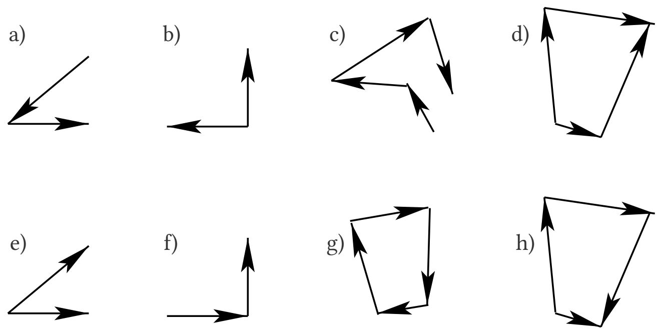
Problem 2.14. **Particles at rest**

There are three forces acting on the center of mass of a body. In which cases does it stay at rest?



Problem 2.15. **Graphical sum of vectors**

Determine the sum of the vectors. In which cases is the resulting vector vertical to the horizontal direction?



Problem 2.16. **Towing a stone**

Three Scottish muscleman[5] try to tow a stone with mass $M = 20\,\text{cwt}$ from a field. Each of them gets his own rope, and he can act a maximal force of 300 lbg as long as the ropes run in directions that differ by at least 30°.

a) Sketch the forces acting on the stone and their sum. By which ratio is the force exerted by three men larger than that of a single man?

[5] In highland games one still uses Imperial Units. A hundredweight (cwt) amounts to eight stones (stone) that each have a mass of 14 pounds (lb). A pound-force (lbg) amounts to the gravitational force acting on a pound. One can solve this problem without converting units.

b) The stone counteracts the pulling of the men by a static friction force $\mu M g$, where g is the gravitational acceleration. What is the maximum value that the friction coefficient μ may take when the men can move the stone?

2.7 The inner product

The position of a particle, the direction of its motion and the angle of attack of forces are constantly changing during the motion of a particle. In Chapter 3 we explore how they are related. The calculations are feasible because the involved vector spaces also have an inner product.

Definition 2.12: Inner Product of vector spaces over $\mathbb{R}$ or $\mathbb{C}$

The *inner product* on a vector space $(\mathsf{V}, \mathbb{R}, \oplus, \odot)$ defines a binary operation $\langle \_ \mid \_ \rangle : \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ with the following properties for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathsf{V}$ and $c \in \mathbb{R}$

a) commutativity: $\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle = \langle \boldsymbol{w} \mid \boldsymbol{v} \rangle$

b) linearity in the first argument: $\langle c\, \boldsymbol{v} \mid \boldsymbol{w} \rangle = c\, \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$
and $\langle \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{w} \rangle = \langle \boldsymbol{u} \mid \boldsymbol{w} \rangle + \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$

c) positivity: $\langle \boldsymbol{v} \mid \boldsymbol{v} \rangle \geq 0$
where equality applies iff $\boldsymbol{v} = \mathbf{0}$, $\langle \boldsymbol{v} \mid \boldsymbol{v} \rangle = 0 \Leftrightarrow \boldsymbol{v} = \mathbf{0}$

For a vector space over $\mathbb{C}$ the requirement a) is replaced by

a) conjugate symmetry: $\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle = \overline{\langle \boldsymbol{w} \mid \boldsymbol{v} \rangle}$

and the constant c is a complex number.

Remark 2.16. The idea underlying these properties is that $\sqrt{\langle \boldsymbol{v} \mid \boldsymbol{v} \rangle}$ can be interpreted as the length of the vector $\boldsymbol{v}$.

Remark 2.17. Conjugate symmetry and linearity for the first argument imply the following relations for the second argument

$$\langle \boldsymbol{v} \mid c\, \boldsymbol{w} \rangle = \overline{\langle c\, \boldsymbol{w} \mid \boldsymbol{v} \rangle} = \bar{c}\, \overline{\langle \boldsymbol{w} \mid \boldsymbol{v} \rangle} = \bar{c}\, \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$$
$$\langle \boldsymbol{u} \mid \boldsymbol{v} + \boldsymbol{w} \rangle = \overline{\langle \boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{u} \rangle} = \overline{\langle \boldsymbol{v} \mid \boldsymbol{u} \rangle} + \overline{\langle \boldsymbol{w} \mid \boldsymbol{u} \rangle} = \langle \boldsymbol{u} \mid \boldsymbol{v} \rangle + \langle \boldsymbol{u} \mid \boldsymbol{w} \rangle$$

Remark 2.18. Certain properties that hold for addition and scalar multiplication do *not* hold for the inner product.

a) There is no inverse: The information about the direction of vectors is lost upon taking the inner product. For instance, when $\langle \boldsymbol{u} \mid \boldsymbol{v} \rangle = 0$ and $\langle \boldsymbol{u} \mid \boldsymbol{w} \rangle = 0$ then one still can not tell the result of $\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$.

b) Associativity does not hold: $\langle \boldsymbol{u} \mid \boldsymbol{v} \rangle\, \boldsymbol{w} \neq \boldsymbol{u}\, \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle$.

Example 2.21: Inner product for real-valued vectors

For real-valued vectors the inner product is commutative, $\langle \boldsymbol{v} \mid \boldsymbol{w} \rangle = \langle \boldsymbol{w} \mid \boldsymbol{v} \rangle$. The inner product is then also be written as $\boldsymbol{v} \cdot \boldsymbol{w}$, and it obeys bilinearity

$$\boldsymbol{u} \cdot (a\boldsymbol{v} + b\boldsymbol{w}) = a(\boldsymbol{u} \cdot \boldsymbol{v}) + b(\boldsymbol{u} \cdot \boldsymbol{w})$$

Theorem 2.2: Geometric Interpretation of the Inner Product for Real-Valued Vectors

For vectors of $\mathbb{R}^D$ the inner product of two vectors $\boldsymbol{a}, \boldsymbol{b}$ takes the value

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\,\cos\theta$$

where $\theta = \angle(\boldsymbol{a}, \boldsymbol{b})$ is the angle between the two vectors, see Figure 2.13.

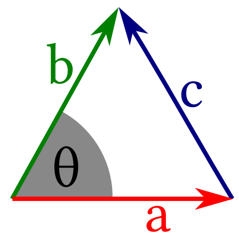


Figure 2.13: Notations for the geometric interpretation of the inner product, Theorem 2.2

Proof. The cosine theorem for triangles with sides of length a, b and c and angle θ opposite to c states that

$$c^2 = a^2 + b^2 - 2\,a\,b\,\cos\theta$$

Let now a, b, and c be the length of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$, as shown in Figure 2.13. Then we have

$$\begin{aligned} a^2 + b^2 - 2\,a\,b\,\cos\theta = c^2 = \boldsymbol{c} \cdot \boldsymbol{c} = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) \\ = \boldsymbol{a} \cdot \boldsymbol{a} - 2\,\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b} = a^2 + b^2 - 2\,\boldsymbol{a} \cdot \boldsymbol{b} \\ \Rightarrow \quad \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\,\cos\theta \end{aligned}$$

□

Remark 2.19. Theorem 2.2 entails that the inner product $\boldsymbol{u} \cdot \boldsymbol{v}$ vanishes when the vectors are orthogonal, $\theta = \pi/2$. Also in general we say that

$$\boldsymbol{v} \text{ and } \boldsymbol{w} \text{ are othogonal iff} \qquad \langle \boldsymbol{v} \mid \boldsymbol{w} \rangle = 0\,.$$

Remark 2.20. The expression for the inner product that is provided Theorem 2.2 does not imply that the inner product is unique. Rather it is a consequence of the cosine theorem that holds iff the geometric interpretation of the vectors applies. This is demonstrated by an example provided in Problem 2.17.

2.7.1 *Self Test*

Problem 2.17. **The inner product is not unique**

Let v_1 and v_2 be two non-orthogonal vectors in a two-dimensional vector space with an inner product $\langle \_ \mid \_ \rangle$, and let λ_1 and λ_2 two positive real numbers. Then the following relation defines another inner product $(\_ \mid \_)$:

$$(\boldsymbol{a} \mid \boldsymbol{b}) = \lambda_1 \langle \boldsymbol{a} \mid \boldsymbol{e}_1 \rangle \langle \boldsymbol{e}_1 \mid \boldsymbol{b} \rangle + \lambda_2 \langle \boldsymbol{a} \mid \boldsymbol{e}_2 \rangle \langle \boldsymbol{e}_2 \mid \boldsymbol{b} \rangle \qquad (2.7.1)$$

a) Verify that the properties a) and b) of an inner product $\langle \_ \mid \_ \rangle$ as given in Definition 2.12 are also be obeyed by $(\_ \mid \_)$.

b) Verify that $(\boldsymbol{a} \mid \boldsymbol{a}) \geq 0$ iff λ_1 and λ_2 two positive real numbers.

c) Verify that $(\boldsymbol{a} \mid \boldsymbol{a}) = 0$ implies $\boldsymbol{a} = \boldsymbol{0}$ iff the vector space is two-dimensional.

Problem 2.18. **Inner products for polynomials**

Let $\boldsymbol{p} = \left[\sum_{i=0}^{D} p_i\, x^i\right]$ and $\boldsymbol{q} = \left[\sum_{i=0}^{D} q_i\, x^i\right]$ be elements of the vector space of N-dimensional polynomials. Verify that the following rules define inner products on this space.

a) $\langle \boldsymbol{p} \mid \boldsymbol{q} \rangle = \sum_{i=0}^{N} \bar{p}_i\, q_i$

b) $\langle \boldsymbol{p} \mid \boldsymbol{q} \rangle_{[a,b]} = \int_a^b \mathrm{d}x \, \overline{\left[\sum_{i=0}^{D} p_i\, x^i\right]} \, \left[\sum_{i=0}^{D} q_i\, x^i\right] \quad$ for $a < b \in \mathbb{R}$

c) Show that $\boldsymbol{p} = [1]$ and $\boldsymbol{q} = [x]$ are orthogonal with respect to the inner product defined in a). Under which condition are they also orthogonal for the inner product defined in b)?

2.8 *Cartesian coordinates*

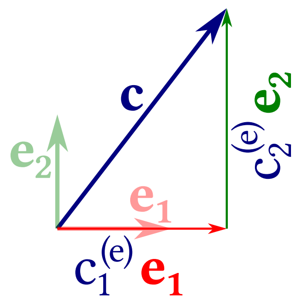


Figure 2.14: Representation of the vector $\boldsymbol{c}$ in terms of the orthogonal unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2)$.

Theorem 2.2 entails an extremely elegant possibility to deal with vectors. We first illustrate the idea based on a two-dimensional example, Figure 2.14, and then we develop the general theory:

Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be two orthogonal vectors that have unit length,

$$\langle \boldsymbol{e}_1 | \boldsymbol{e}_1 \rangle = \langle \boldsymbol{e}_2 \mid \boldsymbol{e}_2 \rangle = 1 \quad \text{and} \quad \langle \boldsymbol{e}_1 | \boldsymbol{e}_2 \rangle = 0$$

For every vector $\boldsymbol{c}$ in the plane described by these two vectors, we can then find two numbers $c_1^{(e)}$ and $c_2^{(e)}$ such that

$$\boldsymbol{c} = c_1^{(e)}\, \boldsymbol{e}_1 + c_2^{(e)}\, \boldsymbol{e}_2$$

Now the choice of the vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ entails that triangle with edge $\boldsymbol{c}$, $c_1^{(e)}\, \boldsymbol{e}_1$, and $c_2^{(e)}\, \boldsymbol{e}_2$ is right-angled and that

$$c_i^{(e)} = |\boldsymbol{c}| \, \cos \angle(\boldsymbol{c}, \boldsymbol{e}_i) = \langle \boldsymbol{c} \mid \boldsymbol{e}_i \rangle \quad \text{for } i \in \{1, 2\}$$
$$\Rightarrow \quad \boldsymbol{c} = \langle \boldsymbol{c} \mid \boldsymbol{e}_1 \rangle\, \boldsymbol{e}_1 + \langle \boldsymbol{c} \mid \boldsymbol{e}_2 \rangle\, \boldsymbol{e}_2$$

This strategy to represent vectors applies in all dimensions.

Definition 2.13: Basis and Coordinates

Let $\mathsf{B} = \{\boldsymbol{e}_i,\ i \in \{1, \dots, D\}$ be a set of D pairwise orthogonal unit vectors

$$\forall i,j \in \{1,\ldots,D\}: \quad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

in a vector space $(\mathsf{V}, \mathbb{F}, +, \cdot)$ with inner product $\langle \_|\_ \rangle$. We say that B forms a *basis* for a D-dimensional vector space iff

$$\forall \boldsymbol{v} \in \mathsf{V}\ \exists v_i,\ i \in \{1,\ldots,D\}: \quad \boldsymbol{v} = \sum_{i=1}^{D} v_i^{(e)}\ \boldsymbol{e}_i$$

In that case we also have $v_i^{(e)} = \langle \boldsymbol{v} \mid \boldsymbol{e}_i \rangle$, $i \in \{1,\ldots,D\}$ and these numbers are called the *coordinates* of the vector $\boldsymbol{v}$. The number of vectors D in the basis of the vector space is denoted as *dimension of the vector space*.

Remark 2.21. The choice of a basis, and hence also of the coordinates, is not unique. Figure 2.15 shows the representation of a vector in terms of two different bases $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2)$. We suppress the superscript that indicates the basis when the choice of the basis is clear from the context.

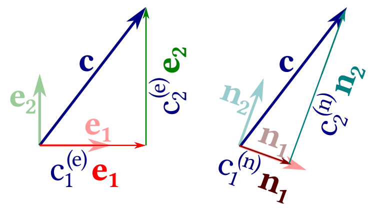


Figure 2.15: Representation of the vector $\boldsymbol{c}$ of Figure 2.14 in terms of the bases $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2)$.

Remark 2.22. For a given basis the representation in terms of coordinates is unique.

Proof. 1. The coordinates a_i of a vector $\boldsymbol{a}$ are explicitly given by $a_i = \langle a \mid \boldsymbol{e}_i \rangle$. This provides unique numbers for a given basis set.

2. Assume now that two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same coordinate representation. Then the vector-space properties imply

$$\left.\begin{array}{l} \boldsymbol{a} = \sum_i c_i\, \boldsymbol{e}_i \\ \boldsymbol{b} = \sum_i c_i\, \boldsymbol{e}_i \end{array}\right\} \Rightarrow \boldsymbol{a} - \boldsymbol{b} = \left(\sum_i c_i\, \boldsymbol{e}_i\right) - \left(\sum_i c_i\, \boldsymbol{e}_i\right)$$

$$= \sum_i (c_i - c_i)\, \boldsymbol{e}_i = \sum_i 0\, \boldsymbol{e}_i = \boldsymbol{0}$$

$$\Rightarrow \quad \boldsymbol{a} = \boldsymbol{b}$$

Hence, they must be identical. □

Remark 2.23 (Kronecker δ_{ij}). It is convenient to introduce the abbreviation δ_{ij} for

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

where i, j are elements of some index set. This symbol is denoted as *Kronecker* δ. With the Kronecker symbol the condition on orthogonal unit vectors of a basis can more concisely be written as

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

Moreover, for $i, j \in \{1,\ldots,D\}$ the numbers, δ_{ij}, describe a $D \times D$ matrix which is the neutral element for multiplication with another $D \times D$ matrix, and also with a vector of $\mathbb{R}^D$, when it is interpreted as a $D \times 1$ matrix.

Theorem 2.3: Scalar product on $\mathbb{R}^D$ and $\mathbb{C}^D$

The axioms of vector spaces and the inner product imply that

$$\text{on } \mathbb{R}^D : \qquad \langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = \sum_{i=1}^{D} \langle \boldsymbol{a} \mid i \rangle \, \langle i \mid \boldsymbol{b} \rangle = \sum_{i=1}^{D} a_i b_i$$

$$\text{on } \mathbb{C}^D : \qquad \langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = \sum_{i=1}^{D} \langle \boldsymbol{a} \mid i \rangle \, \langle i \mid \boldsymbol{b} \rangle = \sum_{i=1}^{D} a_i \bar{b}_i$$

where the bar indicates complex conjugation of complex numbers. This can be written as follows when representing the coordinates as a 1D array of numbers

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_D \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_D \end{pmatrix} = a_1 \, \bar{b}_1 + a_2 \, \bar{b}_2 + \cdots + a_D \, \bar{b}_D$$

where the complex conjugation does not apply for real numbers. This latter form of the inner product is denoted as *scalar product*.

Proof. We first note that the case of real numbers can be interpreted as special case of the complex numbers with a vanishing complex part. Hence, we only provide the proof for the complex case.

We use the representations $\boldsymbol{a} = \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \boldsymbol{e}_i$ and $\boldsymbol{b} = \sum_j \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle \, \boldsymbol{e}_j$, and work step by step from the left to the aspired result:

$$\begin{aligned}
\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle &= \left\langle \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \boldsymbol{e}_i \,\middle|\, \sum_j \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle \, \boldsymbol{e}_j \right\rangle \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \left\langle \boldsymbol{e}_i \,\middle|\, \sum_j \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle \, \boldsymbol{e}_j \right\rangle \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \sum_j \overline{\langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle} \, \langle \boldsymbol{e}_i \mid \boldsymbol{e}_j \rangle \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \sum_j \langle \boldsymbol{e}_j \mid \boldsymbol{b} \rangle \, \delta_{ij} \\
&= \sum_i \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle \, .
\end{aligned}$$

Due to $a_i = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle$ and $\bar{b}_i = \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle$ we therefore have

$$\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = \sum_i a_i \, \bar{b}_i$$

□

Remark 2.24. Einstein pointed out that the sums over pairs of identical indices arise ubiquitously in calculations like to proof of Theorem 2.3. He therefore adopted the convention that one always sums over pairs of identical indices, and does no longer explicitly write that down. This leads to substantially clearer representation of the calculation. For instance, the proof looks then as follows:

$$\begin{aligned}
\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle &= \Big\langle \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \boldsymbol{e}_i \,\Big|\, \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle \, \boldsymbol{e}_j \Big\rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \Big\langle \boldsymbol{e}_i \,\Big|\, \langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle \, \boldsymbol{e}_j \Big\rangle \\
&= \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \overline{\langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle} \, \langle \boldsymbol{e}_i \mid \boldsymbol{e}_j \rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \overline{\langle \boldsymbol{b} \mid \boldsymbol{e}_j \rangle} \, \delta_{ij} \\
&= \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \, \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle \\
\Rightarrow \quad \langle \boldsymbol{a} \mid \boldsymbol{b} \rangle &= a_i \, \bar{b}_i
\end{aligned}$$

Remark 2.25. Dirac pointed out that the vector product $\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle$ takes the form of the multiplication of a $1 \times D$ matrix for $\boldsymbol{a}$ and a $D \times 1$ matrix for $\boldsymbol{b}$. He suggested to symbolically write down these vectors as a *bra vector* $\langle a|$ and a *ket vector* $|b\rangle$. When put together as a bra-(c)-ket $\langle a|b\rangle$ one recovers the inner product, and introducing $|\boldsymbol{e}_i\rangle\langle \boldsymbol{e}_i|$ and observing Einstein notation comes down to inserting a unit matrix. For instance for 2×2 vectors

$$\langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = (a_1, a_2) \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} = (a_1, a_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle$$

Conceptually this is a very useful observation because it provides an easy rule to sort out what changes in the equations when one represents a problem in terms of a different basis.

Example 2.22: Changing coordinates from basis $(\boldsymbol{e}_i)$ to basis $(\boldsymbol{n}_i)$

We observe Dirac's observation that the expressions $|e_i\rangle\langle e_i|$ and $|n_i\rangle\langle n_i|$ sandwiched between a bra and a ket amounts to multiplication with one. Hence, the coordinates change according to

$$a_i^{(n)} = \langle \boldsymbol{a} \mid \boldsymbol{n}_i \rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_j \rangle \langle \boldsymbol{e}_j | \boldsymbol{n}_i \rangle = a_j^{(e)} \langle \boldsymbol{e}_j | \boldsymbol{n}_i \rangle$$

which amounts to multiplying the vector with entries $(a_j^{(e)}, j = 1, \ldots, D)$ with the $D \times D$ matrix T with entries $t_{ji} = \langle \boldsymbol{e}_j | \boldsymbol{n}_i \rangle$.
On the other hand, for the inner products we have

$$\begin{aligned}
a_i^{(e)} \bar{b}_i^{(e)} &= \langle \boldsymbol{a} \mid \boldsymbol{b} \rangle = \langle \boldsymbol{a} \mid \boldsymbol{e}_i \rangle \langle \boldsymbol{e}_i \mid \boldsymbol{b} \rangle \\
&= \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \langle \boldsymbol{n}_j \mid \boldsymbol{e}_i \rangle \langle \boldsymbol{e}_i \mid \boldsymbol{n}_k \rangle \langle \boldsymbol{n}_k | \boldsymbol{b} \rangle = \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \langle \boldsymbol{n}_j \mid \boldsymbol{n}_k \rangle \langle \boldsymbol{n}_k | \boldsymbol{b} \rangle \\
&= \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \, \delta_{jk} \, \langle \boldsymbol{n}_k | \boldsymbol{b} \rangle = \langle \boldsymbol{a} \mid \boldsymbol{n}_j \rangle \langle \boldsymbol{n}_j | \boldsymbol{b} \rangle = a_i^{(n)} \bar{b}_i^{(n)}
\end{aligned}$$

Its value does not change, even though the coordinates take entirely different values.

add worked example for an explicit coordinate transformation

2.8.1 *Self Test*

Problem 2.19. **Cartesian coordinates in the plane**

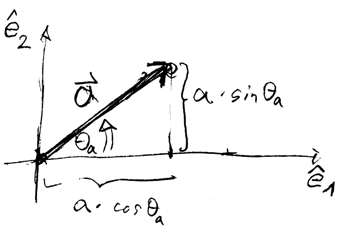
a) Mark the following points in a Cartesian coordinate system:

$$(0, 0) \quad (0, 3) \quad (2, 5) \quad (4, 3) \quad (4, 0)$$

Add the points (0, 0) (4, 3) (0, 3) (4, 0), and connect the points in the given order. What do you see?

b) What do you find when drawing a line segment connecting the following points?

$$(0, 0) \quad (1, 4) \quad (2, 0) \quad (-1, 3) \quad (3, 3) \quad (0, 0)$$



Problem 2.20. **Geometric and algebraic form of the scalar product** The sketch in the margin shows a vector $\boldsymbol{a}$ in the plane, and its representation as a linear combination of two orthonormal vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$,

$$\boldsymbol{a} = a \, \cos\theta_a \, \hat{\boldsymbol{e}}_1 + a \, \sin\theta_a \, \hat{\boldsymbol{e}}_2$$

Here, a is the length of the vector $\boldsymbol{a}$,
and $\theta_1 = \angle(\hat{\boldsymbol{e}}_1, \boldsymbol{a})$.

a) Analogously to $\boldsymbol{a}$ we consider another vector $\boldsymbol{b}$ with a representation

$$\boldsymbol{b} = b \, \cos\theta_b \, \hat{\boldsymbol{e}}_1 + b \, \sin\theta_b \, \hat{\boldsymbol{e}}_2$$

Employ the rules of scalar products, vector addition and multiplication with scalars to show that

$$\boldsymbol{a} \cdot \boldsymbol{b} = a \, b \, \cos(\theta_a - \theta_b)$$

Hint: Work backwards, expressing $\cos(\theta_a - \theta_b)$ in terms of $\cos\theta_a$, $\cos\theta_b$, $\sin\theta_a$, and $\sin\theta_b$.

b) As a shortcut to the explicit calculation of a) one can introduce the coordinates $a_1 = a \, \cos\theta_a$ and $a_2 = a \, \sin\theta_a$, and write $\boldsymbol{a}$ as a tuple of two numbers. Proceeding analogously for $\boldsymbol{b}$ one obtains

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

How does the product $\boldsymbol{a} \cdot \boldsymbol{b}$ look like in terms of these coordinates?

c) How do the arguments in a) and b) change for D dimensional vectors that are represented as linear combinations of a set of orthonormal basis vectors $\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_D$?

What changes when the basis is not orthonormal?
What if it is not even orthogonal?

Problem 2.21. **Scalar product on $\mathbb{R}^D$**

Show that the scalar product on $\mathbb{R}^D$ takes exactly the same form as for the complex case, Theorem 2.3.

However, complex conjugation is not necessary in that case.

Problem 2.22. **Pauli matrices form a basis for a 4D vector space**

Show that the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis of the real vector space of 2×2 Hermitian matrices, $\mathbb{H}$, with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{H} \quad \Leftrightarrow \quad a_{ij} \in \mathbb{C} \ \wedge \ a_{ij} = a_{ji}^*$$

Show to that end

a) The matrices $\sigma_0, \dots, \sigma_4$ are linearly independent.

b) $x_0, \dots, x_4 \in \mathbb{R} \quad \Rightarrow \quad \sum_{i=0}^{4} x_i \, \sigma_i \in \mathbb{H}$

c) $M \in \mathbb{H} \quad \Rightarrow \quad \exists x_0, \dots, x_4 \in \mathbb{R} : M = \sum_{i=0}^{4} x_i \, \sigma_i$

What about linear combinations with coefficients $z_1, \dots, z_4$? Is $\sum_{i=0}^{4} z_i \, \sigma_i$ Hermitian? Do these matrices form a vector space?

2.9 *Cross products — torques*

adapted from rachaelvoorhees from arlington, va / wikimedia CC BY 2.0

The pictures in the margin show the sign of a seesaw, a playground toy that works even for people with vastly different weight and size. Figure 2.16a) shows a balanced scale. When the forces acting on the scale do not add up to zero, we pick up the scale. It moves. The according force balance for the beam of the scale is shown in Figure 2.16c). In general the beam does not stay at rest, when the two masses are not attached at the same distance from the fulcrum. The force balance, Figure 2.16c), still hods, and the beam turns, rather than being lifted. The sum of attached forces tells us if an object is displaced. In analogy we introduce the *torque* to describe whether it turns.

more explanation needed.

When the beam is vertical there is no torque, and it takes its maximum when the beam is horizontal. In the former case the forces act parallel to the beam, and in the latter they act in orthogonal direction. Moreover, a weight that is attached at a larger distance to the fulcrum induces a larger torque, and the torque also increases with mass. This is expressed in the lever rule.

Example 2.23: Torques on a Lever

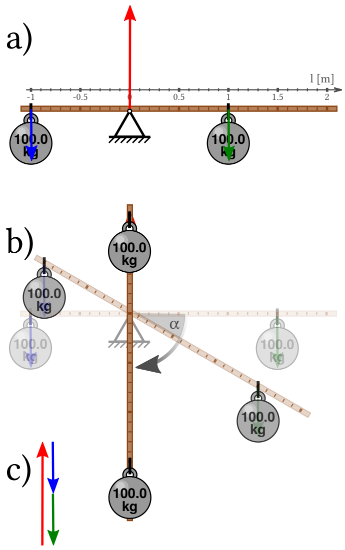
The torque T exerted by a lever is given by the product, $T = l\,F$, of the modulus of the force F acting vertical to the lever and the distance l between the fulcrum and the point where the force is applied, which is called *length of the lever arm*.

When several forces act on the same lever, then the total torque amounts to the sum of the torques induced by the individual forces, $T = \sum_i l_i\,F_i$. For the scale in Figure 2.16a) and b) we find

$$T_a = (1\,\mathrm{m})\,(100\,\mathrm{kg})\,(-g) + (-1\,\mathrm{m})\,(100\,\mathrm{kg})\,(-g) = 0$$

$$T_b = (1.5\,\mathrm{m})\,(100\,\mathrm{kg})\,(-g)\,\cos\alpha + (-1\,\mathrm{m})\,(100\,\mathrm{kg})\,(-g)\,\cos\alpha$$
$$\simeq -500\,\cos\alpha\,\mathrm{kg\,m^2/s^2}$$

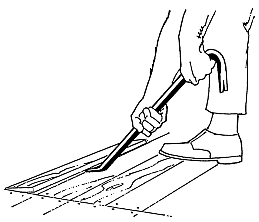
The torque vanishes only when $\alpha = \pi/2$ as shown in the figure, and for the unstable tipping point $\alpha = -\pi/2$.



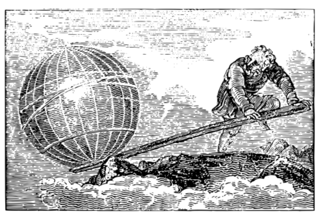
based on from Jahobr/wikimedia CC0 1.0
Figure 2.16: a) The lever is balanced when two equal masses are attached at the same distance from the fulcrum. b) It is at (stable) rest only in a single position when equal weighs are attached at different distances. c) In all positions the sum of the forces on the beam, by the fulcrum and by the two weights, add to zero.

Remark 2.26. Adopting a lever where force is applied on a long arm allows one to move very heavy objects or break very stable objects. Common technological applications are the crowbar and the lever. Archimedes was so impressed by this principle that he is quoted to have remarked “Δοσ μοι που στω και κινω την γην” (Archimedes, 1878), i.e. “Give me but one firm spot on which to stand, and I will move the earth” (Oxford Dictionary of Quotations, 1953)

estimate amplification of force the the crowbar



Pearson Scott Foresman / Public domain
Figure 2.17: Action of a crowbar.



Mechanic’s Magazine cover of Vol II, Knight & Lacey, London, 1824./wikipedia, public domain
Figure 2.18: Illustration of Archimedes’ remark about moving the earth.

Observe the sign of the torque: In Example 2.23 it is positive for counterclockwise motion, and negative for clockwise motion. The axis of rotation is fixed by the fulcrum. However, when acting the crowbar, one applies a horizontal force to get the crowbar under the obstacle. This induces a rotation around a vertical axis. Subsequently, a vertical force is applied to lift the obstacle. It induces a rotation around a horizontal axes. The relation between the directions of the lever arm, the force, and the rotation axis is commonly illustrated by the right-hand rule (Figure 2.19): Here the arm points in the direction of the lever arm, the fingers in the direction of the applied force, and the thump along the rotation axis. This suggests to define torque as a product of two vectors, the arm ℓ and the force $\boldsymbol{F}$ that provide the torque, $\boldsymbol{T}$, which is a vector of length $|\ell|\,|\boldsymbol{F}|\,\sin\angle\ell,\boldsymbol{F}$ in a direction normal to the plane defined by ℓ and $\boldsymbol{F}$. This operation, $\boldsymbol{T} = \ell \times \boldsymbol{F}$ defines the cross product. We explore its properties in a mathematical digression.

2.9.1 *Algebraic properties of cross products*

Definition 2.14: Cross product on $\mathbb{R}^3$

The *cross product* on the vector space $\mathbb{R}^3$ defines a binary operation $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

a) anti-commutativity: $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$

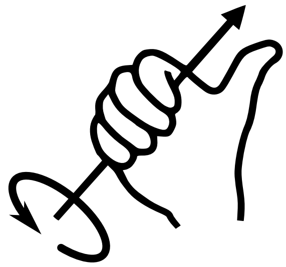
b) distributivity: $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$

c) compatibility with scalar multiplication:
$$(c\,\boldsymbol{u}) \times \boldsymbol{v} = \boldsymbol{u} \times (c\,\boldsymbol{v}) = c\,(\boldsymbol{u} \times \boldsymbol{v})$$

d) symmetry of scalar triple product (Jacobi identity):
$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$$

Moreover for every right-handed set of three orthonormal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ we require

e) normalization: $\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3) = 1$



Schorschi2 at de.wikipediaderivative work: Wizard191, public domain

Figure 2.19: Right-hand rule.

Remark 2.27. The cross product of a vector with itself vanishes

$$\forall \boldsymbol{v} \in \mathbb{R}^3 \; : \; \boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$$

Proof. Vanishing of $\boldsymbol{v} \times \boldsymbol{v}$ is a consequence of anti-commutativity:

$$\boldsymbol{v} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{v} \quad \Rightarrow 2\,\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$$
□

Theorem 2.4: Right-handed orthonormal basis in $\mathbb{R}^3$

Let $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^3$ be orthonormal vectors, $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \delta_{12}$. Then $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ form a right-handed orthonormal basis for $\mathbb{R}^3$, and we have

$$\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \begin{cases} 1 & \text{for} \quad ijk \in \{123, 231, 312\} \\ -1 & \text{for} \quad ijk \in \{132, 213, 321\} \\ 0 & \text{else} \end{cases}$$

Remark 2.28 (Levi-Civita tensor ε_{ijk}). It is convenient to introduce the abbreviation ε_{ijk} for

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for} \quad ijk \in \{123, 231, 312\} \\ -1 & \text{for} \quad ijk \in \{132, 213, 321\} \\ 0 & \text{else} \end{cases}$$

This symbol is denoted as *Levi-Civita tensor* ε_{ijk}. With this symbol the relations between right-handed orthogonal unit vectors of a basis can more concisely be written as

$$\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \varepsilon_{ijk}$$

Moreover, it immediately provides the following representation of the scalar triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ in terms of coordinates u_i, v_j, w_k, $i, j, k \in \{1, 2, 3\}$,

$$\left.\begin{array}{ll} \boldsymbol{u} & = \sum_{i=1}^{3} u_i\, \boldsymbol{e}_i \\ \boldsymbol{v} & = \sum_{j=1}^{3} v_j\, \boldsymbol{e}_j \\ \boldsymbol{w} & = \sum_{k=1}^{3} w_k\, \boldsymbol{e}_k \end{array}\right\} \quad \Rightarrow \quad \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, u_i\, v_j\, w_k$$

or even $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \varepsilon_{ijk}\, u_i\, v_j\, w_k$ with Einstein notation. The symmetry of the triple scalar product is an immediate consequence of the symmetry of the ε-tensor.

Proof. The identity $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \varepsilon_{ijk}\, u_i\, v_j\, w_k$ follows from the compatibility with scalar product and the relation for the basis vectors $\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k)$. The details of the proof are given as Problem 2.23. □

Proof of Theorem 2.4. We show that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ form three orthonormal vectors. By assumption $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthonormal. Hence, we show that $\boldsymbol{e}_3$ is a unit vector that is orthogonal to $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$:

$$\begin{aligned} \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 &= \boldsymbol{e}_1 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = \boldsymbol{e}_2 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_1) = \boldsymbol{e}_2 \cdot \boldsymbol{0} = 0 \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 &= \boldsymbol{e}_2 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot \boldsymbol{0} = 0 \\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 &= \boldsymbol{e}_3 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3) = 1 \end{aligned}$$

□

Remark 2.29 (bac-cab rule). The double cross product can be expressed in terms of scalar products. Commonly this relation is stated in terms of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c} \in \mathbb{R}^3$,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\, (\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\, (\boldsymbol{a} \cdot \boldsymbol{b})$$

and referred to as bac-cab rule.

Proof. We express the three vectors in terms of their coordinates with respect to the orthonormal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$,

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i\, \boldsymbol{e}_i \quad \boldsymbol{b} = \sum_{j=1}^{3} b_j\, \boldsymbol{e}_j \quad \boldsymbol{c} = \sum_{k=1}^{3} c_k\, \boldsymbol{e}_k \quad \text{with} \quad a_i, b_j, c_k \in \mathbb{R}$$

and use the rules defining the cross products and inner products

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \left(\sum_{i=1}^{3} a_i\, \boldsymbol{e}_i \right) \times \left[\left(\sum_{j=1}^{3} b_j\, \boldsymbol{e}_j \right) \times \left(\sum_{k=1}^{3} c_k\, \boldsymbol{e}_k \right) \right] \\ &= \sum_{i,j,k=1}^{3} a_i\, b_j\, c_k \boldsymbol{e}_i \times (\boldsymbol{e}_j \times \boldsymbol{e}_k) \end{aligned}$$

When $j = k$ or when j and k are both different from i then the summand vanishes due to Remark 2.27. For $i = j \neq k$ one has $\boldsymbol{e}_i \times (\boldsymbol{e}_j \times \boldsymbol{e}_k) = -\boldsymbol{e}_k$, and for $i = k \neq j$ one has $\boldsymbol{e}_i \times (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \boldsymbol{e}_j$. Consequently,

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \sum_{i,k=1}^{3} a_i\, b_i\, c_k (-\boldsymbol{e}_k) + \sum_{i,j=1}^{3} a_i\, b_j\, c_i (\boldsymbol{e}_j) \\ &= \boldsymbol{b}\, (\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\, (\boldsymbol{a} \cdot \boldsymbol{b}) \end{aligned}$$

□

Remark 2.30 (Jacobi identity). The cross product obeys the Jacobi identity:

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{v} \times (\boldsymbol{w} \times \boldsymbol{u}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{0}$$

Proof. This can be verified by evaluating the triple cross products by the bac-cab rule. Details are give as Problem 2.24. □

Remark 2.31. In coordinate notation the cross product takes the form

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_2 b_3 - a_3 b_2 \end{pmatrix}$$

Proof. For component k of $\boldsymbol{a} \times \boldsymbol{b}$ we have

$$[\boldsymbol{a} \times \boldsymbol{b}]_k = \hat{\boldsymbol{e}}_k \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \hat{\boldsymbol{e}}_k \cdot \left[\left(\sum_{i=1}^{3} a_i\, \hat{\boldsymbol{e}}_i \right) \times \left(\sum_{j=1}^{3} b_j\, \hat{\boldsymbol{e}}_j \right) \right]$$

$$= \sum_{i,j=1}^{3} a_i\, b_j \;\; \hat{\boldsymbol{e}}_k \cdot \left[\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j \right]_k = \sum_{i,j=1}^{3} a_i\, b_j\, \varepsilon_{ijk}$$

In the remark this is explicitly written out for $k \in \{1, 2, 3\}$. □

2.9.2 *Geometric interpretation of cross products*

The cross product and the scalar triple product have distinct geometrical interpretations. The geometric meaning of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ can best be seen by adopting a basis where the first basis vector is parallel to $\boldsymbol{e}_1 = \boldsymbol{a}/|\boldsymbol{a}|$, and the second basis vector $\boldsymbol{e}_2$ lies orthogonal to $\boldsymbol{e}_1$ in the plane spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$. The third basis vector will then be $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and hence also of $\boldsymbol{e}_1$ and $\boldsymbol{b}$ is denoted as θ. Thus, $\boldsymbol{b}$ can be written as $\boldsymbol{b} = b_1\, \boldsymbol{e}_1 + b_2\, \boldsymbol{e}_2 = |\boldsymbol{b}|\, (\cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2)$, cf. Figure 2.20). For this choice of the basis we find

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= |\boldsymbol{a}|\, \boldsymbol{e}_1 \times (b_1\, \boldsymbol{e}_1 + b_2\, \boldsymbol{e}_2) = |\boldsymbol{a}|\, b_1\, \boldsymbol{e}_1 \times \boldsymbol{e}_1 + |\boldsymbol{a}|\, b_2\, \boldsymbol{e}_1 \times \boldsymbol{e}_2 \\ &= |\boldsymbol{a}|\, |\boldsymbol{b}|\, \sin\theta\, \boldsymbol{e}_3 \end{aligned}$$

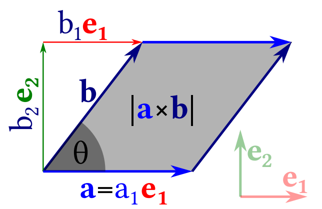


Figure 2.20: Geometric interpretation of the absolute value of the cross product.

Figure 2.20 illustrates that $|\boldsymbol{a}|\, |\boldsymbol{b}|\, \sin\theta$ amounts to the area of the parallelogram spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Hence, the cross product amounts to a vector that is aligned vertically on the parallelogram, with a length that amounts to the area of the parallelogram.

In order to evaluate also the product $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ we introduce the coordinate representation of $\boldsymbol{c}$ as $\boldsymbol{c} = c_1\, \boldsymbol{e}_1 + c_2\, \boldsymbol{e}_2 + c_3\, \boldsymbol{e}_3$ (Figure 2.21), and observe

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} &= |\boldsymbol{a} \times \boldsymbol{b}|\, \boldsymbol{e}_3 \cdot (c_1\, \boldsymbol{e}_1 + c_2\, \boldsymbol{e}_2 + c_3\, \boldsymbol{e}_3) \\ &= |\boldsymbol{a} \times \boldsymbol{b}|\, c_3 = a_1\, b_2\, c_3 \end{aligned}$$

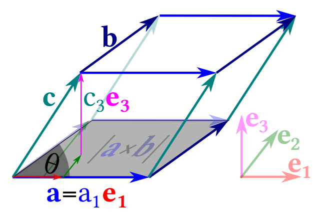


Figure 2.21: Geometric interpretation of the scalar triple product.

This amounts to the product of the area of the parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ multiplied by the height of the parallelepiped spanned by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Due to the special choice of the basis this volume amount to $a_1\, b_2\, c_3$ because all other contributions to the general expression $\sum_{ijk} a_i\, b_j\, c_k\, \varepsilon_{ijk}$ vanish. The symmetry of the scalar triple product, property d) in Definition 2.14, is understood from this perspective as the statement that the volume of the parallelepiped is invariant under (cyclic) renaming of the vectors that define its edges.

As a final remark, we emphasize that the geometric interpretation that we have given to the cross product holds in general — in spite of the special basis adopted in the derivation. It is a distinguishing feature of vector spaces that the scalar numbers that are derived from vectors take the same values every choice of the basis. It is up to the physicist to find the basis that admits the easiest calculations.

2.9.3 *The Torque*

The cross product equips us with the mathematical notions to define the torque on a body.

Definition 2.15: Torque

The *torque* $\boldsymbol{T}$ defines a force that is going to rotate a body around a position $\boldsymbol{q}_0$. Let $\boldsymbol{F}_i$ be the forces that attach the body at the positions $\boldsymbol{q}_i$ with respect to the considered origin. Then the torque is defined as

$$\boldsymbol{T} = \sum_i (\boldsymbol{q}_i - \boldsymbol{q}_0) \times \boldsymbol{F}_i$$

Remark 2.32. The value of the torque depends on the choice of the reference position $\boldsymbol{q}_0$.

Remark 2.33. In general, the torques induced by different forces point in different directions. They are added as vectors. We will further discuss this below in Example 2.24.

Axiom 2.2: Torque balance

Let N forces $\boldsymbol{F}_1, \dots, \boldsymbol{F}_N$ attack a body at the (body-fixed) positions $\boldsymbol{q}_i$. The body does not rotate around the position $\boldsymbol{q}_0$ as long as the sum of the torques induced by the forces add to zero, i.e. iff $\boldsymbol{0} = \boldsymbol{T} = \sum_{i=1}^{N} (\boldsymbol{q}_i - \boldsymbol{q}_0) \times \boldsymbol{F}_i$.

Example 2.24: Sailing boat

When a sailboat is going broad reach, as shown in Figure 2.22, the following forces are acting on the boat:

a) the wind in the sails generates a torque towards the bow around a horizontal axis that lies diagonal to the boat axis

b) the buoyancy of the water generates a torque along a horizontal axis parallel to the boat the counteracts heeling

c) the water drag on the hull generates a torque towards the bow around a horizontal axis that is orthogonal to the boat axis

d) the fin and the rudder generate lift forces that generate a torque around a vertical axis

e) the sailor stacks out in the trapeze to generate an additional torque in order to balance the torques

His aim is to minimize the heeling of the boat and to maximize the speed. The boat capsizes if he does not manage to balance the torques.

Gwicke commonswiki, public domain
Figure 2.22: A sailor stacking out in a trapeze in order to minimize the heeling of his sailboat.

2.9.4 *Self Test*

Problem 2.23. Fill in the details of the proof for Remark 2.28.

Problem 2.24. Fill in the details of the proof for Remark 2.30.

Problem 2.25. **Turning a wheel**

Two forces of magnitude 4 N are acting on a wheel of radius r that can freely rotate around its axis. What magnitude should a third force, $\boldsymbol{F}$, have that is attacking at a distance $r/2$ from the axis, such that there is no net torque acting on the wheel?

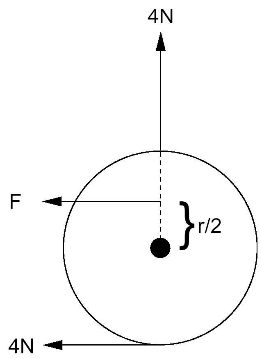


Figure 2.23: Setup for Problem 2.25.

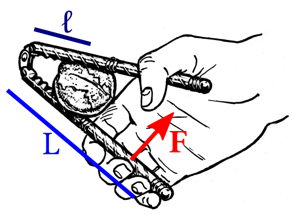
Problem 2.26. **Nutcrackers**

A common type of nutcrackers employs the principle of lever arms to crack nuts with a reasonable amount of force (see Problem 2.26). We idealize the nut as a spring with spring constant $k = 1\,\text{kN/mm}$ and assume that it breaks when it is compressed by $\Delta = 0.6\,\text{mm}$. The nut is mounted at a distance of $l = 3\,\text{cm}$ from the joint of the nutcracker and the hand exerts a force $\boldsymbol{F}$ at a distance L.

a) Demonstrate that a force of magnitude $F = \dfrac{lk\Delta}{L}$ is required to crack the nut.

b) Calculate the numerical value of F.

c) If you try to crack the nut by placing it under a heavy stone: which mass should that stone have in order to crack the nut?



based on Pearson Scott Foresman nutcracker-tool, public domain
Figure 2.24: Setup for Problem 2.26.

2.10 Worked example: Calder's mobiles

We describe here the setup of a traditional mobile where beams are supported by a string in the middle and balanced by attaching masses or further beams at their outer ends. The setup of a mobile can be laid out on a plane surface, as shown in Figure 2.25. The different parts of the mobile should not run into each other. Hence, they must not over overlap in the 2d layout.

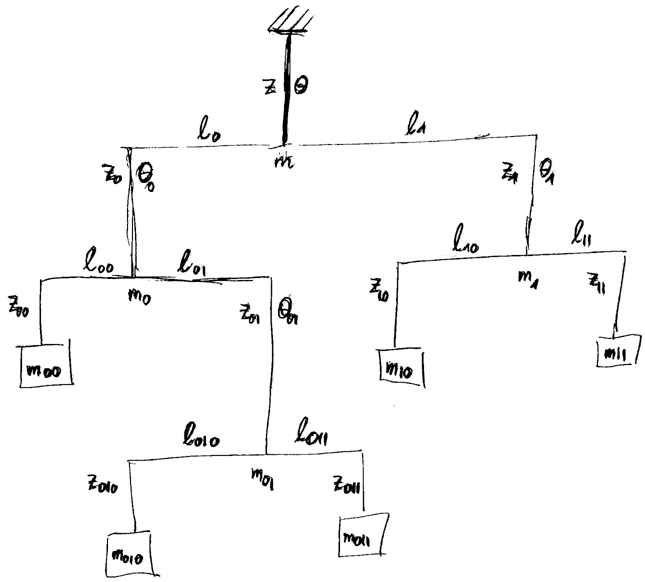


Figure 2.25: Notations for the mathematical description of the motion of a mobile. The mobile is suspended at a string of length z that holds a beam with two sections of length ℓ_0 to the left and ℓ_1 to the right, respectively. The string holds the total mass m of the mobile. When suspended, the beam can rotate by an angle θ out of the plane.

The left arm of the uppermost beam has length ℓ_0, and it holds another beam with an overall additional mass m_0 that can take an angle θ_0 out of the plane in the suspended mobile. Similarly, the right arm has length ℓ_1, and it holds another beam with an overall additional mass m_1 that can take an out-of-plane angle θ_1. The situation further down is described by hierarchical binary indices, as indicated in the figure.

The mobile can represented as a binary tree. Each beam has two arms reaching left (0) and right (1). We assume that the mass of the beams may be neglected, and reach the masses at the far ends of the mobile, by going down from the suspension and marking the track by a sequence of 0 and 1. The leftmost mass, 00, of the mobile in Figure 2.25 is reached by going left, 0, twice. The next one in counterclockwise direction by going left 0, right 1, left 0, and hence denoted as 010, and so forth. Hence, the mobile is build of beams that are labeled by some index I. They support a total mass m_I, and can rotated out of the plane by an angle θ_I. The beam has two arms of length ℓ_{I0} to the left and ℓ_{I1} to the right that support masses m_{I0} and m_{I1} attached to strings of length z_{I0} and z_{I1}. This hierarchical setup of the descriptions allows us to reduce the requirement of stability by a condition that the forces and torques acting on the beams must be balanced. For the forces this implies

$$\boldsymbol{F}_I = \boldsymbol{F}_{I0} + \boldsymbol{F}_{I1} \quad \Rightarrow \quad m_I = m_{I0} + m_{I1}$$

and for the torques we find

$$\ell_{I0} m_{I0} g = \ell_{I1} m_{I1} g \quad \Rightarrow \quad \ell_{I0} m_{I0} = \ell_{I1} m_{I1}$$

When we take all masses to take the same value m in Figure 2.25, we hence find

$$\ell_{010} = \ell_{011} \qquad \ell_{10} = \ell_{11} \qquad \ell_{00} = 2\,\ell_{01} \qquad 3\,\ell_0 = 2\,\ell_1$$

Moreover, vector calculus provides an effective means to specify the positions of the masses. We select the support of the mobile as origin of the coordinate system. The support of the uppermost beam is at position $(0, 0, -z)$. Then the far ends of the uppermost beam are at positions $\boldsymbol{l}_o = (-\ell_0 \cos\theta, -\ell_0 \sin\theta, -z)$ and $\boldsymbol{l}_1 = (\ell_1 \cos\theta, -\ell_1 \sin\theta, -z)$, respectively. Moreover, from the left end we reach the far ends of the next beam by the displacement vectors $\boldsymbol{l}_{00} = (-\ell_{00} \cos\theta_0, -\ell_{00} \sin\theta_0, -z_0)$ and $\boldsymbol{l}_{01} = (\ell_{01} \cos\theta_0, \ell_{01} \sin\theta_0, -z_0)$. Hence, the positions of the first two masses can be represented by the following sums of vectors

$$\boldsymbol{q}_{00} = \boldsymbol{l}_0 + \boldsymbol{l}_{00} - \begin{pmatrix} 0 \\ 0 \\ z_{00} \end{pmatrix} = \begin{pmatrix} -\ell_0 \cos\theta - \ell_{00} \cos\theta_0 \\ -\ell_0 \sin\theta - \ell_{00} \sin\theta_0 \\ -z - z_0 - z_{00} \end{pmatrix}$$

$$\boldsymbol{q}_{010} = \boldsymbol{l}_0 + \boldsymbol{l}_{01} + \boldsymbol{l}_{010} - \begin{pmatrix} 0 \\ 0 \\ z_{010} \end{pmatrix} = \begin{pmatrix} -\ell_0 \cos\theta + \ell_{01} \cos\theta_0 - \ell_{010} \cos\theta_{01} \\ -\ell_0 \sin\theta + \ell_{01} \sin\theta_0 - \ell_{010} \sin\theta_{01} \\ -z - z_0 - z_{01} - z_{010} \end{pmatrix}$$

We urge the reader to also work out the expressions for the positions of the other masses.

add discussion and stability analysis for bended beams

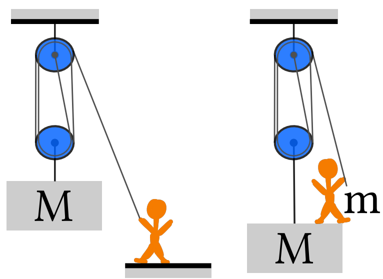
2.11 *Problems*

2.11.1 *Rehearsing Concepts*

Problem 2.27. **Tackling tackles and pulling pulleys**

a) Which forces are required to hold the balance in the left and the right sketch?

b) Let the sketched person and the weight have masses of $m = 75\,\text{kg}$ and $M = 300\,\text{kg}$, respectively. Which power is required then to haul the line at a speed of $1\,\text{m/s}$.
Hint: The power is defined here as the change of) $Mg\,z(t)$ and $(M + m)\,g\,z(t)$, per unit time, respectively. Verify by dimensional analysis that this is a meaningful definition.



2.11.2 *Practicing Concepts*

Problem 2.28. **Angles between three balanced forces**

We consider three masses m_1, m_2, and m_3. With three ropes they are attached to a ring at position $\boldsymbol{q}_0$. The ropes with the attached masses hang over the edge of a table at the fixed positions $\boldsymbol{q}_1 = (x_1, 0)$, $\boldsymbol{q}_2 = (0, y_2)$, and $\boldsymbol{q}_3 = (w, y_3)$. Here, w denotes the width of the table board. We now determine the angles θ_{ij} between the ropes from $\boldsymbol{q}_0$ to $\boldsymbol{q}_i$ and $\boldsymbol{q}_j$, respectively.

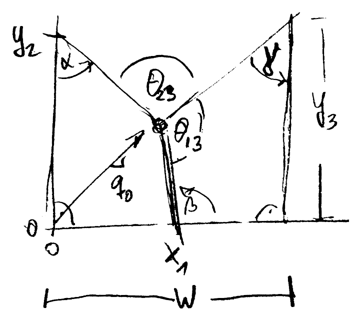


Figure 2.26: Setup of Problem 2.28.

a) Let $\hat{\boldsymbol{e}}_i = (\boldsymbol{q}_i - \boldsymbol{q}_0)/|\boldsymbol{q}_i - \boldsymbol{q}_0|$ be the unit vectors pointing from the ring to the positions where the ropes hang over the table edge, and θ_{ij} be the angle between $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}_j$. Argue why

$$\mathbf{0} = \sum_{i=1}^{3} m_i \, \hat{\boldsymbol{e}}_i$$

Multiplying this equation with $\hat{\boldsymbol{e}}_1, \dots \hat{\boldsymbol{e}}_3$ provides three equations that are linear in $\cos\theta_{ij}$. The first one is $0 = M_1 + M_2 \cos\theta_{12} + M_3 \cos\theta_{13}$. Find the other two equation, and solve the equations as follows.
From the equation that is given above you find $\cos\theta_{12}$ in terms of $\cos\theta_{13}$.
Inserting this into the other equation involving $\cos\theta_{12}$ (and rearranging terms) provides $\cos\theta_{23}$ in terms of $\cos\theta_{13}$.
Inserting this into the third equation provides

$$\cos\theta_{13} = \frac{M_2^2 - M_1^2 - M_3^2}{2\, M_1\, M_3}$$

b) Which angle θ_{23} do you find when $M_1 = M_2 = M_3$? The three forces have the same absolute value in this case. Which symmetry argument does then also provide the value of the angle?

c) Determine also the other two angles θ_{13} and θ_{12}. They can also be found from a symmetry argument without calculation.
Hint: The angles do not care which mass you denote as 1, 2, and 3.

d) Note that we found the angles θ_{ij} without referring to the positions $\boldsymbol{q}_1, \dots \boldsymbol{q}_3$! Make a sketch what this implies for the position of the ring, and how $\boldsymbol{q}_0$ changes qualitatively upon changing a mass.

The calculation of the position $\boldsymbol{q}_0$ can then be attacked by observing that

$$\boldsymbol{q}_0 = \boldsymbol{q}_1 + l_1 \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = \boldsymbol{q}_2 + l_2 \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix} = \boldsymbol{q}_3 + l_3 \begin{pmatrix} -\sin\gamma \\ -\cos\gamma \end{pmatrix}$$

where l_i is the distance of the ring to the position where rope i hangs over the table. Further, the fact that the angles of quadrilaterals add to 2π provides

$$\alpha = \theta_{23} - \gamma \qquad \text{and} \qquad \beta = \frac{3\pi}{2} - \gamma - \theta_{13}$$

Altogether these are 8 equations to determine the two components of $\boldsymbol{q}_0$, $l_1, \dots l_3$, and the angles α, β, and γ. Determine $\boldsymbol{q}_0$.

Problem 2.29. **Torques acting on a ladder**

The sketch in the margin shows the setup of a ladder leaning to the roof of a hut. The indicated angle from the downwards vertical to the ladder is denoted as θ. There is a gravitational force of magnitude Mg acting of a ladder of mass M. At the point where it leans to the roof there is a normal force of magnitude F_r acting from the roof to the ladder. At the ladder feet there is a normal force to the ground of magnitude F_g, and a tangential friction force of magnitude γF_f. This is again the sketch to the ladder leaning to the roof of a hut. The angle from the downwards vertical to the ladder is denoted θ. There is a gravitational force of magnitude Mg acting of a ladder. At the point where it leans to the roof there is a normal force of magnitude F_r. At the ladder feet there is a normal force to the ground of magnitude F_g, and a tangential friction force of magnitude F_f.

change to problem given on homework sheet 3.

a) In principle there also is a friction force $\gamma_r F_r$ acting at the contact from the ladder to the roof. Why is it admissible to neglect this force?
Remark: There are at least two good arguments.

b) Determine the vertical and horizontal force balance for the ladder. Is there a unique solution?

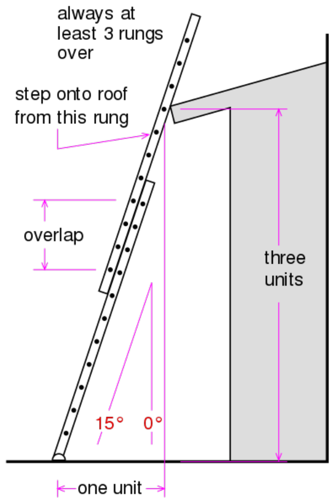
c) The feet of the ladder start sliding when F_f exceeds the maximum static friction force γF_g. What does this condition entail for the angle θ?
Assume that $\gamma \simeq 0.3$ What does this imply for the critical angle θ_c.

d) Where does the mass of the ladder enter the discussion? Do you see why?

e) Determine the torque acting on the ladder. Does it matter whether you consider the torque with respect to the contact point to the roof, the center of mass, or the foot of the ladder?

f) The ladder slides when the modulus of the friction force F_f exceeds a maximum value $\mu_S F_g$ where μ is the static friction coefficient for of the ladder feet on the ground. For metal feet on a wooden ground it takes a value of $\mu_S \simeq 2$. What does that tell about the angels where the ladder starts to slide?

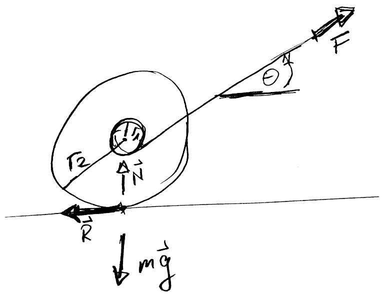
g) Why does a ladder commonly starts sliding when when a man has climbed to the top? Is there anything one can do against it? Is that even true, or just an urban legend?



original: Bradley, vector: Sarang / wikimedia public domain
Figure 2.27: Setup for Problem 2.29: leaning a ladder to a roof.

Problem 2.30. **Walking a yoyo**

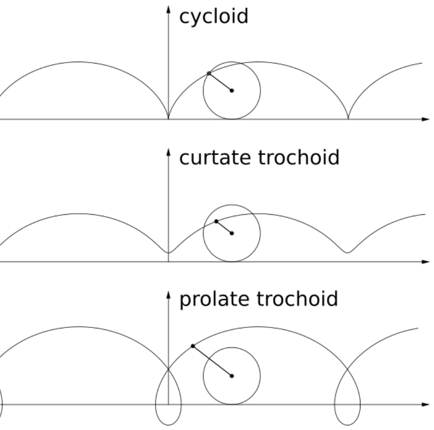
The sketch to the right shows a yoyo of mass m standing on the ground. It is held at a chord that extends to the top right. There are four forces acting on the yoyo: gravity mg, a normal force $\boldsymbol{N}$ from the ground, a friction force $\boldsymbol{R}$ at the contact to the ground, and the force $\boldsymbol{F}$ due to the chord. The chord is wrapped around an axle of radius r_1. The outer radius of the yoyo is r_2.



a) Which conditions must hold such that there is no net force acting on the center of mass of the yoyo?

b) For which angle θ does the torque vanish?

c) Perform an experiment: What happens for larger and for smaller angels θ? How does the yoyo respond when fix the height where you keep the chord and pull continuously?



based on Kmhkmh Zykloiden, CC BY 4.0

Problem 2.31. **Retro-reflector paths on bike wheels**
The more traffic you encounter when it becomes dark the more important it becomes to make your bikes visible. Retro-reflectors fixed in the sparks enhance the visibility to the sides. They trace a path of a curtate trochoid that is characterized by the ratio ρ of the reflectors distance d to the wheel axis and the wheel radius r. A small stone in the profile traces a cycloid ($\rho = 1$). Animations of the trajectories can be found at
`https://en.wikipedia.org/wiki/Trochoid` and `http://katgym.by.lo-net2.de/c.wolfseher/web/zykloiden/zykloiden.html`.

A trochoid is most easily described in two steps: Let $\boldsymbol{M}(\theta)$ be the position of the center of the disk, and $\boldsymbol{D}(\theta)$ the vector from the center to the position $\boldsymbol{q}(\theta)$ that we follow (i.e. the position of the retro-reflector) such that $\boldsymbol{q}(\theta) = \boldsymbol{M}(\theta) + \boldsymbol{D}(\theta)$.

a) The point of contact of the wheel with the street at the initial time t_0 is the origin of the coordinate system. Moreover, we single out one spark and denote the change of its angle with respect to its initial position as θ. Note that negative angles θ describe forward motion of the wheel!

Sketch the setup and show that

$$\boldsymbol{M}(\theta) = \begin{pmatrix} -r\theta \\ r \end{pmatrix}, \qquad \boldsymbol{D}(\theta) = \begin{pmatrix} -d\ \sin(\varphi + \theta) \\ d\ \cos(\varphi + \theta) \end{pmatrix}.$$

What is the meaning of φ in this equation?

check signs of components of $\boldsymbol{D}$

b) The length of the track of a trochoid can be determined by integrating the modulus of its velocity over time, $L = \int_{t_0}^{t} \mathrm{d}t\ |\dot{\boldsymbol{q}}(\theta(t))|$. Show that therefore

$$L = r \int_0^{\theta} \mathrm{d}\theta\ \sqrt{1 + \rho^2 + 2\rho\ \cos(\varphi + \theta)}$$

c) Consider now the case of a cycloid and use $\cos(2x) = \cos^2 x - \sin^2 x$ to show that the expression for L can then be written as

$$L = 2\,r \int_0^{\theta} \mathrm{d}\theta\ \left|\cos\frac{\varphi + \theta}{2}\right|$$

How long is one period of the track traced out by a stone picked up by the wheel profile?

2.11.3 *Mathematical Foundation*

Problem 2.32. **The natural numbers modulo n are a group**

We consider here groups G_n where the combined action of group elements can be represented as a sum of two numbers modulo $n \in \mathbb{N}$. In other words, for the elements of G_n can be represented by the numbers $\{0, \dots, n-1\}$, and for all $a, b \in G_n$ we define $a \circ b = (a+b) \mathrm{mod} n$.

a) Show that G_n is a group.

b) Show that G_n represents the rotations that interchange the vertices of a regular n-sided polygon.

Problem 2.33. **Groups with four elements**

In Problem 2.32 we encountered the group G_n. Here, we will study another group with four elements. The neutral element will be denoted as n.

a) Show that the group has at least one non-trivial element e that is self-inverse, $e \circ e = n$.
Remark: Non-trivial means here that $e \neq n$.

b) Show that the group is isomorphic to G_4 if there is exactly one non-trivial element that is self-inverse. In other words: the group elements can be represented in that case by the numbers $\{0, \dots, 3\}$, and the operation of the group on two of its elements yields the same result as the action of G_4 on the corresponding numbers.

c) Show that the group is isomorphic to G_4 if there is at least one element that is not self-inverse.

d) Determine the group table for the case where all group elements are self-inverse. Show that it is unique, and that it is isomorphic to the symmetry group of rectangles (cf. Problem 2.6).

e) Proof that all groups with four elements are commutative by representing the group elements in terms of generating elements. Do not refer to the group table.

Problem 2.34. **Conic Sections**

A conic section describes the line of intersection of a double cone C and a plane P in three dimensions. In the margin we show the shape of conic sections for different inclinations that are characterized by the eccentricity ϵ. Depending on the inclination of the plane one observes

- a circle, when the axis of the cone is orthogonal to P, i.e. for $\epsilon = 0$,
- an ellipse, when the plane is slightly tilted, $\epsilon < 1$,

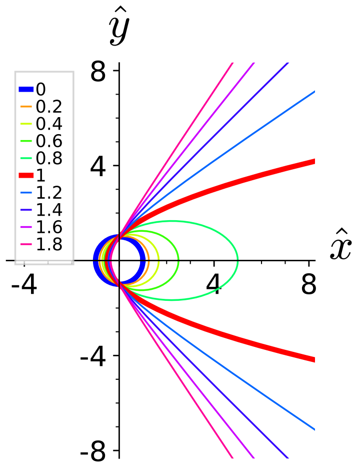


Figure 2.28: Conic sections for different eccentricity ϵ. i.e., the ratio of the slope of the plane P and the surface of the double cone, as observed in a plane that contains the axis of the double cone and is orthogonal to P.

- a parabola, when its inclination matches with the opening angle of the cone, $\epsilon = 1$, and
- a hyperbola, when it intersects with both sides of the double cone, $\epsilon > 1$.

a) Sketch the different types of intersection of the double cone and the plane.

b) Determine the vector $\boldsymbol{a}$ that points from the vertex of the double cone to the point where the plane intersects the axis of the double cone.

c) Describe the points in the intersection as sum of $\boldsymbol{a}$ and a vector $\boldsymbol{b}$ that lies in the plane.

d) Determine the length of the vector $\boldsymbol{b}$ as function of the angle θ that characterizes the direction of $\boldsymbol{b}$ in P. How can this expression be used to plot the functions shown in Figure 4.19.

Problem 2.35. **Linear dependence of three vectors in 2D**

In the lecture I pointed out that every vector $\boldsymbol{v} = (v_1, v_2)$ of a two-dimensional vector space can be represented as a *unique* linear combination of two linearly independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{v} = \alpha\, \boldsymbol{a} + \beta\, \boldsymbol{b}$$

In this exercise we revisit this statement for $\mathbb{R}^2$ with the standard forms of vector addition and multiplication by scalars.

a) Provide a triple of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{v}$ such that $\boldsymbol{v}$ can *not* be represented as a scalar combination of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b) To be specific we henceforth fix

$$\boldsymbol{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Determine the numbers α and β such that

$$\boldsymbol{v} = \alpha\, \boldsymbol{a} + \beta\, \boldsymbol{b}$$

c) Consider now also a third vector

$$\boldsymbol{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and find two different choices for (α, β, γ) such that

$$\boldsymbol{v} = \alpha\, \boldsymbol{a} + \beta\, \boldsymbol{b} + \gamma \boldsymbol{c}$$

What is the general constraints on (α, β, γ) such that $\boldsymbol{v} = \alpha\, \boldsymbol{a} + \beta\, \boldsymbol{b} + \gamma \boldsymbol{c}$.
What does this imply on the number of solutions?

d) Discuss now the linear dependence of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ by exploring the solutions of

$$\mathbf{0} = \alpha\, \boldsymbol{a} + \beta\, \boldsymbol{b} + \gamma \boldsymbol{c}$$

How are the constraints for the null vector related to those obtained in part c)?

Problem 2.36. **Algebraic number fields**

Consider the set $\mathbb{K} = \mathbb{Q} + \mathrm{I}\mathbb{Q}$ with $\mathrm{I}^2 \in \mathbb{Q}$. We define the operations $+$ and $\cdot$ in analogy to those of the complex numbers (cf. Example 2.13): For $z_1 = x_1 + \mathrm{I}y_1$ and $z_2 = x_2 + \mathrm{I}y_2$ we have $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ and

$$\begin{aligned} \forall z_1, z_2 \in \mathbb{K} : z_1 + z_2 &= (x_1 + x_2) + \mathrm{I}\,(y_1 + y_2) \\ z_1 \cdot z_2 &= (x_1\, x_2 + \mathrm{I}^2\, y_1 y_2) + \mathrm{I}\,(x_1\, y_2 + y_1\, x_2) \end{aligned}$$

$$\forall c \in \mathbb{Q}, z = (x + \mathrm{i}y) \in \mathbb{K} : cz = c\, x + \mathrm{I}\, y$$

a) Let I be a rational number, $\mathrm{I} \in \mathbb{Q}$. Show that $\mathbb{K} = \mathbb{Q}$.

b) Consider $\mathrm{I} = \sqrt{10}$. Show that $\mathbb{K}$ is a field that is different from $\mathbb{Q}$.

c) Consider $\mathrm{I} = \sqrt{8}$. In this case $\mathbb{K}$ is *not* a field! Why?

Find the general rule: For which natural numbers n does $\mathrm{I} = \sqrt{n}$ provide a non-trivial field?
Remark: Non-trivial means here different from $\mathbb{Q}$.

Problem 2.37. **Bases for polynomials**

We consider the set of polynomials $\mathbb{P}_N$ of degree N with real coefficients p_n, $n \in \{0, \dots, N\}$,

$$\mathbb{P}_N := \left\{ \boldsymbol{p} = \left(\sum_{k=0}^{N} p_k\, x^k \right) \quad \text{mit} \;\; p_k \in \mathbb{R}, k \in \{0, \dots, N\} \right\}$$

a) Demonstrate that $(\mathbb{P}_N, \mathbb{R}, +, \cdot)$ is a vector space when one adopts the operations

$$\forall \quad \boldsymbol{p} = \left(\sum_{k=0}^{N} p_k\, x^k \right) \in \mathbb{P}_N\,, \quad \boldsymbol{q} = \left(\sum_{k=0}^{N} q_k\, x^k \right) \in \mathbb{P}_N\,, \text{ and } \; c \in \mathbb{R} \;:$$

$$\boldsymbol{p} + \boldsymbol{q} = \left(\sum_{k=0}^{N} (p_k + q_k)\, x^k \right) \quad \text{and} \quad c \cdot \boldsymbol{p} = \left(\sum_{k=0}^{N} (c\, p_k)\, x^k \right) .$$

(b) Demonstrate that

$$\boldsymbol{p} \cdot \boldsymbol{q} = \left(\int_0^1 \mathrm{d}x \left(\sum_{k=0}^{N} p_k\, x^k \right) \left(\sum_{j=0}^{N} q_j\, x^j \right) \right) ,$$

establishes a scalar product on this vector space.

(c) Demonstrate that the three polynomials $\boldsymbol{b}_0 = (1)$, $\boldsymbol{b}_1 = (x)$ and $\boldsymbol{b}_2 = (x^2)$ form a basis of the vector space $\mathbb{P}_2$: For each polynomial $\boldsymbol{p}$ in $\mathbb{P}_2$ there are real numbers x_k, $k \in \{0, 1, 2\}$, such that $\boldsymbol{p} = x_0\, \boldsymbol{b}_0 + x_1\, \boldsymbol{b}_1 + x_2\, \boldsymbol{b}_2$. However, in general we have $x_i \neq \boldsymbol{p} \cdot \boldsymbol{b}_i$. Why is that?
Hint: Is this an orthonormal basis?

(d) Demonstrate that the three vectors $\hat{\boldsymbol{e}}_0 = (1)$, $\hat{\boldsymbol{e}}_1 = \sqrt{3}\,(2\,x - 1)$ and $\hat{\boldsymbol{e}}_2 = \sqrt{5}\,(6\,x^2 - 6\,x + 1)$ are orthonormal.

(e) Demonstrate that every vector $\boldsymbol{p} \in \mathbb{P}_2$ can be written as a scalar combination of $(\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$,

$$\boldsymbol{p} = (\boldsymbol{p} \cdot \hat{\boldsymbol{e}}_0)\, \hat{\boldsymbol{e}}_0 + (\boldsymbol{p} \cdot \hat{\boldsymbol{e}}_1)\, \hat{\boldsymbol{e}}_1 + (\boldsymbol{p} \cdot \hat{\boldsymbol{e}}_2)\, \hat{\boldsymbol{e}}_2\,.$$

Hence, $(\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ form an orthonormal basis of $\mathbb{P}_2$.

*(f) Find a constant c and a vector $\hat{\boldsymbol{n}}_1$, such that $\hat{\boldsymbol{n}}_0 = (c\,x)$ and $\hat{\boldsymbol{n}}_1$ form an orthonormal basis of $\mathbb{P}_1$.

Problem 2.38. **Systems of linear equations**

A system of N linear equations of M variables $x_1, \dots x_M$ comprises N equations of the form

$$\begin{aligned} b_1 &= a_{11}\, x_1 + a_{12}\, x_2 + \dots + a_{1M}\, x_M \\ b_2 &= a_{21}\, x_1 + a_{22}\, x_2 + \dots + a_{2M}\, x_M \\ \vdots\; &\;\; \vdots \\ b_N &= a_{N1}\, x_1 + a_{N2}\, x_2 + \dots + a_{NM}\, x_M \end{aligned}$$

where $b_i, a_{ij} \in \mathbb{R}$ for $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$.

a) Demonstrate that the linear equations $(\mathbb{L}_M, \mathbb{R}, +, \cdot)$ form a vector space when one adopts the operations

$$\begin{aligned} \forall \quad \boldsymbol{p} &= \left[p_0 = p_1\, x_1 + p_2\, x_2 + \dots + p_M\, x_M\right] \in \mathbb{L}_N\,, \\ \boldsymbol{q} &= \left[q_0 = q_1\, x_1 + q_2\, x_2 + \dots + q_M\, x_M\right] \in \mathbb{L}_N\,, \\ c &\in \mathbb{R} \;: \end{aligned}$$

$$\begin{aligned} \boldsymbol{p} + \boldsymbol{q} &= \left[p_0 + q_0 = (p_1 + q_1)\, x_1 + (p_2 + q_2)\, x_2 + \dots + (p_M + q_M)\, x_M\right] \\ c \cdot \boldsymbol{p} &= \left[c\, p_0 = c\, p_1\, x_1 + c\, p_2\, x_2 + \dots + c\, p_M\, x_M\right]. \end{aligned}$$

How do these operations relate to the operations performed in Gauss elimination to solve the system of linear equations?

b) The system of linear equations can also be stated in the following form

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} x_2 + \dots + \begin{pmatrix} a_{1M} \\ a_{2M} \\ \vdots \\ a_{NM} \end{pmatrix} x_M$$

$$\Leftrightarrow \qquad \boldsymbol{b} = x_1\, \boldsymbol{a}_1 + x_2\, \boldsymbol{a}_2 + \dots + x_M\, \boldsymbol{a}_M$$

where $\boldsymbol{b}$ is expressed as a linear combination of $\boldsymbol{a}_1, \ldots \boldsymbol{a}_M$ by means of the numbers $x_1, \ldots, x_M$. What do the conditions on linear independence and representation of vectors by means of a basis tell about the existence and uniqueness of the solutions of a system of linear equations.

2.11.4 *Transfer and Bonus Problems, Riddles*

Problem 2.39. **Crossing a river**

A ferry is towed at the bank of a river of width $B = 100$ m that is flowing at a velocity $v_F = 4$ m/s to the right. At time $t = 0$ s it departs and is heading with a constant velocity $v_B = 10$ km/h to the opposite bank.

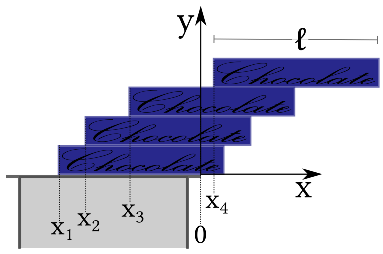
a) When will it arrive at the other bank when it always heads straight to the other side? (In other words, at any time its velocity is perpendicular to the river bank.)

   How far will it drift downstream on its journey?

b) In which direction (i.e. angle of velocity relative to the downstream velocity of the river) must the ferryman head to reach exactly at the opposite side of the river?

   Determine first the general solution. What happens when you try to evaluate it for the given velocities?

Problem 2.40. **Piling bricks**

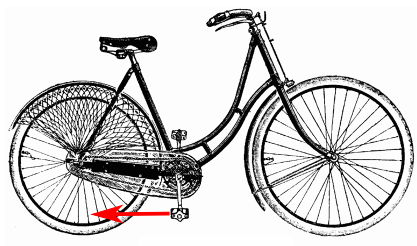


At Easter and Christmas Germans consume enormous amounts of chocolate. If you happen to come across a considerable pile of chocolate bars (or beer mats, or books, or anything else of that form) I recommend the following experiment:

a) We consider N bars of length l piled on a table. What is the maximum amount that the topmost bar can reach beyond the edge of the table.

b) The sketch above shows the special case $N = 4$.
   However, what about the limit $N \to \infty$?

Problem 2.41. **Where does the bike go?**



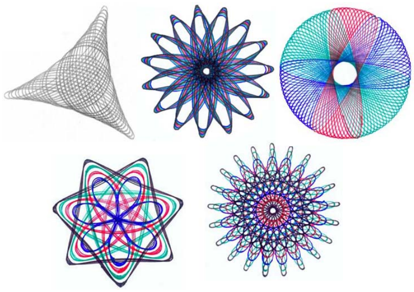
adapted from picture "Damenfahrrad von 1900" in article "Fahrrad" of Lueger (1926–1931)

Consider the picture of the bicycle to the left. The red arrow indicates a force that is acting on the paddle in backward direction.

Will the bicycle move forwards or backwards?

Take a bike and do the experiment!

Problem 2.42. **Hypotrochoids, roulettes, and the Spirograph**


wikimedia, public domain

A roulette is the curve traced by a point (called the generator or pole) attached to a disk or other geometric object when that object rolls without slipping along a fixed track. A pole on the circumference of a disk that rolls on a straight line generates a cycloid. A pole inside that disk generates a trochoid. If the disk rolls along the inside or outside of a circular track it generates a hypotrochoid. The latter curves can be drawn with a spirograph, a beautiful drawing toy based on gears that illustrates the mathematical concepts of the least common multiple (LCM) and the lowest common denominator (LCD).

a) Consider the track of a pole attached to a disk with n cogs that rolls inside a circular curve with $m > n$ cogs. Why does the resulting curve form a closed line? How many revolutions does the disk make till the curve closes? What is the symmetry of the resulting roulette? (The curves to the top left is an examples with three-fold symmetry, and the one to the bottom left has seven-fold symmetry.)

b) Adapt the description for the curves developed in Problem 2.31 such that you can describe hypotrochoids.

c) Test your result by writing a Python program that plots the curves for given m and n.

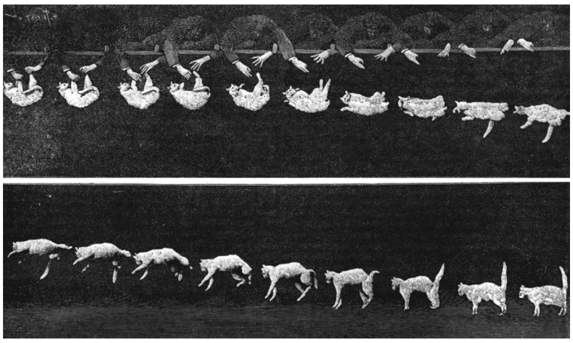
2.12 *Further reading*

The second chapter of Großmann (2012) provides a clear and concise introduction to the mathematical framework of vectors with an emphasis on applications to physics problems.

A nice discussion of force and torque balances with many worked exercises can be found in Chapter 2 of Morin (2007).

3
Newton's Laws

In Chapter 2 we explored how several forces that act on a body can be subsumed into a net total force and torque. The body stays in rest, say at position $\boldsymbol{q}_0$, when the net force and torque vanish. Now we explore how the forces induce motion and how the position of the body evolves in time, $\boldsymbol{q}(t)$, when it is prepared with an initial condition $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ at the initial time t_0.



Photographs of a Tumbling Cat. Nature 51, 80–81 (1894)

At the end of this chapter we will be able to discuss the likelihood for injuries in different types of accidents, be it men or cat or mice. Why do the cats go away unharmed in most cases when they fall from a balcony, while an old professor should definitely avoid such a fall. As a worked example we will discuss water rockets.

3.1 Motivation and outline: What is causing motion?

Every now and then I make the experience that I sit in a train, reading a book. Then I look out of the window, realize that we are passing a train, feeling happy that we are further approaching my final destination; and then I realize that the train is moving and my train is still in the station. Indeed, the motion of objects in my compartment is exactly identical, no matter whether it is at rest or moves with a constant velocity; be it zero in the station, at 15 m/s in a local commuter train, or 75 m/s in a Japanese high-speed train. However, changes of velocity matter. I forcefully experience the change of speed of the train during an emergency break, and coffee is spilled when it takes too sharp a turn.

Modern physics was born when Galileo and Newton formalized this experience by saying that bodies (e.g. the set of bodies in the compartment of a train) move in a straight line with a constant velocity as long as there is no net force acting on the bodies, and that the change of its velocity is proportional to the applied force.

Outline

In the first part of this chapter we will relate temporal changes of positions and velocities to time derivatives. Subsequently, we can formulate equations of motion that relate these changes to forces. The last part of the chapter deals with strategies to find solutions by making use of conservation laws.

mass	m
position	$\boldsymbol{q}(t)$
velocity	$\dot{\boldsymbol{q}}(t), \boldsymbol{v}(t)$
acceleration	$\ddot{\boldsymbol{q}}(t)$
forces	$\boldsymbol{F}_\alpha(\boldsymbol{q}, t)$

Table 3.1: Notations adopted to describe the motion of a particle. A single dot denotes the time derivative, and double dot the second derivative with respect to time.

3.2 Time derivatives of vectors

In this section we consider the motion of a particle with mass m that is at position $\boldsymbol{q}(t)$ at time t. Its average velocity $\boldsymbol{v}_{\text{av}}(t, \Delta t)$ during the time interval $[t, t + \Delta t]$ is

$$\boldsymbol{v}_{\text{av}}(t, \Delta t) = \frac{\boldsymbol{q}(t + \Delta t) - \boldsymbol{q}(t)}{\Delta t}$$

When the limit $\lim_{\Delta t \to 0} \boldsymbol{v}_{\text{av}}(t, \Delta t)$ exists[1] we can define the velocity of the particle at time t,

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}(t + \Delta t) - \boldsymbol{q}(t)}{\Delta t} \tag{3.2.1}$$

[1] The discussion of this limit for general functions is a core topic of vector calculus. For our present purpose the intuitive understanding based on the idea that $\boldsymbol{q}(t + \Delta t) \simeq \boldsymbol{q}(t) + \Delta t \; \boldsymbol{v}(t)$ provides the right idea. To provide a hint for the origin of the mathematical subtleties we point out that the approximation works unless there is an *instantaneous* collision with a wall at some point in the time interval $]t, t + \Delta t[$. In physics we try our luck, and fix the problem when we face it. Indeed, upon a close look there are no instantaneous collisions in physics, see Problem 3.17.

The velocity is then the time derivative of the position, and in an immediate generalization of the time derivative of scalar functions we also write

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t}$$

Finally, we point out that the components of the time derivative of a vector amount to the derivatives of the components.

Theorem 3.1: Time derivatives of vectors

Let $\boldsymbol{a}(t)$ be a vector with time-dependent components $a_i(t)$ with respect to orthonormal basis $\{\hat{\boldsymbol{e}}_i,\ i = 1 \cdots D\}$ that is fixed in time.
Then $\dot{\boldsymbol{a}}(t) = \sum_i \dot{a}_i(t)\, \hat{\boldsymbol{e}}_i$. The components of $\dot{\boldsymbol{a}}(t)$ amount to the time derivatives of the components of $\boldsymbol{a}(t)$.

Proof. For each time we have $\boldsymbol{a}(t) = \sum_i a_i(t)\, \hat{\boldsymbol{e}}_i$ where it is understood that the sum runs over $i = 1 \cdots D$. We insert this into the definition, Equation (3.2.1), of the the time derivative and use the linearity of scalar products with vectors to obtain

$$\begin{aligned}
\dot{\boldsymbol{a}}(t) &= \lim_{\Delta t \to 0} \frac{\boldsymbol{a}(t+\Delta t) - \boldsymbol{a}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sum_i a_i(t+\Delta t)\, \hat{\boldsymbol{e}}_i - \sum_i a_i(t)\, \hat{\boldsymbol{e}}_i}{\Delta t} \\
&= \lim_{\Delta t \to 0} \sum_i \hat{\boldsymbol{e}}_i \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} = \sum_i \hat{\boldsymbol{e}}_i \lim_{\Delta t \to 0} \frac{a_i(t+\Delta t) - a_i(t)}{\Delta t} \\
&= \sum_i \hat{\boldsymbol{e}}_i\, \dot{a}_i(t)
\end{aligned}$$

The subtle step here, from a mathematical point of view, is the swapping of the limit and the sum in the second line of the argument. Courses on vector calculus will spell out the assumptions needed to justify this step (or, more interestingly from a physics perspective, under which conditions it fails). □

The change of the velocity will be denoted as acceleration. Based on an analogous argument as for the velocity, it will be written as a time derivative

Definition 3.1: Acceleration

The time derivative of the velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{q}}(t)$ is denoted as *acceleration*, and written as

$$\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{q}}(t)$$

In the next section it will be related to the action of forces $\boldsymbol{F}(\boldsymbol{q}, t)$ acting on a particle that resides at the position $\boldsymbol{q}$ at time t.

3.2.1 *Self Test*

Problem 3.1. **Derivatives of elementary functions**

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^x = \mathrm{e}^x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \ln x = x^{-1}$$

Use only the three rules for derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)+g(x)\big) = f'(x)+g'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) = f'(x)\,g(x)+f(x)\,g'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big) = g'(x)\,f'\big(g(x)\big)$$

to work out the following derivatives

a) $\sinh x = \frac{1}{2}\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)$

b) $\cos x = \sin(\pi/2+x)$

c) $x^{a} = \mathrm{e}^{a\ln x}$ for $a\in\mathbb{R}$
What does this imply for the derivative of $f(x)=x^{-1}$?

d) Use the result from (c) to proof the quotient rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)\,g(x)-f(x)\,g'(x)}{\big(g(x)\big)^{2}}$$

e) $\tan x = \dfrac{\sin x}{\cos x}$ and $\tanh x = \dfrac{\sinh x}{\cosh x}$

f) Find the derivative of $\ln x$ solely based on $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x}=\mathrm{e}^{x}$.
Hint: Use that $x=\mathrm{e}^{\ln x}$ and take the derivative of both sides.

Problem 3.2. **Integrals of elementary functions**

In a moment we will also perform integrals to determine the work performed on a body when it is moving subject to a force. Practice you skills by evaluating the following integrals.

a) $\displaystyle\int_{-1}^{1}\mathrm{d}x\,(a+x)^{2}$ c) $\displaystyle\int_{0}^{\infty}\mathrm{d}x\,\mathrm{e}^{-x/L}$ f) $\displaystyle\int_{0}^{\infty}\mathrm{d}x\,x\,\mathrm{e}^{-x^{2}/(2Dt)}$

b) $\displaystyle\int_{-5}^{5}\mathrm{d}q\,(a+b\,q^{3})$ d) $\displaystyle\int_{-L}^{L}\mathrm{d}y\,\mathrm{e}^{-y/\xi}$ g) $\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}}\mathrm{d}\ell\,\ell\,\mathrm{e}^{-\ell^{2}/(2Dt)}$

$\displaystyle\int_{0}^{B}\mathrm{d}k\,\tanh^{2}(kx)$ e) $\displaystyle\int_{0}^{L}\mathrm{d}z\,\frac{z}{a+b\,z^{2}}$ $\displaystyle\int_{-\sqrt{Dt}}^{\sqrt{Dt}}\mathrm{d}z\,x\,\mathrm{e}^{-z\,x^{2}}$

Except for the integration variable all quantities are considered to be constant.

Hint: Sometimes symmetries can substantially reduce the work needed to evaluate an integral.

3.3 *Newton's axioms and equations of motion (EOM)*

In Section 4.1 we referred to a train compartment to point out that physical observations will be the same — irrespective of the velocity of its motion, as long as it is constant. A setting where we perform an experiment is denoted as reference frame, and reference frames that move with constant velocity are called inertial systems.

Definition 3.2: Reference Frames and Inertial Systems

A *reference frame* $(\boldsymbol{Q}, \{\hat{\boldsymbol{e}}_i(t),\ i = 1 \cdots D\})$ is an agreement about the, in general time dependent, position of the origin $\boldsymbol{Q}(t)$ of the coordinate system and a set of orthonormal basis vectors $\{\hat{\boldsymbol{e}}_i(t),\ \ i\ =\ 1 \cdots D\}$, that are adopted to indicate the positions of particles in a physical model.
The reference frame refers to an *inertial system* when it does not rotate and when it moves with a constant velocity, i.e. if and only if $\ddot{\mathbf{Q}} = \mathbf{0}$ and $\dot{\hat{\boldsymbol{e}}}_i = \mathbf{0}$ for all $i \in \{1 \cdots D\}$.

Remark 3.1. The requirement $\dot{\hat{\boldsymbol{e}}}_i = \mathbf{0}$ implies that the orientation of the basis vectors $\hat{\boldsymbol{e}}_i$ does not change, i.e. the reference frame does not rotate.

Remark 3.2. The *rest frame* for a particle is a reference frame where the particle velocity takes the constant velocity $\mathbf{0}$.

Remark 3.3. Let $\boldsymbol{q} = (q_1, \dots q_D)$ be the coordinates of a particles, as specified in in the inertial frame $(\boldsymbol{Q}, \{\hat{\boldsymbol{e}}_i\})$, and $\boldsymbol{x} = (x_1, \dots x_D)$ its position given in the inertial frame $(\boldsymbol{X}, \{\hat{\boldsymbol{n}}_i\})$. Then

$$\boldsymbol{q} = \boldsymbol{Q} + \sum_{i=1}^{D} q_i\, \hat{\boldsymbol{e}}_i = \boldsymbol{X} + \sum_{i=1}^{D} x_i\, \hat{\boldsymbol{n}}_i \,.$$

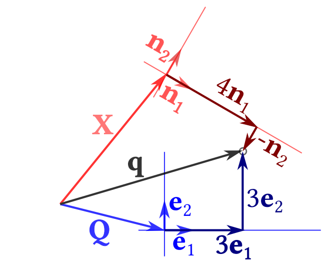


Figure 3.1: Graphical illustration of the description of a position from the perspective of two different reference frames, $\boldsymbol{q} = \boldsymbol{Q} + 3\,\hat{\boldsymbol{e}}_1 + 3\,\hat{\boldsymbol{e}}_2 = \boldsymbol{X} + 4\,\hat{\boldsymbol{n}}_1 - \hat{\boldsymbol{n}}_2$ with the notations of Remark 3.3.

3.3.1 *1st Law*

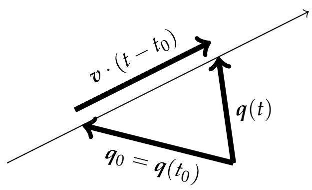
As long as a reference frame moves with a constant velocity, it feels like at rest. Physical measurements can only detect acceleration. This is expressed by

Axiom 3.1: Newton's 1st law

breakable]Newton!1st law | textbf The velocity of a particle moving in an inertial system is constant, unless a (net) force is acting on the particle,

$$\begin{aligned} \forall t \geq t_0 : \boldsymbol{F}(t) = \mathbf{0} \quad &\Leftrightarrow \quad \dot{\boldsymbol{q}}(t) = \boldsymbol{v} = \text{const} \\ &\Leftrightarrow \quad \boldsymbol{q}(t) = \boldsymbol{q}_0 + \boldsymbol{v}\,(t - t_0) \end{aligned}$$

as sketched in the margin.



The particle moves then in a straight line with a constant speed. Indeed, when a particle moves with the constant velocity $\boldsymbol{v} = \dot{\boldsymbol{q}}(t)$ in the reference frame $(\boldsymbol{Q}_1, \{\hat{\boldsymbol{e}}_i(t), i\ =\ 1 \cdots D\}$ then it is at rest in the alternative reference frame $(\boldsymbol{Q}_2, \{\hat{\boldsymbol{e}}_i(t), i\ =\ 1 \cdots D\}$ where $\boldsymbol{Q}_2\ =\ \boldsymbol{Q}_1 + \boldsymbol{v}\ t$. Therefore, in the latter coordinate system the particle is at rest, and it will remain at rest when it is not perturbed by a net external force. After all,

$$\boldsymbol{q} = \boldsymbol{Q}_1 + \boldsymbol{v}\,t = \boldsymbol{Q}_2 + \mathbf{0}\,.$$

3.3.2 2nd Law

Newton's second law spells out how the velocity of the particle changes when there is a force.

Axiom 3.2: Newton's 2nd law

The change, $\ddot{\boldsymbol{q}}(t)$, of the velocity of a particle, $\dot{\boldsymbol{q}}(t)$, at position, $\boldsymbol{q}(t)$, is proportional to the sum of the forces $\boldsymbol{F}_\alpha$ acting on the particle, and the proportionality factor is the particle mass m,

$$m\,\ddot{\boldsymbol{q}}(t) = \sum_\alpha \boldsymbol{F}_\alpha(t)\,.$$

Remark 3.4. In general the time dependence of the forces can be decomposed into three contributions

a) An implicit time dependence, $\boldsymbol{F}(\boldsymbol{q}(t))$, when the force depends on the position, $\boldsymbol{q}(t)$ of the particle. For instance, for a Hookian spring with spring constant k one has, $\boldsymbol{F}(\boldsymbol{q}) = -k\,\boldsymbol{q}$.[2]

b) An implicit time dependence, $\boldsymbol{F}(\dot{\boldsymbol{q}}(t))$, when the force depends on the velocity, $\dot{\boldsymbol{q}}(t)$ of the particle.
For instance, the sliding friction for a particle with mass m and friction coefficient γ is, $\boldsymbol{F}(\dot{\boldsymbol{q}}) = -m\,\gamma\,\dot{\boldsymbol{q}}$.

c) An explicit time dependence when the force is changing in time. For instance, when pushing a child sitting on a swing one will only push when the swing is moving in forward direction.

[2] The spring constant k is a positive constant of dimension Newton per meter that characterizes the strength of the spring, and the minus sign makes it explicit that the Hoookian force is a restoring force pushing the particle back towards $\boldsymbol{q} = \boldsymbol{0}$.

Typically, one explicitly sorts out these dependencies and writes

$$m\,\ddot{\boldsymbol{q}}(t) = \sum_\alpha \boldsymbol{F}_\alpha(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)$$

The resulting relation between the acceleration and the force is called equation of motion of the particle.

Definition 3.3: Equation of Motion (EOM)

Newton's second law establishes a relation between the position $\boldsymbol{q}(t)$ of a particle of mass m, its velocity $\dot{\boldsymbol{q}}(t)$, and acceleration $\ddot{\boldsymbol{q}}(t)$,

$$m\,\ddot{\boldsymbol{q}}(t) = \boldsymbol{F}(\dot{\boldsymbol{q}}(t), \boldsymbol{q}(t), t)$$

that is referred to as the *equation of motion* (EOM) of the particle.
The motion of N particles residing at the positions $\boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t) \in \mathbb{R}^D$ and interacting with each other amounts to $N\,D$ coupled equations

$$\begin{aligned} m\,\ddot{\boldsymbol{q}}_1(t) &= \boldsymbol{F}_1\left(\dot{\boldsymbol{q}}_1(t), \dots, \dot{\boldsymbol{q}}_N(t), \boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t), t\right) \\ \vdots \quad &= \quad \vdots \\ m\,\ddot{\boldsymbol{q}}_N(t) &= \boldsymbol{F}_N\left(\dot{\boldsymbol{q}}_1(t), \dots, \dot{\boldsymbol{q}}_N(t), \boldsymbol{q}_1(t), \dots, \boldsymbol{q}_N(t), t\right) \end{aligned}$$

The primary aim of Theoretical Mechanics is to determine the solution of the EOM for given *initial conditions* (cf. Definition 1.6),

$$\Gamma_0 = \Big(\boldsymbol{q}_1(t_0), \dots, \boldsymbol{q}_N(t_0), \dot{\boldsymbol{q}}_1(t_0), \dots, \dot{\boldsymbol{q}}_N(t_0)\Big)$$

for the positions and velocities of the particles at time t_0. Bundles of phase-space trajectories characterize the motion of sets of trajectories, and they can be analyzed to determine how the behavior of a system changes upon varying the parameters of the setup.

Example 3.1: Particle moving in the gravitational field

The gravitational field induces a constant force $m\,\boldsymbol{g}$ on a particle with mass m. Let it have velocity v_0 at time t_0 when it is taking off from the position $\boldsymbol{q}_0$. Then Newton's 2nd law states that $\ddot{\boldsymbol{q}}(t) = \boldsymbol{g}$, and this equation must be solved subject to the initial conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}(t_0) = \boldsymbol{v}$. By working out the derivatives one readily checks that this is given for

$$\boldsymbol{q}(t) = \boldsymbol{q}_0 + \boldsymbol{v}\,(t - t_0) + \frac{1}{2}\,\boldsymbol{g}\,(t - t_0)^2$$

Example 3.2: Particle moving in a circle

Let a particle of mass m move with constant speed in a circle of radius R such that its position can be written as

$$\boldsymbol{q}(t) = \begin{pmatrix} R\ \cos(\omega t) \\ R\ \sin(\omega t) \end{pmatrix}$$

with a constant angular velocity ω. Then its velocity and acceleration take the form

$$\dot{\boldsymbol{q}}(t) = \begin{pmatrix} -\omega\,R\ \sin(\omega t) \\ \omega\,R\ \cos(\omega t) \end{pmatrix}$$

$$\text{and} \quad \ddot{\boldsymbol{q}}(t) = \begin{pmatrix} -\omega^2\,R\ \cos(\omega t) \\ -\omega^2\,R\ \sin(\omega t) \end{pmatrix} = -\omega^2\,\boldsymbol{q}(t)$$

The speed is constant, taking the value $\sqrt{\dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}} = \omega\,R$. The force is antiparallel to $\boldsymbol{q}$ with magnitude $m\,\omega^2\,R$. Moreover, $\dot{\boldsymbol{q}} \cdot \boldsymbol{F} = 0$ at all times. Hence, the force only changes the direction of motion, and not the speed.

3.3.3 *3rd Law*

Newton's third law states that the reference frame does not matter for the description of the evolution of two particles, even when they interact with each other — i.e. when they exert forces on each other. Consider for instance the motion of two particles of the same mass m that reside at the positions $\boldsymbol{q}_1(t)$ and $\boldsymbol{q}_2(t)$. We decide to observe them from a position right in the middle between the two particles $\boldsymbol{Q} = \big(\boldsymbol{q}_1(t) + \boldsymbol{q}_2(t)\big)/2$. In the absence of external forces this is an inertial frame, such that $\ddot{Q} = \mathbf{0}$ according to Newton's first law. However, Newton's second law implies that also

$$\mathbf{0} = 2m\ddot{Q} = m\,\ddot{q}_1 + m\,\ddot{q}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2$$

where $\boldsymbol{F}_1 = m\,\ddot{q}_1$ and $\boldsymbol{F}_2 = m\,\ddot{q}_2$ are the forces acting on particle 1 and 2, respectively. Up to a change of sign the forces are the same, $\boldsymbol{F}_1 = -\boldsymbol{F}_2$. This action-reaction principle is stipulated by

Axiom 3.3: Newton's 3rd law

Forces act in pairs:

actio when a body A is pushing a body B with force $\boldsymbol{F}_{A\to B}$

reactio then B is pushing A with force $\boldsymbol{F}_{B\to A} = -\boldsymbol{F}_{A\to B}$,

and these forces are always balanced, $\boldsymbol{F}_{A\to B} + \boldsymbol{F}_{B\to A} = \mathbf{0}$.

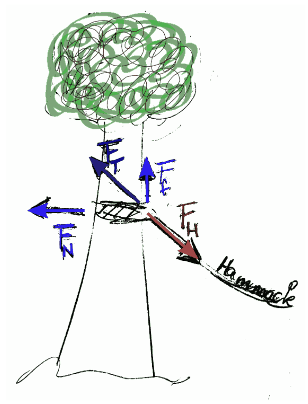


Figure 3.2: Graphical illustrations of forces involved in hanging a hammock on a tree, Example 3.3.

Example 3.3: Fixing a hammock at a tree

When you lie in a hammock that is fixed at a tree, your hammock exerts a force $\boldsymbol{F}_H$ on the tree (actio). The hammock stays where it is because the tree pulls back with exactly the same force $-\boldsymbol{F}_T$, up to a change of sign (reactio), and, in turn, this force can be written as the sum of two components accounting for the normal force F_N of the tree on the rope and a friction force F_f that prevents the rope from sliding down the tree.

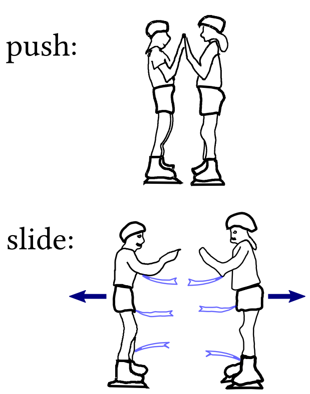


Figure 3.3: Graphical illustrations of motion of the two ice-skaters of Example 3.4.

Example 3.4: Ice skaters

- When two ice skaters of the same mass push each other starting from a position at rest, then they will move in opposite directions with the same speed (unless they brake).
- When they have masses m_1 and m_2 their velocities will be related by $m_1\,\boldsymbol{v}_1 + m_2\,\boldsymbol{v}_2 = \mathbf{0}$ because $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \mathbf{0}$ initially, and $m_1\,\dot{\boldsymbol{v}}_1 + m_2\,\dot{\boldsymbol{v}}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \mathbf{0}$ at any instant of time. As long as they push, the velocities are non-zero and speed increases. When they slide there is no force any longer, and they go at constant speed—except for the impact of friction of the skates on the ice.

Example 3.5: Water Rocket

A water rocket receives its thrust by the repulsive force in response of accelerating and releasing a water jet. Let M the mass of a rocket at a given time, and V_R its speed. To determine the acceleration of the rocket we consider a short time interval Δt where water of mass ΔM is ejected with speed v_f. In the absence of gravitation the momentum balance implies that at any given time the momentum of the rocket $M(t)\, V_R(t)$ must amount to the sum of the water $\Delta M \; (v_R(t) \; - \; v_f$ emitted during a short time Δt and the momentum of the rocket $M(t \; + \; \Delta t)\, V_R(t \; + \; \Delta t)$ after that time,

$$M\,V_R = (M - \Delta M)\,(V_R + \Delta V_R) + \Delta M\,(V_R - v_f)$$
$$\Leftrightarrow \qquad 0 = M\,\Delta V_R - \Delta M\,v_f - \Delta M\,\Delta V_R$$

Now we observe that $\Delta M = a\,\rho, v_f\,\Delta t$ where a is the cross section of the ejected jet, and ρ the mass density of the ejected water:

$$M\,\frac{\Delta V_R}{\Delta t} = a\,\rho, v_f^2 + a\,\rho, v_f\,\frac{\Delta V_R}{\Delta t}\,\Delta t$$

and in the limit of small time increments $\Delta t \to 0$ we obtain the force F_R that is accelerating the rocket

$$F_R = M\,\dot{V}_R = a\,\rho\,v_f^2$$

The rocket trajectory results from interplay of gravity and F_R. One case will be discussed as worked example at the end of this chapter, in Section 3.5. Solving the general case has been suggested as an instructive computer-based example for teaching mechanics (Gale, 1970; Finney, 2000). Instructions about how to build and discuss the rocket in school is available from the NASA and the instructables community.

Michal Richard Trowbridge / wikimedia CC BY-SA 3.0
Figure 3.4: Launching a water rocket, as introduced in Example 3.5.

3.3.4 *Punchline*

Newton's equations are stated nowadays in terms of derivatives, a concept in calculus that has been pioneered by Leibniz.[3] In this language they take the following form for a particle of mass m that is at position $\boldsymbol{q}(t)$ at time t,

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{v}(t)$$
$$\dot{\boldsymbol{v}}(t) = \frac{1}{m}\,\boldsymbol{F}_{\text{tot}}(\boldsymbol{q}(t), \boldsymbol{v}(t), t)$$

[3] Even though these principles of calculus were independently understood by Newton which lead to a very long fight for authorship and fame.

Prior to Newton, physical theories adopted the Aristotelian point of view that $\boldsymbol{v}$ is proportional to the force. Indeed in those days many scientists were regularly inspecting mines, and from the perspective of pushing mine carts is is quite natural to assert that their velocity is proportional to the pushing force. Galileo's achievement is to add the 'tot' of the force side of the equation, pointing out that there also is a friction force acting on the mine cart. Newton's achievement is to add the 'dot' on the left side of the equation, stating that the velocity stays constant when the pushing force and the friction force balance.

Example 3.6: Pushing a mine cart

The motion of the mine cart is one-dimensional along its track such that the position, q, velocity, x, and forces are one-dimensional, i. e. scalar functions. Once the mine cart is moving it experiences a friction force $\boldsymbol{F}_f = -\gamma\, \boldsymbol{v}$, that (to a first approximation) is proportional to its velocity, $\boldsymbol{v}$. Now, let the mine worker push with a constant force F_M such that

$$m\, \ddot{q} = m\, \dot{v} = F_{\text{tot}} = F_M - \gamma\, v\,.$$

The mine cart travels with constant velocity $\dot{v} = 0$, when the attacking forces balance, i. e. for $v_c = F_M / m\, \gamma$.
For a different initial velocity, $v(t_0) = v_0$, one finds an exponential approach to the asymptotic velocity,

$$v(t) = v_c + (v_0 - v_c)\; \mathrm{e}^{-\gamma\,(t-t_0)}$$

After all, $v(t_0) = v_c + (v_0 - v_c) = v_0$ and

$$\begin{aligned}\dot{v}(t) &= (v_0 - v_c)\; (-\gamma)\, \mathrm{e}^{-\gamma\,(t-t_0)} \\ &= \big(-\gamma\, (v(t) - v_c) = -\gamma\, v(t) + F_M\big) / m\end{aligned}$$

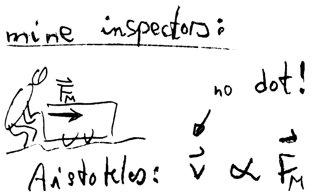


Figure 3.5: The pre-Newtonian understanding of the relation between force and velocity of a body.

clarify: Aristotelian mechanics + criticism

The advantage of the Newtonian approach above earlier modeling attempts is that it makes a quantitative prediction about the asymptotic velocity, and that it also addresses the regime where the velocity is changing, e. g. when the mine cart is taking up speed.

3.3.5 *Self Test*

Problem 3.3. **Terminal velocity for turbulent drag**

Rather than a friction of the type of the mine cart, a golf ball experiences a drag force

$$\boldsymbol{F}_d = -\frac{\rho |u|^2}{2} c_d A\, \hat{\boldsymbol{u}}$$

where A is the cross section of the ball, ρ the density of air, $\boldsymbol{u}$ the velocity of the golf ball, and $c_d \simeq 0.5$ the drag coefficient.

a) The drag coefficient is a dimensionless number that depends on the shape of the object that experiences drag. For the rest the expression for the drag force follows from dimensional analysis. Verify this claim.

b) A slightly more informed derivation of $\boldsymbol{F}_d$ introduces also the diameter D of the golf ball and states that drag arises because the ball has to push air out of its way. When moving it has to push air out of the way at a rate $A\, u$. The air was at rest initially and must move roughly with a velocity u to get out of the way. Subsequently, its kinetic energy is lost. Check out, how this leads to the expression provided for $\boldsymbol{F}_d$.

c) What is the terminal velocity of a golf ball that is falling out of the pocket of a careless hang glider?

d) Use dimensional analysis to estimate the distance after which the ball acquires its terminal velocity, and how long it takes to reach the velocity.

Problem 3.4. **Orbit of the Moon around Earth**

The Moon is circling around Earth due to the gravitational force of modulus

$$F_{ME} = \frac{G M_E M_M}{R_{ME}^2}$$

where $G = \frac{2}{3} \times 10^{-12} \mathrm{m}^3/\mathrm{kg\,s}$ is the gravitational constant, $M_E \simeq 6 \times 10^{24}$ kg and $M_M \simeq \frac{3}{4} \times 10^{-23}$kg are the masses of Moon and Earth, respectively, and $R_{ME} = \frac{7}{4} \times 10^6$m is the distance from Earth to Moon.

a) Calculate the force that Moon is experiencing due to the Earth. Compare it to the gravitational acceleration $g \simeq 10\,\mathrm{m/s^2}$ scaled by $(R_{ME}/R_E)^2$ where $R_E = 2\pi \times 10^6$m is the Earth radius. Why would one scale by this factor?

b) Assume that the Moon trajectory is circular and identify F_{ME} with the centripetal force that keeps the moon on its orbit. What does this tell about the dependence of the period T of the motion on G, R_{ME} and the masses.

c) Evaluate T and compare it to the duration of a month.

Problem 3.5. **Escape velocities**

The escape velocity is the minimum speed of a projectile that would allow it to escape into outer space when friction due to the atmosphere is neglected.

a) Estimate the escape velocity of Earth based on the gravitational force law F_{ME} given in Problem 3.4, the gravitational acceleration $g = 10\,\mathrm{m/s^2}$ on Earth, and the fact that the Earth circumference was set to $2\pi R_E = 4 \times 10^4$ km.

b) Recall the relation between gravity on Earth and Moon given in Problem 1.8, and estimate also the escape velocity from Moon.

c) **After you performed the calculations:**
Compare your estimates to the values provided by Wikipedia.

Problem 3.6. **Pulling a cow**

A child is pulling a toy cow with a force of $F = 5\,\mathrm{N}$. The cow has a mass of $m = 100\,\mathrm{g}$ and the chord has an angle $\theta = \pi/5$ with the horizontal.

[3] For this angle one has $\tan\theta \approx 3/4$.

a) Describe the motion of the cow when there is no friction. In the beginning the cow is at rest.

b) What changes when there is friction with a friction coefficient of $\gamma = 0.2$, i.e. a horizontal friction force of magnitude $-\gamma m g$ acting on the cow.

c) Is the assumption realistic that the force remains constant and will always act in the same direction? What might go wrong?

3.4 *Constants of motion (CM)*

In the previous section we saw that Newton's laws can be expressed as equations relating the second derivative of the position of a particle to the forces acting on the particle. The forces are determined as part of setting up the physical model. Subsequently, determining the time dependence of the position is a mathematical problem. Often it can be solved by finding constraints on the solution that must hold for all times. Such a constraint is called a

Definition 3.4: Constant of motion

A function $\mathcal{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is a *constant of motion (CM)* iff its time derivative vanishes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0$$

It provides us with an opportunity to take a closer look at the expressions that emerge when taking derivatives of functions with arguments that are vectors. In order to evaluate the time derivative of $\mathcal{C}$ we write $\boldsymbol{q} = (q_1, \ldots\ldots, q_D)$, and apply the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(q_1(t), \cdots, q_D(t), \dot{q}_1(t), \cdots, \dot{q}_D(t), t) \\ &= \sum_{i=1}^{D} \frac{\mathrm{d}q_i}{\mathrm{d}t}\frac{\partial \mathcal{C}}{\partial q_i} + \sum_{i=1}^{D} \frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\frac{\partial \mathcal{C}}{\partial \dot{q}_i} + \frac{\partial \mathcal{C}}{\partial t}\end{aligned} \tag{3.4.1}$$

In this expression the operation ∂ is called '*partial*', and the derivative $\partial\mathcal{C}/\partial q_i$ is denoted as partial derivative of $\mathcal{C}$ with respect to q_i. For the purpose of calculating the partial derivative, we consider $\mathcal{C}$ to be a function of only the single argument q_i. For sake of a more compact notation we also write $\partial_{q_i}\mathcal{C}$ rather than $\partial\mathcal{C}/\partial q_i$. Moreover, when it is not clear from the context which conditions are adopted, they can explicitly be stated as subscript of a vertical bar to the right of the derivative (or even square brackets).

Example 3.7: Partial derivatives

For $f(x,y) = x/\sqrt{x^2+y^2}$ and $R = \sqrt{x^2+y^2}$ we have

$$\partial_x f(x,y)|_y = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{R^3}$$
$$\partial_x f(x,y)|_R = \partial_x \left[\frac{x}{R}\right]_R = \frac{1}{R}$$

A compact notation that allows us to state the expression of Equation (3.4.1) in a more transparent way is achieved as follows: We observe that the expressions in the sums amount to writing out in components a scalar product of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ with vectors that are obtained by the partial derivatives. These vectors are denoted *gradients* with respect to $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, and they will be written as

In the literature one also finds the alternative notations

$$\nabla_q \mathcal{C} = \frac{\partial \mathcal{C}}{\partial \boldsymbol{q}} = \partial_q \mathcal{C}$$

$$\nabla_q \mathcal{C} = \begin{pmatrix} \partial_{q_1}\mathcal{C} \\ \vdots \\ \partial_{q_D}\mathcal{C} \end{pmatrix} \qquad \text{and} \qquad \nabla_{\dot{q}} \mathcal{C} = \begin{pmatrix} \partial_{\dot{q}_1}\mathcal{C} \\ \vdots \\ \partial_{\dot{q}_D}\mathcal{C} \end{pmatrix} \tag{3.4.2}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) = \dot{\boldsymbol{q}} \cdot \nabla_q \mathcal{C} + \ddot{\boldsymbol{q}} \cdot \nabla_{\dot{q}} \mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

In terms of the phase-space coordinates $\boldsymbol{\Gamma} = (\boldsymbol{q}, \dot{\boldsymbol{q}})$ one can also adopt the even more compact notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) = \dot{\boldsymbol{\Gamma}} \cdot \nabla_\Gamma \mathcal{C} + \frac{\partial \mathcal{C}}{\partial t}$$

or even

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\boldsymbol{\Gamma}(t), t) = \dot{\boldsymbol{\Gamma}} \cdot \nabla \mathcal{C}(\boldsymbol{\Gamma}(t), t) + \frac{\partial \mathcal{C}}{\partial t}(\boldsymbol{\Gamma}(t), t)$$

where the index of the nabla operator has been dropped with the understanding that it is clear from the context what the operator refers to.

We make use of these derivatives while introducing some important physical quantities that are constants of the motion in specific settings.

3.4.1 *The kinetic energy*

When no forces are acting on a particle, $\boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$, it moves with constant velocity. All functions that depend only on the velocity will then be constant. In particular this holds for the kinetic energy, T, that will play a very important role in the following.

Theorem 3.2: Conservation of kinetic energy

The *kinetic energy* $T = \frac{m}{2}\,\dot{\boldsymbol{q}}^2$ of a particle is conserved iff no net force acts on the particle, i. e. iff $\boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$.

Proof.
$$\frac{\mathrm{d}}{\mathrm{d}t}T = \frac{m}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\sum_i \dot{q}_i \cdot \dot{q}_i = m\,\sum_i \dot{q}_i \cdot \ddot{q}_i$$
$$= m\dot{\boldsymbol{q}} \cdot \ddot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \cdot (m\,\ddot{\boldsymbol{q}}) = \dot{\boldsymbol{q}} \cdot \boldsymbol{F}_{\text{tot}} = 0$$

In the last two steps we used Newton's 2nd law, and the assumption that $\boldsymbol{F}_{\text{tot}} = \boldsymbol{0}$. □

3.4.2 *Work and total energy*

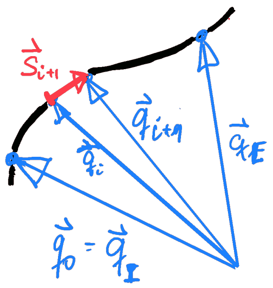


Figure 3.6: Breaking a particle track $\boldsymbol{q}(t)$ into a sequence of discrete points $\boldsymbol{q}_i$ with segments $\boldsymbol{s}_{i+1} = \boldsymbol{q}_{i+1} - \boldsymbol{q}_i$.

From a physics perspective, work is performed when a body is moved in the presence of an external force.

- When the force $\boldsymbol{F}$ is constant along a straight path of displacement $\boldsymbol{s} = \boldsymbol{q}_E - \boldsymbol{q}_I$, from a position $\boldsymbol{q}_I$ to the position $\boldsymbol{q}_E$, then the work W amounts to the scalar product $W = \boldsymbol{F} \cdot \boldsymbol{s}$.

- When the force depends on the position along the path, we parameterize the motion along the path by time, $\boldsymbol{q}(t)$, with $\boldsymbol{q}(t_I) = \boldsymbol{q}_I$ and $\boldsymbol{q}(t_E) = \boldsymbol{q}_E$ and break it into sufficiently small pieces $\boldsymbol{s}_i = \boldsymbol{q}(t_i) - \boldsymbol{q}(t_i - \Delta t)$ where the force $\boldsymbol{F}_i = \boldsymbol{F}(t_i)$ and the velocity of the particle $\dot{q}(t_i)$ may be assumed to be constant, such that $\dot{\boldsymbol{q}}(t_i) = \big(\boldsymbol{q}(t_i) - \boldsymbol{q}(t_{i-1})\big)/\Delta t$. Then

$$W = \sum_i \boldsymbol{F}_i \cdot \boldsymbol{s}_i = \lim_{\Delta t \to 0} \sum_i \boldsymbol{F}_i \cdot \dot{\boldsymbol{q}}\,\Delta t = \int_{t_0}^{t_1} \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)\,\mathrm{d}t = \int_{\boldsymbol{q}(t)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$$

  The last equality should be understood here as a definition of the final expression that is interpreted here in the spirit of the substitution rule of integration.

Definition 3.5: Work and Line Integrals

The *work*, W, of a particle that performs a path $\boldsymbol{q}$ under the influence of a force $\boldsymbol{F}(t)$ amounts to the result of the *line integral*
$$W = \int_{\boldsymbol{q}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$$
When the path is parameterized by time, then W amounts to the time integral of dissipated power $P(t) = \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)$,
$$W = \int \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)\,\mathrm{d}t = \int P(t)\,\mathrm{d}t$$

Remark 3.5. The scalar product $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q}$ or $P(t) = \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)$ singles out only the action of the force parallel to the trajectory. The perpendicular components do not perform work. Hence, a force that is always acting perpendicular to the velocity, i. e. perpendicular to the path of the particle, does not perform any work,

$$W = \int \boldsymbol{F}(t) \cdot \dot{\boldsymbol{q}}(t)\,\mathrm{d}t = \int 0\,\mathrm{d}t = 0$$

It only changes the direction of motion.

Remark 3.6. The result of the integral does not rely on the parameterization of the path by time. For instance mathematicians prefer to use the length ℓ of the path. The speed of the particle is then $\dot\ell(t) = |\dot{\boldsymbol{q}}(t)|$ and one finds

$$W = \int \boldsymbol{F}(t) \cdot \mathrm{d}\boldsymbol{s} = \int \boldsymbol{F}(t(\ell)) \cdot \dot{\boldsymbol{q}}(t(\ell)) \, \frac{\mathrm{d}\ell}{\dot\ell} = \int \boldsymbol{F}(\ell) \cdot \frac{\boldsymbol{q}(\ell)}{\mathrm{d}\ell} \, \mathrm{d}\ell$$

where $\mathrm{d}\hat{\boldsymbol{q}}/\mathrm{d}\ell$ is a unit vector pointing in the direction of the trajectory.

Remark 3.7. Line integrals are also used to determine the length, L, of a path in space. After all, the length amounts to the time integral of the speed, $\dot\ell(t)$, and one has

$$L = \int \mathrm{d}\ell = \int \mathrm{d}t \, \dot\ell(t) = \int \mathrm{d}t \, \frac{\dot{\boldsymbol{q}}^2(t)}{\dot\ell(t)} = \int \mathrm{d}\boldsymbol{q} \cdot \frac{\dot{\boldsymbol{q}}(t)}{\dot\ell(t)} = \int \mathrm{d}\boldsymbol{q} \cdot \frac{\mathrm{d}\boldsymbol{q}(\ell)}{\mathrm{d}\ell}$$

This is further illustrated in Problem 3.22.

The calculation of work simplifies dramatically when the force can be written as gradient of another function, Φ.

Definition 3.6: Potentials and Conservative Forces

A force $\boldsymbol{F}(\boldsymbol{q})$ that can be expressed as the negative gradient of a function $\Phi(\boldsymbol{q})$,

$$\boldsymbol{F}(\boldsymbol{q}) = -\nabla\Phi(\boldsymbol{q}) = -\begin{pmatrix} \partial_{q_1}\Phi(q_1, \dots, q_D) \\ \vdots \\ \partial_{q_D}\Phi(q_1, \dots, q_D) \end{pmatrix}$$

is called a *conservative force* and the function Φ is the *potential* associated to the force.

Remark 3.8. Conservative forces only depend on position, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{q})$. They neither explicitly depend on time nor on the velocity $\boldsymbol{q}$.

Remark 3.9. Conservative forces only depend on position, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{q})$. They neither explicitly depend on time nor on the velocity $\boldsymbol{q}$.

Example 3.8: Conservative forces: (counter-)examples

- Gravitational acceleration g is is constant in space. Hence, gravity is a conservative force.
- Friction of a cube sliding over a table is proportional to the particle speed v. Therefore, friction is *not* a conservative force.
- Setting the rope into motion for rope skipping requires an oscillatory force. Due to its time time dependence such a force is not conservative.

Rope skipping on the poster of the movie “Doubletime”, wikimedia, CC BY 2.0

Theorem 3.3: Work for conservative forces

For conservative forces, $\boldsymbol{F} = -\nabla\Phi(\boldsymbol{q})$, the work for a path $\boldsymbol{q}(t)$ from $\boldsymbol{q}_0$ to $\boldsymbol{q}_1$ amounts to the difference of the potential evaluated at the initial and at the final point of the path

$$W = \int_{\boldsymbol{q}(t)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{q} = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1)$$

Proof.

$$\begin{aligned} W &= \int_{t_0}^{t_1} \boldsymbol{F} \cdot \dot{\boldsymbol{q}}\, \mathrm{d}t = -\int_{t_0}^{t_1} \nabla\Phi \cdot\, \dot{\boldsymbol{q}}\, \mathrm{d}t \\ &= -\int_{t_0}^{t_1} \sum_i \frac{\partial\Phi}{\partial q_i}\, \frac{\partial q_i}{\partial t}\, \mathrm{d}t = -\int_{t_0}^{t_1} \frac{\mathrm{d}\,\Phi}{\mathrm{d}t}\, \mathrm{d}t \\ &= -\big(\Phi(\boldsymbol{q}(t_1)) - \Phi(\boldsymbol{q}(t_0))\big) = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1) \end{aligned}$$ □

Remark 3.10. The work performed along a closed path vanishes for conservative forces. After all, in that case $\boldsymbol{q}_1 = \boldsymbol{q}_0$ such that $W = \Phi(\boldsymbol{q}_0) - \Phi(\boldsymbol{q}_1) = 0$.

Remark 3.11. The potential in itself is not an observable.[4] One can only observe the work, which is the potential difference between two positions, and the force, which is the negative gradient of the potential. Therefore, the potential is only defined up to adding a constant.

[4] An *observable* is a quantity that can be measured by direct observation.

Example 3.9: Gravitational Potential

For a particle of mass m gravity on the Earth surface gives rise to a force of magnitude $\boldsymbol{F}(x, y, z) = -m\, g\, \hat{\boldsymbol{z}}$ that can be derived from the potential $\boldsymbol{\Phi}(x, y, z) = m\, g\, z$,

$$-\nabla\Phi_1(x, y, z) = \begin{pmatrix} -\partial_x\Phi(x, y, z) \\ -\partial_y\Phi(x, y, z) \\ -\partial_z\Phi(x, y, z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -m\, g \end{pmatrix} = \boldsymbol{F}(x, y, z)$$

Far away, at a position $\boldsymbol{q} = (q_1, q_2, q_3)$ from the center of Earth, gravity induces a force $\boldsymbol{F}(\boldsymbol{q}) = -G\, M_E\, m\, \boldsymbol{q}/|\boldsymbol{q}|^3$ on a body of mass m. This force can be obtained as

$$-\nabla\phi_2(\boldsymbol{q}) = \nabla \frac{G\, M_E\, m}{\sqrt{q_1^2 + q_2^2 + q_3^2}} = G\, M_E\, m \begin{pmatrix} \partial_{q_1} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \\ \partial_{q_2} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \\ \partial_{q_3} \frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} \end{pmatrix}$$

$$= G\, M_E\, m \begin{pmatrix} \frac{-q_1}{[q_1^2+q_2^2+q_3^2]^{3/2}} \\ \frac{-q_2}{[q_1^2+q_2^2+q_3^2]^{3/2}} \\ \frac{-q_3}{[q_1^2+q_2^2+q_3^2]^{3/2}} \end{pmatrix} = \frac{-G\, M_E\, m}{[q_1^2 + q_2^2 + q_3^2]^{3/2}}\, \boldsymbol{q} = \boldsymbol{F}(\boldsymbol{q})$$

Remark 3.12. According to Theorem 3.3 differences of the value of the potential between two positions amount to the work performed in the potential. Different approaches to calculate the value of this scalar observable must yield identical results. Therefore, the functional dependence of the potential must not depend on the choice of the coordinate system. This invariance requires that the potential can always be expressed in terms of scalar products. For the potentials in Example 3.9 this is achieved by writing

$$\Phi_1(\boldsymbol{q}) = m\,\boldsymbol{g}\cdot\boldsymbol{q} \quad \text{with} \quad \boldsymbol{g} = (0, 0, -g)$$
$$\Phi_2(\boldsymbol{q}) = -G\,M_E\,m/\sqrt{\boldsymbol{q}\cdot\boldsymbol{q}}$$

Remark 3.13. One can make use of the properties of scalar products to reduce the computational work to determine the force for a given potential by working out the component i of the gradient where i is can be any index of the vector. For conciseness we also write then ∂_i for the partial derivative with respect to component q_i of the argument $\boldsymbol{q}$ of $\Phi(\boldsymbol{q})$.

For the potentials in Example 3.9 this works as follows

$$-\partial_i\Phi_1(\boldsymbol{q}) = -m\,\partial_i \sum_j g_j q_j = -m \sum_j g_j \delta_{ij} = -m g_i$$
$$-\partial_i\Phi_2(\boldsymbol{q}) = G\,M_E\,m\,\partial_i \Big[\sum_j q_j^2\Big]^{-1/2} = \frac{-G\,M_E\,m\,q_i}{\big[\sum_j q_j^2\big]^{3/2}}$$

In particular in the second case the advantage is evident.

Example 3.10: Falling men and cat

When a cat, that has a mass of $m = 3\,\text{kg}$, falls from a balcony in the fourth floor, i. e. from a height $H \simeq 4 \times 3\,\text{m} = 12\,\text{m}$, the initial potential energy

$$V_{\text{cat}} = mgH = 3\,\text{kg} \times 10\,\text{m}/\text{s}^2 \times 12\,\text{m} = 360\,\text{kg}\,\text{m}^2/\text{s}^2$$

will be transformed into kinetic energy and then dissipated when the cat hits the ground.
To get an idea about this energy we compare it to the energy dissipated when a man of mass $M = 80\,\text{kg}$, falls out of his bed that has a height of $h = 50\,\text{cm}$,

$$V_{\text{man}} = Mgh = 80\,\text{kg} \times 10\,\text{m}/\text{s}^2 \times 0.5\,\text{m} = 400\,\text{kg}\,\text{m}^2/\text{s}^2$$

From the point of view of the dissipated energy the fall of the cat is not as bad as it looks at first sight.

Conservative forces are called conservative forces because motion in such a potential conserves the sum of the potential energy and the kinetic energy.

Theorem 3.4: Conservation of the total energy

The *total energy* $E = T + \Phi$ of a particle is conserved if it moves in a conservative force field $\boldsymbol{F} = -\nabla\Phi$.

Proof.
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}\Phi}{\mathrm{d}t} = m\,\dot{\boldsymbol{q}}\cdot\ddot{\boldsymbol{q}} + \nabla\Phi\cdot\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}\cdot\underbrace{\left(m\ddot{\boldsymbol{q}} - \boldsymbol{F}\right)}_{=\,\boldsymbol{0}} = \boldsymbol{0}$$

In the third equality we used that the force is conservative, and in the final step, we used Newton's second law which states that $m\ddot{\boldsymbol{q}} = \boldsymbol{F}$. □

Example 3.11: Accidents at work and on the street

A paramedic emergency ambulance receives two calls from an accident site:
i. a craftsmen fell from a roof of height H
ii. a teenager hit a tree with his motorcycle with a speed v
For which height does the energy of the craftsman approximately match the one of the motor cyclist when he drove
in the city, $v_C = 50\,\mathrm{km/h}$,
outside the city, $v_L = 100\,\mathrm{km/h}$,
on a German autobahn with $v_A = 150\,\mathrm{km/h}$
or was really speeding with $v_S = 200\,\mathrm{km/h}$.
We assume that they both have comparable mass.
Energy conservation entails that we have to compare the potential energy V_{worker} of the craftsman on the roof and the kinetic energy of the teenager on the motorcycle T_{teenager},

$$mgH = V_{\text{worker}} = T_{\text{teenager}} = \frac{m}{2}\,v^2 \quad\Leftrightarrow\quad H = \frac{v^2}{2g}$$

Hence we find

v	50 km/h	100 km/h	150 km/h	200 km/h
H	12 m	50 m	110 m	200 m
floor	4	16	36	64

Most likely, the teenager will encounter more severe injuries, unless the craftsman is working on a really high building.

3.4.3 *Momentum*

Theorem 3.5: Conservation of momentum

The *momentum* $\boldsymbol{P} = \sum_{i=1}^{N} m_i\dot{\boldsymbol{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\boldsymbol{q}_i(t)$ is conserved if no net force $\boldsymbol{F}_{\text{tot}}$ acts on the system.

Proof. The time derivative of the total momentum is $\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P} = \sum_{i=1}^{N} m_i\ddot{\boldsymbol{q}}_i(t)$ where $m_i\ddot{\boldsymbol{q}}_i(t)$ amounts to the force on particle i. This force amounts to the sum of an external force $\boldsymbol{F}_i$ on particle i and the forces $\boldsymbol{f}_{ji}$ exerted by other particles j on i. The net force amounts to the sum of the external forces, $\boldsymbol{0} = \boldsymbol{F}_{\text{tot}} = \sum_i \boldsymbol{F}_i$. Newton's third law requires that $\boldsymbol{f}_{ji} = -\boldsymbol{f}_{ij}$, and we will set $\boldsymbol{f}_{ii} = \boldsymbol{0}$ to simplify notations of indices in the sums. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P} = \sum_{i=1}^{N}\left(\boldsymbol{F}_i + \sum_{j=1}^{N}\boldsymbol{f}_{ji}\right) = \sum_{i=1}^{N}\boldsymbol{F}_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{f}_{ij}$$

$$= \boldsymbol{F}_{\mathrm{tot}} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\boldsymbol{f}_{ij} + \boldsymbol{f}_{ji}) = \boldsymbol{0}$$

□

Фабрицио, CC BY-SA 4.0

Figure 3.7: Newton's cradle. When the excited ball to the right is released it will come down, hit the rightmost ball that is hanging down at rest. The momentum is transferred to the leftmost ball, and that is moving up (almost) as far to the left as the initial ball was excited to the right. Its motion reverses, and by the same sequence of events the motion proceeds from left to right.

Example 3.12: One-dimensional collisions

We consider two steel balls that can freely move along a line. They have masses m_1 and m_2 and reside at positions x_1 and x_2, respectively. Initially ball two is at rest in the origin, and ball one is approaching from the right with a constant speed v_1. What is the speed of the balls after the collision? Before and after the collision the particles feel no forces such that their velocity is constant. We assume that the collision is elastic such that energy is preserved. Hence,

$$\begin{array}{lrcl} & \text{before collision} & = & \text{after collision} \\ \text{momentum:} & m_1 v_1 & = & m_1 v_1' + m_2 v_2' \\ \text{energy:} & \frac{m_1}{2} v_1^2 & = & \frac{m_1}{2}(v_1')^2 + \frac{m_2}{2}(v_2')^2 \end{array}$$

where the prime indicates the post-collision velocities. These velocities can best be determined by writing the momentum and energy balance in the form

$$m_1(v_1 - v_1') = m_2 v_2' \quad \text{and} \quad m_1(v_1^2 - v_1'^2) = m_2 v_2'^2$$

and dividing the second by the first equation. This provides

$$v_1 + v_1' = v_2'$$

Together with the momentum balance it provides

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 \qquad \text{and} \qquad v_2' = \frac{2 m_1}{m_1 + m_2} v_1$$

In particular, when the two particles have the same mass one obtains that $v_1' = 0$ and $v_2' = v_1$ which is beautifully exemplified by the dynamics of Newton's cradle.

3.4.4 *Angular Momentum*

In the immediate vicinity of the collisions the balls in Newton's cradle perform a motion along a horizontal line, as discussed in Example 3.12. However, during the excursions to the left and right they follow a circular track where the chains act as arms and their suspension as fulcrum of the circular motion. In such settings it is often desirable to also consider the evolution of the angular momentum.

Theorem 3.6: Conservation of angular momentum

The *angular momentum* $\boldsymbol{L} = \sum_{i=1}^{N} m_i \boldsymbol{q}_i(t) \times \dot{\boldsymbol{q}}_i(t)$ of a set of N particles with masses m_i that reside at the positions $\boldsymbol{q}_i(t)$ is conserved if no external forces act on the system and if the interaction forces between pairs of particles act parallel to the line connecting the particles.

Proof.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{L} &= \sum_{i=1}^{N} m_i \Big(\dot{\boldsymbol{q}}_i(t) \times \dot{\boldsymbol{q}}_i(t) + \boldsymbol{q}_i(t) \times \ddot{\boldsymbol{q}}_i(t) \Big) \\ &= \sum_{i<j} \Big(\boldsymbol{q}_i(t) \times \boldsymbol{f}_{ij} + \boldsymbol{q}_j(t) \times \boldsymbol{f}_{ji} \Big) \\ &= \sum_{i<j} \Big(\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t) \Big) \times \boldsymbol{f}_{ij} = \boldsymbol{0} \end{aligned}$$

where we used that $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$ due to Newton's third law, and that $(\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t))$ is parallel to $\boldsymbol{f}_{ij}$ by assumption on the particle interactions. □

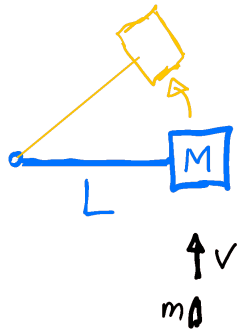


Figure 3.8: Notations adopted in the measurement of the speed v of a bullet of mass m that is hitting a rotor of mass M attached to an arm of length L; see Example 3.13.

Example 3.13: Determine the speed of a bullet.

In a CSI lab one tests the speed of a bullet by shooting it into a rotor where a mass $M = 1\,\mathrm{kg}$ can move horizontally with minimal fraction on an arm with length $L = 1\,\mathrm{m}$. For a bullet of a mass $m = 8\,\mathrm{g}$ we find a rotation frequency $f = 0.16\,\mathrm{Hz}$. What is the muzzle velocity v of the gun? During the collision the bullet gets stuck in the rotor mass. Before and after the collision the angular momentum thus is

$$\begin{aligned} & m\,L\,v = (m+M)\,L^2\,\omega = (m+M)\,L^2\,2\pi\,f \\ \Leftrightarrow \qquad & v = \frac{m+M}{m}\,2\pi\,f\,L = \frac{1008}{8} \times 2\pi \times 0.16\,\mathrm{m/s} \simeq 125\,\mathrm{m/s} \end{aligned}$$

3.4.5 *Self Test*

Problem 3.7. **Derivatives of common composite expressions**

Evaluate the following derivatives.

a) $\frac{\mathrm{d}}{\mathrm{d}x}(a+x)^b$ d) $\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta(t)$ g) $\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{a+b\,z^2}$

b) $\frac{\partial}{\partial x}(x+b\,y)^2$ e) $\frac{\mathrm{d}}{\mathrm{d}t}\big(\sin\theta(t)\ \cos\theta(t)\big)$ h) $\frac{\partial}{\partial x_3}\left[\sum_{j=1}^{6} x_j^2\right]^{-1/2}$

c) $\frac{\mathrm{d}}{\mathrm{d}x}(x+y(x))^2$ f) $\frac{\mathrm{d}}{\mathrm{d}t}\sin(2\theta(t))$ i) $\frac{\partial}{\partial y_1}\ln(\boldsymbol{x}\cdot\boldsymbol{y})$

In these expressions a and b are real constants, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are 6-dimensional vectors.

Problem 3.8. **A conservation law for the harmonic oscillator**

Demonstrate that

$$I = \dot{x}_1(t)\,\dot{x}_2(t) + \omega^2\,x_1(t)\,x_2(t)$$

is a constant of motion of a two-dimensional harmonic oscillator with equation of motion

$$\ddot{\boldsymbol{x}}(t) = -\omega^2\,\boldsymbol{x}(t) \quad \text{where} \quad \omega \in \mathbb{R} \quad \text{and} \quad \boldsymbol{x}(t) = \big(x_1(t), x_2(t)\big) \in \mathbb{R}^2$$

Problem 3.9. **Anvil shooting**

Rex Hammock from USA/ wikimedia CC BY-SA 2.0

Anvil shooting is a tradition in some US communities to celebrate St. Clement's Day, honoring Pope Clement I, the patron saint of blacksmiths and metalworkers. Typical anvils have a mass of about 150 kg and they are shot up to a height of 60 m. Which energy must the gun powder release to the anvil for such a feat?

Problem 3.10. **Running mothers**

In the Clara Zetkin Park one regularly encounters blessings[5] of dozens of mothers jogging in the park while pushing baby carriages. Troops of kangaroo mothers rather carry their youngs in pouches.

[5] Look up "terms of venery" if you ever run out of collective nouns.

a) Estimate the energy consumption spend in pushing the carriages as opposed to carrying the newborn.
The carriages suffer from friction. Let the friction coefficient be $\gamma = 0.3$.
When carrying the baby the kangaroo must lift it up in every jump and the associated potential energy is dissipated.

b) How does the running speed matter in this discussion?

c) How does the mass of the babies/youngs make a difference?

Problem 3.11. **The sledgehammer experiment**

In his magnificent book "Thinking Physics" Lewis Carroll Epstein (2009) sets out a class room experiment that he used to perform in his physics class: He placed an anvil on his chest and asked a student from the audience to hit the anvil with a sledge hammer as hard as he could manage. What will happen?

Epstein changed the way of presentation of this experiment when a very nervous student missed the anvil and hit his hand. Have a look into the book for the full story.

Problem 3.12. **The rotating chair experiment**

deerstop, wikimedia, CC0

The spin increases when an ice dancer pulls inwards arms and legs. This is illustrated in the picture of Yuko Kawaguti in the margin, and the physical principle has beautifully been demonstrated in a wikimedia movie by Oliver Zajkov from the Physics Institute at the University of Skopje.

a) Assume that a less careful experimenter starts his motion with a spin of 1 Hz, holding 5 kg barbells with stretched-out arms 1 m away from the rotation axis. Estimate his spin rate when he pulls in his arms till the barbells reach a distance of 20 cm from the rotation axis.

b) Which trajectory will they take when the careless experimenter gets dizzy and looses hold on the barbells?

3.5 *Worked example: Flight of an Earth-bound rocket*

In order to illustrate the applications of Newton's laws we discuss now the flight of a rocket. We will deal with the case a) where the rocket is moving in vertical direction, b) where the fuel is ejected with a constant speed v_f (or zero when it is exhausted), and c) where the rocket does not reach heights with a noticeable change of the gravitational acceleration. At the end of this section we discuss the impact of relaxing these assumptions, and point to the literature for a further discussion. An example of the considered motion is the vertical flight of the water rocket introduced in Example 3.5.

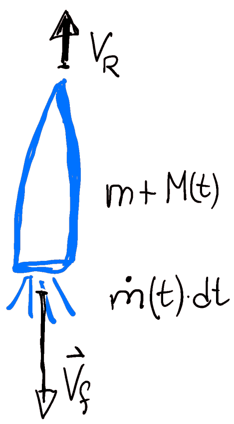


Figure 3.9: Notations adopted for the discussion of the flight of a rocket in Section 3.5.

Let V_R be the speed of the rocket. It is positive when the rocket goes up, and negative when it falls down. On the way down, its mass will be m. Initially, it has a mass $m + M_0$, where M_0 is mass of the fuel (cf. Figure 3.9). As long as the rocket is firing, Newton's third law implies that

$$F_R = \big(m + M(t)\big)\, \dot V_R = a\,\rho\, v_f^2 - \big(m + M(t)\big)\, g$$

The first force on the right-hand side of this equation accounts for the recoil from ejection of the fuel (cf. Example 3.5) and the latter to gravitational acceleration. We also observed in Example 3.5 that the mass $M(t)$ of the remaining fuel at time t obeys the differential equation $\dot M = -a\,\rho\, v_f$ such that[6]

$$M(t) = M_0 - a\,\rho\, v_f\, t\,.$$

At some time T all fuel is consumed, and we have

$$0 = M_0 - a\,\rho\, v_f\, T \quad\Rightarrow\quad T = \frac{M_0}{a\,\rho\, v_f}\,.$$

Moreover, for the rocket acceleration we find

$$\dot V_R(t) = \frac{F_R}{m + M(t)} = -g + \frac{v_f/T}{\mu - t/T} \quad \text{with} \quad \mu = \frac{m + M_0}{M_0}$$

The rocket speed is obtained by integrating the acceleration from the initial time, where the rocket is at rest, till time t. The

[6] One easily checks that this expression is correct for the initial mass, $M(0) = M_0$ and its derivative agrees with $\dot M(t)$. The same applies also for the expressions for the speed and height of the rocket discussed below. Problem 4.2 gives clues how the solutions are determined systematically. In Chapter 4 we discuss systematic approaches to find the solution.

integral takes a simpler form when one adopts the dimensionless integration variable, $\tau = t/\mu T$,

$$V_R(t) = \mu T \int_0^{t/\mu T} \mathrm{d}\tau\, \dot V_R(t) = -g\,t - v_f \int_0^{t/\mu T} \mathrm{d}\tau\, \frac{1}{1-\tau}$$
$$= -g\,t - v_f \ln\left(1 - \frac{t}{\mu T}\right) \tag{3.5.1a}$$

Thus, at time T the rocket has acquired the speed

$$V_R(T) = -g\,T + v_f \ln\frac{\mu-1}{\mu}\,. \tag{3.5.1b}$$

The rocket height $z(t)$ is obtained by observing that $\dot z(t) = V_R(t)$, which in turn is given by Equation (3.5.1a). The solution where the rocket starts at height zero is the given by

$$z(t) = \mu T \int_0^{t/\mu T} \mathrm{d}\tau\, V_R(t) = -\frac{g\,t^2}{2} - \mu T\, v_f \int_0^{t/\mu T} \mathrm{d}\tau\, \ln(1-\tau)$$
$$= -\frac{g\,t^2}{2} - \mu T\, v_f\, \left[(\tau - 1)\,\left(-1 + \ln(1-\tau)\right)\right]_0^{t/\mu T}$$
$$= -\frac{g\,t^2}{2} + v_f\,t + v_f\,T\,\left(\mu - \frac{t}{T}\right)\,\ln\left(1 - \frac{t}{\mu T}\right)\,. \tag{3.5.2a}$$

At time T this simplifies to

$$z(T) = -\frac{g\,T^2}{2} + v_f\,T\,\left[1 + (\mu - 1)\,\ln\frac{\mu-1}{\mu}\right] \tag{3.5.2b}$$

Starting from that position the rocket will perform a ballistic flight with initial velocity $V_R(T)$ that will add to its height another height increment of $V_R^2(T)/2g$. The additional height increment ΔH before the rocket reaches the crest of its height is found by energy conservation and Equation (3.5.1b)

$$m\,g\,\Delta H = \frac{m}{2}\,V_R^2(T)$$
$$\Rightarrow \quad \Delta H = \frac{V_R^2(T)}{2\,g} = \left[\frac{g\,T^2}{2} - T\,v_f\,\ln\frac{\mu-1}{\mu} + \frac{v_f^2}{2\,g}\,\left(\ln\frac{\mu-1}{\mu}\right)^2\right] \tag{3.5.3}$$

Combining Equations (3.5.2b) and (3.5.3) yields the total height, H, reached by the rocket,

$$H = z(T) + \Delta H = \left[-\frac{g\,T^2}{2} + v_f\,T + (\mu-1)\,v_f\,T\,\ln\frac{\mu-1}{\mu}\right]$$
$$+ \left[\frac{g\,T^2}{2} + v_f\,T\,\ln\frac{\mu-1}{\mu} + \frac{v_f^2}{2\,g}\,\left(\ln\frac{\mu}{1+\mu}\right)^2\right]$$
$$= v_f\,T\,\left[1 + \mu\,\ln\frac{\mu-1}{\mu}\right] + \frac{v_f^2}{2\,g}\,\left(\ln\frac{\mu-1}{\mu}\right)^2 \tag{3.5.4}$$

For $m > 0$ we have $\mu > 1$, and the expression in the square bracket is always negative, as one can see based on the inequality $\ln x \le x - 1$ shown Figure 3.10,

$$1 + \mu\,\ln(1+\mu^{-1}) \le 1 + \mu\,\left(1 + \mu^{-1} - 1\right) = 0$$

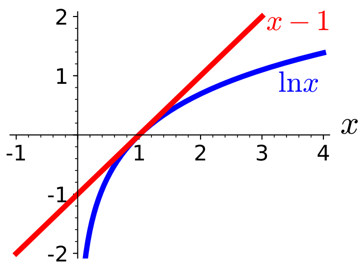


Figure 3.10: The function $x - 1$ (red) is always larger (or equal) than $\ln x$ (blue).

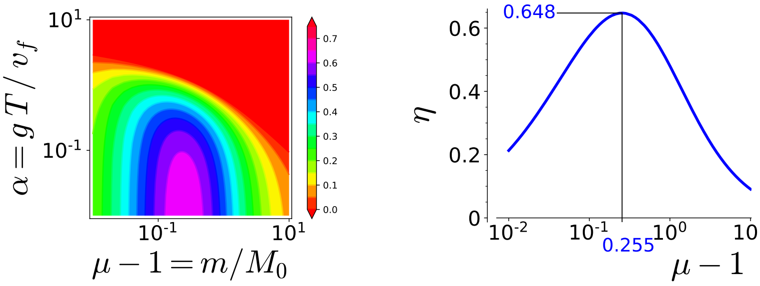


Figure 3.11: (left) Contour line for the efficiency, Equation (3.5.5), as function of the mass ratio $m/M_0 = \mu - 1$ and the dimensionless inverse rocket acceleration $\alpha = gT/v_f$. The maximum is taken for $\alpha = 0$. (right) Plot of the μ dependence of the efficiency for $\alpha = 0$. The maximum efficiency of $\eta_{\text{opt}} \simeq 0.648$ is obtained for $\mu_c \simeq 1.255$.

The best strategy to achieve a large height is to go for a small T in order to suppress the first term in Equation (3.5.4) and large v_f to achieve large values of the second term.

When energy efficiency is a concern, e. g. when the rocket is used for a measurement of the atmosphere at height H, one might be interested to reach the height H with minimum energy cost. This means one is interested to minimize the ratio of the potential energy of the rocket at height H and the the energy $M_0 v_f^2/2$ burned to deliver the freight,

$$
\begin{aligned}
\eta = \frac{m\,g\,H}{M_0 v_f^2/2} &= \frac{2\,m\,g\,T}{M_0\,v_f} \left[1 + \mu \,\ln\frac{\mu-1}{\mu}\right] + \frac{m}{M_0} \left(\ln\frac{\mu}{1+\mu}\right)^2 \\
&= \frac{2\,g\,T}{v_f}\,(\mu-1) \left[1 + \mu \,\ln\frac{\mu-1}{\mu}\right] + (\mu-1) \left(\ln\frac{\mu-1}{\mu}\right)^2
\end{aligned}
\tag{3.5.5}
$$

The efficiency is a function of μ and of the dimensionless number $\alpha = g\,T/v_f$. The contour lines of $\eta(\mu,\alpha)$ are plotted in the left panel of Figure 3.11.

Definition 3.7: Contour lines and isosurfaces

The *contour lines* of a two variable function $f(x,y)$ are those lines in the (x,y)-plane, where $f(x,y)$ takes some constant value. More generally these lines are also called *isolines*, the two-dimensional surfaces where a three-variable function $g(x,y,z)$ in the (x,y,z)-space takes constant values are called isosurfaces, and the $N-1$-dimensional hypersurfaces of $\mathbb{R}^N$ where the function $h(\boldsymbol{q})$ with $\boldsymbol{q} \in \mathbb{R}^N$ takes a constant values will also be denoted as isosurfaces.

Example 3.14: Isosurfaces of the 3D Gaussian distribution

The 3D Gaussian distribution

$$ P(x,y,z) = \frac{1}{(2\pi D\,t)^{3/2}}\,\exp\left(-\frac{x^2+y^2+z^2}{2Dt}\right) $$

describes the distribution of dye molecules at time t when a tiny droplet of dye is added without motion in a large container of water (Brownian motion). At any given instant of time the surfaces where the concentration take constant values C amount to

$$C = (2\pi D\,t)^{-3/2}\,\exp\big(-(x^2+y^2+z^2)/(2Dt)\big)$$
$$\Leftrightarrow\quad x^2+y^2+z^2 = -2Dt\,\ln\big(C\,(2\pi D\,t)^{3/2}\big) = R^2$$

where R^2 is an abbreviation of the (positive) constant on the right-hand side of the equation. Hence, the isosurface I_R for a given R amounts to a sphere of radius R,

$$I_R = \{(x,y,y) \in \mathbb{R}:\ x^2+y^2+z^2 = R^2\}\,.$$

The contour lines of the efficiency reveal that the maximum efficiency is obtained for $\alpha = 0$, which can expected since the expression in square brackets in Equation (3.5.5) is negative. From a physics perspective it means that high efficiencies require a large fuel expulsion speed v_f. The maximum efficiency amounts to the maximum of $\eta(\mu,\alpha=0) = \mu\ \ln^2[\mu/(1+\mu)]$, which amounts to the root μ_c of the equation $2/(1+\mu) + \ln[\mu/(1+\mu)]$. Numerically it is found to be $\mu_c \simeq 0.255$. Hence, the maximum efficiency is obtained when the mass of the fuel M_0 is roughly four times larger than the mass of the empty rocket. The maximum efficiency amounts then to

$$\eta_{\max} = \frac{4\,\mu_c}{(1+\mu_c)^2} \simeq 0.648\,.$$

Irrespective of the rocket design one can not transform more than 2/3 of the energy of the fuel into potential energy of the rocket. The remaining energy is dissipated in the kinetic energy of the exhaust.

Further discussion of the trajectories of rockets can be found in Finney (2000); Gale (1970); Seifert *et al.* (1947). A discussion of water rockets that addresses the change of speed v_f of the ejected water was given in Kagan *et al.* (1995); Gommes (2010).

3.6 *Problems*

3.6.1 *Practicing Concepts*

Problem 3.13. **Car on an air-cushion**

We consider a car of mass $m = 20\,\text{g}$ moving – to a very good approximation without friction – on an air-cushion track. There is a string attached to the car that moves over a roll and hangs vertically down on the side opposite to the car.

a) Sketch the setup and the relevant parameters.

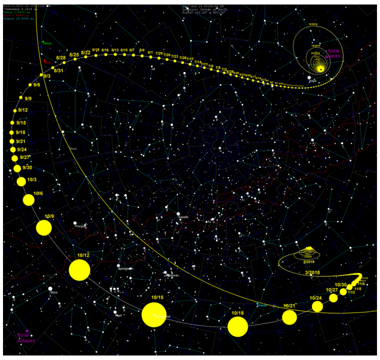
b) Which acceleration is acting on the car when the string is vertically pulled down with a force of $F = 2\,\text{N}$. Determine the velocity $v(t)$ and its position $x(t)$.

c) Determine the force acting on a 200 g chocolate bar, in order to get a feeling for the size of the force that was considered in (b).

d) Now we fix the chocolate bar at the other side of the string. The velocity of the car can then be obtained based on energy conservation

$$E = E_{\text{kin}} + E_{\text{pot}} = \frac{m+M}{2}\, v^2 + Mgh = \text{konst},$$

where M is the mass of the chocolate bar. Is the acceleration the same of different as in the cases (b) and (c)? Provide an argument for your conclusion.



Tomruen/wikimedia CC BY-SA 4.0
Figure 3.12: 'Oumuamua trajectory as seen by an observer on Earth.

Problem 3.14. **'Oumuamua**

On 19 October 2017 astronomers at the Haleakala Observatory in Hawaii discovered 'Oumuamua, the first interstellar object observed in our solar system. It approached the solar system with a speed of about $v_I = 26\,\text{km/s}$ and reached a maximum speed of $v_P = 87.71\,\text{km/s}$ at its perihelion, i. e. upon closest approach to the sun on 9 September 2017.

a) Show that at the perihelion the speed and 'Oumuamua's smallest distance to the sun, D, obey the relation

$$\frac{v_P^2 - v_I^2}{2} = \frac{M_S\, G}{D}$$

while for the Earth we always have

$$\frac{4\pi^2 R}{T^2} \simeq \frac{M_S\, G}{R^2}$$

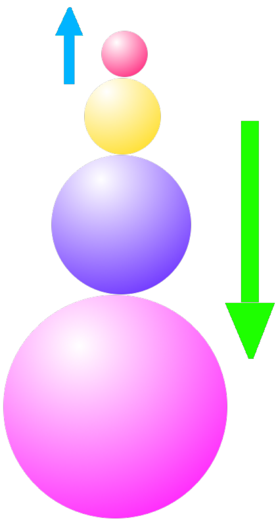
Here, M_S is the mass of Sun, R is the Earth-Sun distance, and $T = 1$ year is the period of Earth around Sun.

b) Show that this entails that $\frac{D}{R} = \frac{2\,v_E^2}{v_P^2 - v_I^2}$, where $v_E = 2\pi R/T$ is the speed of Earth around sun.

c) Use the relation obtained in (b) to determine D in astronomical units, and compare your estimate with the observed value $D = 0.25534(7)\,\text{AU}$.

Problem 3.15. **Galilean cannon**

In the margin we show a sketch of a Galilean cannon. Assume that the mass mass ratio of neighboring balls with always two, and that they perform elastic collisions.



SteveBaker/wikimedia, CC BY-SA 3.0

a) Initially they are stacked exactly vertically such that their distance is negligible. Let the distance between the ground the lowermost ball be 1 m. How will the distance of the balls evolve prior to the collision of the lowermost ball with the ground?

b) After the collision with the ground the balls will move up again. Determine the maximum height that is reached by each of the balls.

Problem 3.16. **Motion in a harmonic central force field**

A particle of mass m and at position $\boldsymbol{r}(t)$ is moving under the influence of a central force field

$$\boldsymbol{F}(\boldsymbol{r}) = -k\,\boldsymbol{r}\,.$$

a) We want to use the force to build a particle trap,[7] i. e. to make sure that the particle trajectories $\boldsymbol{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\boldsymbol{r}(t)| < B$ for all times t. What is the requirement on the sign of the constant k to achieve this aim?

b) Determine the energy of the particle and show that its energy is conserved.

c) Demonstrate that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times m\,\dot{\boldsymbol{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field is no longer coincide with the origin of the coordinate system in that case.

[7] Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

Problem 3.17. **Collision with an elastic bumper**

Consider two balls of radius R with masses m_1 and m_2 that are moving along a line. Their positions will be denoted as x_1 and x_2 in such a way that they touch when $x_1 = x_2$ and they do not feel each other when $x_1 < x_2$. When they run into each other, the balls can slightly be deformed such that the distance between their centers takes the value $2R - d$, and they experience a harmonic repulsive forces $\pm k\,d$. We will say then that $d = x_2 - x_1 < 0$.

a) Newton's equations for the collision of the two balls take the form

$$m_1\,\ddot{x}_1(t) = -k\,d(t) \qquad\qquad m_2\,\ddot{x}_2(t) = k\,d(t)$$

Show that this implies

$$\ddot{d} = -\omega^2\,d$$

for some positive constant ω. How does ω depend on the spring constant k and on the masses m_1 and m_2?

b) Let $d(t) = -d_M\,\sin\big(\omega\,(t - t_0)\big)$ describe the deformation of the balls for a collision at $t = t_0$, and contact in the time interval $t_0 \leq t \leq t_R$. Verify that it is a solution of the equation of motion. At which time t_R will the particles release (i.e. there is no overlap any longer)? What is the maximum potential energy stored in the harmonic potential?

c) We consider initial conditions where particle 1 arrives with a constant velocity v_0 from the left, and particle 2 is at rest. What is the total kinetic energy in this situation? Assume that at most a fraction α of the kinetic energy is transferred to potential energy. What is the relation between v_0 and the maximum deformation d_M?

d) The velocity of the two particles at times $t_0 \le t \le t_R$ can now be obtained by solving the integrals

$$m_i \, \dot{x}_i(t) = m_i \, x_i(t_0) + (-1)^i \int_{t_0}^{t} \mathrm{d}t' \, k \, d(t') \,, \quad \text{with } i \in \{1,2\}$$

Why does this hold? Which values does $x_i(t_0)$ take? Solve the integral and show that

$$\dot{x}_1 = v_0 \, \left[1 + \sqrt{\alpha\beta} \left(\cos(\omega\,(t - t_0)) - 1\right)\right]$$
$$\dot{x}_2 = v_0 \, \frac{m_1}{m_2} \, \sqrt{\alpha\beta} \left(\cos(\omega\,(t - t_0)) - 1\right)$$

How does β depend on the masses?

e) Verify that at release we have

$$\dot{x}_1 = v_0 \, (1 - 2\sqrt{\alpha\beta})$$
$$\dot{x}_2 = v_0 \, \frac{2\,m_1}{m_2} \, \sqrt{\alpha\beta})$$

Verify that these expressions comply to momentum conservation. Verify that the expressions obey energy conservation iff $\alpha = \beta = m_2/(m_1 + m_2)$.

f) What does this imply for particles of identical masses, $m_1 = m_2$? How does your result fit to the motion observed in Newton's cradle? What does it tell about the assumption of instantaneous collisions of balls that is frequently adopted in theoretical physics?

Problem 3.18. **Inelastic collisions, ballistics, and cinema heroes**

Let us take a look at how cinema heroes shoot.

"Free Metal Jacket" movie poster (wikimedia fair use)

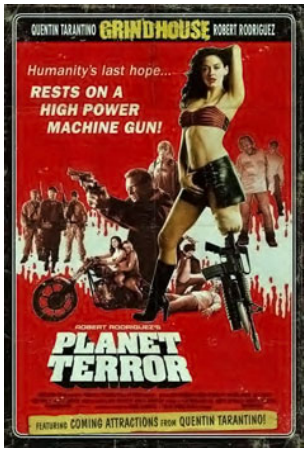
a) The title of Stanley Kubrick's movie Full Metal Jacket refers to full metal jacket bullets, i. e. projectiles as they were used in the M16 assault rifle used in the Vietnam war. Its bullets have a mass of 10 g and they set a 1 kg wooden block revolving at a 1 m arm into a 8 Hz motion. What is the velocity of the bullets?

The bullets of a 9 mm Luger pistol have a mass of 8 g and they are fired with a muzzle velocity of $350\,\mathrm{m\,s^{-1}}$. What is the resulting angular speed $\dot{\theta}$ of the wooden block?

b) Alternatively one can preform this measurement by shooting the bullet into a swing where a wooden block of mass M is attached to ropes of length ℓ. Initially it is at rest. Consider angular momentum conservation to determine its velocity immediately after impact. What does this tell about the kinetic energy immediately after the impact, and what about the maximum height of reached by the swing in its subsequent motion?

Let L be 2 m. Which mass is required to let the swing go up to the height of its spindle?

What does this tell about the recoil of the pistol and the rifle? What do you think now about the shooting scenes that you might recall from Rambo movies or grindhouse movies like Planet Terror.



Dutch Movie poster of Planet Terror. (wikimedia fair use license)

3.6.2 *Mathematical Foundation*

Problem 3.19. **Solving integrals by partial integration**

Evaluate the following integrals by partial integration

$$\int \mathrm{d}x\, f(x)\, g'(x) = f(x)\, g(x) - \int \mathrm{d}x\, f'(x)\, g(x)$$

a) $\int_a^b \mathrm{d}x\, x\, \mathrm{e}^{kx}$

b) $\int_a^b \mathrm{d}x\, x^2\, \mathrm{e}^{kx}$

c) $\int_0^\infty \mathrm{d}x\, x^3\, \mathrm{e}^{-x^2}$

d) $\int_a^b \mathrm{d}x\, x^n\, \mathrm{e}^{kx},\ n \in \mathbb{N}$

The integral d) can only be given as a sum over $j = 0, \dots, n$.

add: problems for line integrals, in particular parameterization with length

add: calculation of the length of a path

add: problems for contour lines

Problem 3.20. **Substitution with trigonometric and hyperbolic functions**

Evaluate the following integrals by employing the suggested substitution, based on the substitution rule

$$\int_{f(x_I)}^{f(x_F)} \mathrm{d}f\, g(f) = \int_{x_I}^{x_F} \mathrm{d}x\, \frac{\mathrm{d}f(x)}{\mathrm{d}x}\, g(f(x))$$

with a function $f(x)$ that is bijective on the integration interval $[x_I, x_F]$. A graphical illustration of the rule is given in Figure 3.13.

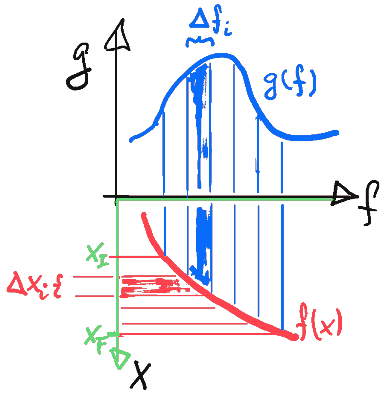


Figure 3.13: Illustration of the substitute rule for integrals that may be represented in terms of a Riemann sum ():

$$\int_{f(x_I)}^{f(x_F)} \mathrm{d}f\, g(f) \simeq \sum_i \Delta f_i\, g(f_i) \simeq \sum_i \Delta x_i\, \frac{\Delta f_i}{\Delta x_i}\, g(f(x_i)) \simeq \int_{x_I}^{x_F} \mathrm{d}x\, \frac{\mathrm{d}f(x)}{\mathrm{d}x}\, g(f(x))$$

provide reference

a) $\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}}$ by substituting $x = \sin\theta$

b) $\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{1+x^2}}$ by substituting $x = \sinh z$

c) $\int_a^b \mathrm{d}x\, \frac{1}{1+x^2}$ by substituting $x = \tan\theta$

d) $\int_a^b \mathrm{d}x\, \frac{1}{1-x^2}$ by substituting $x = \tanh z$

Problem 3.21. **Gradients and contour lines**

a) Contour lines in the (x, y)-plane are lines $y(x)$ or $x(y)$ where a functions $f(x, y)$ takes a constant value (cf. Definition 3.7). Sketch the contour lines of the functions

$$f_1(x,y) = (x^2 + y^2)^{-1} \qquad \text{and} \qquad f_2(x,y) = -x^2\, y^2$$

b) Determine the gradients $\nabla f_1(x, y)$ and $\nabla f_2(x, y)$.
Hint: The gradient $\nabla f(x, y)$ of a function $f(x, y)$ is the vector $(\partial_x f(x, y), \partial_y f(x, y))$ that contains the two partial derivatives of the (scalar) function $f(x, y)$ (cf. Equation (3.4.2)).

c) Indicate the direction and magnitude of the gradient by appropriate arrows in the sketch showing the contour lines. In which direction is the gradient pointing?

DJ Spooky playing venyl records at the Sundance Film Festival (2003), Eddie Codel (Ekai) via Wikimedia Commons, CC BY-SA 3.0

Problem 3.22. **Length of the groove of venyl records**

The sound information in venyl records is stored in small undulations of the walls of a groove that runs in a spiral from the outer rim of the disc, at a distance R_o from the centre, towards its center where it stops at a inner radius $R_i < R_o$. Neighboring lanes of the groove have a fixed distance d. The sound is picked up by a needle that glides through the groove when the disc is turned. Let $\theta = 0$ be the angle where the needle first touched the disc at time t_0, and $\theta(t)$ be the overall traversed angle at time $t \geq t_0$. In the frame of the disc we will denote the position of the needle as

$$\boldsymbol{q}(\theta) = R(\theta)\, \hat{\boldsymbol{r}}(\theta)$$

Here $\hat{\boldsymbol{r}}(\theta)$ is the radial unit vector of polar coordinates taken with respect to the center of the disc, and $R(\theta)$ is the distance from the center.

a) Verify that $R(\theta) = R_a - \varepsilon\, \theta$. What is the relation between ε and d?

b) What is the speed v of the needle while it glides through the groove?
Show that one can express v as follows

$$v = \dot{\theta}\, \sqrt{R^2 + f(R)}$$

Determine $f(R)$.

⋆ c) Demonstrate that the length L of the groove that is traversed while turning from θ_I to θ_E can be written as

$$L = \int_i^e \mathrm{d}\theta\, \sqrt{1 + \theta^2}$$

How do i and e depend on θ_I and θ_E?

d) Observe that

$$\sqrt{1+\theta^2} = \frac{1}{\sqrt{1+\theta^2}} + \frac{x^2}{\sqrt{1+\theta^2}}$$

where the first term is the derivative of $\operatorname{arcsinh}(x)$ and the latter term is related to $\sqrt{1+\theta^2}$ by partial integration. Use this information to evaluate L.

⋆ e) Compare your result with the following estimate:
i) The groove covers an area of size $A = \pi R_o^2 - \pi R_i^2$.
Why does this hold? How are R_o and R_i related to θ_I und θ_E?
ii) The area can also be estimated by multiplying the length L of the groove with the groove distance, d, such that also $A \approx L \times d$.
How well does the resulting estimate of L agree with the result of the explicit calculation obtained in d).

f) An LP has an outer radius of $R_o = 15\,\text{cm}$ and the groove stops at the inner radius of about $R_i = 8\,\text{cm}$. It is played with an angular speed of $33\frac{1}{3}$rpm (revolutions per minute), and each side is playing for about 22 min. What is the length of the groove and what is the distance d between its neighboring revolutions?

3.6.3 *Transfer and Bonus Problems, Riddles*

Problem 3.23. **Moeschenbroeks double-cone experiment**


User:FA2010, Public domain

In the margin we show Moeschenbroeks double-cone experiment. The setup involves three angles:

1. The opening angle α between the two rails.
2. The angle ϕ of the rail surface with the horizontal.
3. The opening angle θ of the cone.

When it is released from the depicted position the cone might move to the right, to the left, and it could stay where it is. How does the selected direction of motion depend on the choice of the three angles?

Problem 3.24. **Coulomb potential and external electric forces**

We consider the Hydrogen atom to be a classical system as suggested by the Bohr-Sommerfeld model. Let the proton be at the center of the coordinate system and the electron at the position $\boldsymbol{r}$. The interaction between the proton and the electron is described by the Coulomb potential $\alpha/|\boldsymbol{r}|$. In addition to this interaction there is a constant electric force acting, that is described by the potential $\boldsymbol{F} \cdot \boldsymbol{r}$. Altogether the motion of the electron is therefore described by the potential

$$U = -\frac{\alpha}{|\boldsymbol{r}|} - \boldsymbol{F} \cdot \boldsymbol{r}$$

a) Sketch the system and the relevant parameters.

b) Which force is acting on the particle? How do its equation of motion look like?

c) Verify that the energy is conserved.

d) Show that also the following quantity is a constant of motion,

$$I = \boldsymbol{F} \cdot (\dot{\boldsymbol{r}} \times \boldsymbol{L}) - \alpha \, \frac{\boldsymbol{F} \cdot \boldsymbol{r}}{|\boldsymbol{r}|} + \frac{1}{2} \, (\boldsymbol{F} \times \boldsymbol{r})^2$$

Here $\boldsymbol{L}$ is the angular momentum of the particle with respect to the origin of the coordinate system.

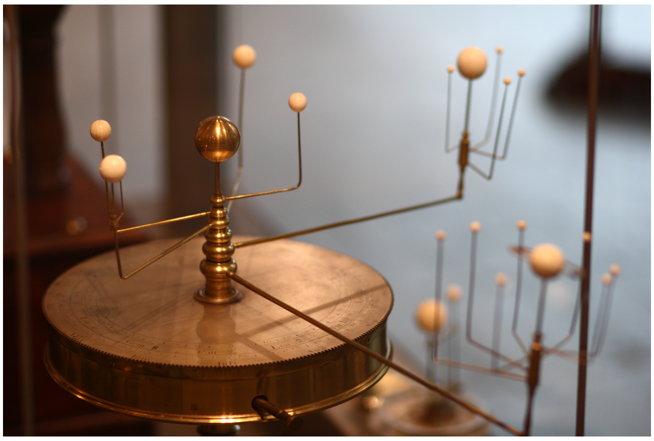
3.7 Further reading

Sommerfeld's (1994) classical discussion of Newton's axioms dates back to the 1940s, but still is a one of the most superb expositions of the topic.

A comprehensive discussion of the flight of water bottle rockets has been given in Finney (2000), and it has been augmented by a discussion of subtle corrections involving the thermodynamic expansion of air in Gommes (2010).

4
Motion of Point Particles

In Chapter 3 we learned how to set up a physical model based on finding the forces acting on a body, and thus determining the acceleration of its motion. For a particle of mass m and position $\boldsymbol{q}$ Newton's second law relates its acceleration $\ddot{\boldsymbol{q}}$ to the force that is acting on the particle. In Chapter 2 we saw that the total force $\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ acting on the particle may depend on $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and t. The resulting relation between the acceleration and the force is called equation of motion of the particle, Definition 3.3. In the present chapter we will discuss approaches that will allow us to systematically find the solutions of EOMs. Moreover, we will explore what type of behavior is encountered for different types of initial conditions.



Mechanical planetarium used to teach astronomy at Harvard

At the end of this chapter we will discuss the motion of planets around the sun, moons around their planets, and will be able to figure out which rules determine the intricate trajectory of 'Oumuamua shown in Figure 3.12.

4.1 Motivation and outline: EOM are ODEs

From the mathematical point of view the equation of motion is an *ordinary differential equation* (ODE).

Definition 4.1: Ordinary Differential Equation (ODE)

An *ordinary differential equation (ODE)* of n^{th} order for a function $f(t)$ expresses the n^{th} derivative of the function,

$$f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)$$

as a function of time and the lower derivatives of the function, $f^{(n-1)}(t), \dots, f^{(1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$, $f^{(0)}(t) = f(t)$,

$$f^{(n)}(t) = F(f^{(n-1)}(t), \dots, f(t), t) .$$

Here, f and F may be scalar or vector valued functions.

Remark 4.1. The EOM for a particle at position $\boldsymbol{q} \in \mathbb{R}^3$ is a second order ODE where the second time derivative $\ddot{\boldsymbol{q}}(t)$ of the vector valued function $\boldsymbol{q}(t)$ (the position of the particle) is related to $\boldsymbol{F}/m$, which is a vector that depends on $\dot{\boldsymbol{q}}$, $\boldsymbol{q}$ and t; cf. Definition 3.3.

Remark 4.2. A differential equation is called an *ordinary differential equation*, when all derivatives are taken with respect to the same variable. When discussing the physics of waves, e. g. for the full description of Tsunami waves mentioned in Example 1.11, to deal with electromagnetic waves or gravitational waves, one has to deal with differential equations involving space and time derivatives. These type of equations are called *partial differential equations (PDE)*. In Leipzig they are addressed in the course "Theoretical Physics II".

Commonly, the forces in an EOM for a particle only depend on particle positions and velocities, and not explicitly on time. The forces only depend on the particle configuration, and they will be the same irrespective of whether I measure them today or when my grand-daughter determines them with her grand-children at the dawn of the next century.

Definition 4.2: Autonomous Equations of Motion

An ODE is called *autonomous* when its right-hand side does not explicitly depend on time. In particular an autonomous EOM takes the form

$$m\,\ddot{\boldsymbol{q}}(t) = \boldsymbol{F}(\dot{\boldsymbol{q}}(t), \boldsymbol{q}(t)) .$$

The forthcoming discussion of ODEs makes use of the very important observation that every ODE can be stated as first order ODE in some abstract phase space. We introduce this idea for N particles with masses m_i, $i = 1 \cdots N$ that are moving in D dimensions. According to Definition 3.3 their motion is described by a system of $N\,D$ differential equations for the coordinates of the D dimensional vectors $\boldsymbol{q}_i = (q_{i,\alpha}, \alpha = 1 \cdots D)$

$$\ddot{q}_{i,\alpha} = \frac{1}{m_i}\, F_{i,\alpha}(\{\dot{\boldsymbol{q}}_i, \boldsymbol{q}_i\}_{i=1\cdots N}, t)\,, \quad i = 1 \cdots N\,, \;\; \alpha = 1 \cdots D$$

To avoid clutter in the equations we did not explicitly state here the time dependence of $\ddot{q}_{i,\alpha}(t)$, $\dot{q}_{i,\alpha}(t)$, and $q_{i,\alpha}(t)$.

By introducing the variables $\boldsymbol{v}_i = \dot{\boldsymbol{q}}_i$ the EOMs can be written as a set of $2\,D\,N$ first order ODEs

$$\begin{aligned} \dot{q}_{i,\alpha} &= v_{i,\alpha} \\ \dot{v}_{i,\alpha} &= \frac{1}{m_i}\, F_{i,\alpha}(\{\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i\}_{i=1\cdots N}, t) \end{aligned}$$

For an autonomous system this can be written in a more compact form by introducing the $2DN$ dimensional phase-space coordinate $\boldsymbol{\Gamma}$ and the flow $\boldsymbol{\mathcal{V}}$ as follows

$$\begin{aligned} \boldsymbol{\Gamma} &= \big(q_{1,1} \cdots q_{1,D}\,,\; q_{2,1} \cdots q_{N,D}\,,\; \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\; \dot{q}_{2,1} \cdots \dot{q}_{N,D}\big) \\ \boldsymbol{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\; v_{2,1} \cdots v_{N,D}\,,\; \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\; \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\right) \\ \dot{\boldsymbol{\Gamma}} &= \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}) \qquad \text{for autonomous systems.} \end{aligned}$$

Moreover, a non-autonomous system can always be expressed as an autonomous, first order ODE where $\boldsymbol{\Gamma}$ and $\boldsymbol{\mathcal{V}}$ denote points in a $2DN + 1$ dimensional phase space,

$$\begin{aligned} \boldsymbol{\Gamma} &= \big(q_{1,1} \cdots q_{1,D}\,,\; q_{2,1} \cdots q_{N,D}\,,\; \dot{q}_{1,1} \cdots \dot{q}_{1,D}\,,\; \dot{q}_{2,1} \cdots \dot{q}_{N,D}\,,\; t\big) \\ \boldsymbol{\mathcal{V}} &= \left(v_{1,1} \cdots v_{1,D}\,,\; v_{2,1} \cdots v_{N,D}\,,\; \frac{F_{1,1}}{m_1} \cdots \frac{F_{1,D}}{m_1}\,,\; \frac{F_{2,1}}{m_2} \cdots \frac{F_{N,D}}{m_N}\,,\; 1\right) \\ \dot{\boldsymbol{\Gamma}} &= \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}) \qquad \text{for non-autonomous systems.} \end{aligned}$$

In phase space, $\boldsymbol{\Gamma}$ denotes a point that characterizes the state of our system, and $\boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma})$ provides the *unique* direction and velocity of the temporal change of this state. In an approximation, that is accurate for sufficiently small Δt, we have

$$\boldsymbol{\Gamma}(t + \Delta t) \simeq \boldsymbol{\Gamma}(t) + \Delta t\; \boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma}(t))$$

In phase space the ODE therefore can be represented as a field of vectors $\boldsymbol{\mathcal{V}}(\boldsymbol{\Gamma})$ that represent signposts signifying which direction a trajectory will take when it continues from this point, and how fast it will proceed.

Definition 4.3: Phase-Space Plot

A *phase-space plot* provides an overview of all solutions of an ODE by marking the direction of motion of the trajectories in phase space by arrows, and showing the evolution of a representative set of trajectories by solid lines. At times such a plot is therefore also denoted as the *phase-space portrait* of the solutions of an ODE.

Remark 4.3. For an autonomous system with a single DOF

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = m^{-1}\, F\big(v(t), x(t)\big)$$

the phase-space portrait is a two-dimensional plot with arrows $\big(v, F(v,x)/m\big)$ at the positions (x, v) in the plane, and trajectories $v(x)$. One can only see the shape of the trajectories, and not their time dependence.

Example 4.1: Phase-space plot for the harmonic oscillator

The EOM of the harmonic oscillator is $\ddot{x}(t) = -\omega^2\, x(t)$ where ω can be absorbed into the time scale by adopting the dimensionless time $\tau = \omega t$. Its dimensionless energy is then given by

$$E = \frac{v^2}{2} + \frac{x^2}{2} \quad \text{with} \quad \begin{cases} \dot{x} &= v \\ \dot{v} &= -x \end{cases}$$

The energy is conserved because

$$\frac{\mathrm{d}}{\mathrm{d}t} E = x\,\dot{x} + v\,\dot{v} = x\,v - v\,x = 0$$

Therefore trajectories in phase space amount to contour lines of the the energy function $E(x, v)$. This is shown in Figure 4.1 where the energy is marked by color coding and the direction of the flow is provided by arrows.

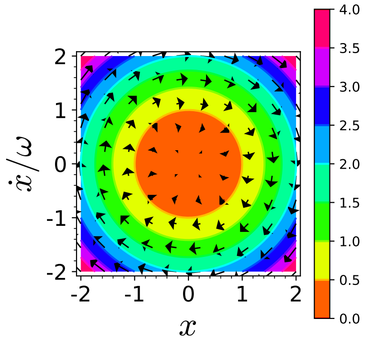


Figure 4.1: Color plot of contour lines of the energy of the harmonic oscillator, and the flow field of its EOM.

Outline

The forthcoming discussion in the present chapter will provide

i. a classification of ODEs with an emphasis on strategies to find solutions for specific initial conditions, and
ii. further discussion of phase-space plots used to characterize sets of solutions.

The methods will be introduced and motivated based on elementary physical problems that will serve as examples of particular relevance in physics.

4.2 *Integrating ODEs — Free flight*

We first discuss the motion of a single particle moving in a gravitational field that gives rise to the constant gravitational acceleration $\boldsymbol{g}$. Hence, the particle position $\boldsymbol{q}(t)$ obeys the EOM

$$\ddot{\boldsymbol{q}} = \boldsymbol{g} \tag{4.2.1}$$

The right hand side of this equation is constant. It neither depends on $\dot{\boldsymbol{q}}$, $\boldsymbol{q}$, nor explicitly on t. This has two remarkable consequences that we will exploit whenever possible.

4.2.1 Decoupling of the motion of different DOF

Each component q_α of $\boldsymbol{q}$ can be solved independently of the other DOF

$$\dot{q}_\alpha = g_\alpha$$

Rather than dealing with a vector-valued ODE, one can therefore solve D scalar ODEs which turns out to be a much simpler task. Indeed, we will see in our further discussion that the solution of vector-valued ODEs will often proceed via a coordinate transformation that decouples the different DOF.

4.2.2 Solving ODEs by integration

The ODE, Equation (4.2.1), can be solved by integration

Algorithm 4.1: Integrating ODEs

An ODE for $\boldsymbol{f}(t)$ can be solved by *integration* when its right-hand side does not depend on on $\boldsymbol{f}(t)$ and its derivatives, i. e. when it takes the form

$$\dot{\boldsymbol{f}}(t) = \boldsymbol{g}(t)$$

For the initial condition $\boldsymbol{f}(t_0) = \boldsymbol{f}_0$ one can then expresses the solution of the ODE in terms of an integral,[1]

$$\boldsymbol{f}(t) = \boldsymbol{f}_0 + \int_{t_0}^{t} \mathrm{d}t' \, \dot{\boldsymbol{f}}(t') = \boldsymbol{f}_0 + \int_{t_0}^{t} \mathrm{d}t' \, \boldsymbol{g}(t')$$

For an autonomous ODE, where $\boldsymbol{g}(t) = \boldsymbol{c} = \text{const}$, one thus obtains

$$\boldsymbol{f}(t) = \boldsymbol{f}_0 + \boldsymbol{c} \cdot (t - t_0)$$

[1] The prime of the integration variable t' indicates here that the integration variable t' must not be confused with the boundary t of the integral. Physicists often drop the prime with the understanding that this simplifies notation and there is no danger of confusion (after this note has been made).

The idea underlying the algorithm can be understood by reading the equations in reverse order and taking into account the substitution rule for integration,

$$\int_{t_0}^{t} \mathrm{d}t' \, \boldsymbol{g}(t') = \int_{t_0}^{t} \mathrm{d}t' \, \dot{\boldsymbol{f}}(t') = \int_{\boldsymbol{f}(t_0)}^{\boldsymbol{f}(t)} \mathrm{d}\boldsymbol{f} = \boldsymbol{f}(t) - \boldsymbol{f}(t_0)$$

4.2.3 Integrating the EOM for free flight

For the free flight only the constant acceleration $\boldsymbol{g}$ due to gravity is acting on the particle such that $\ddot{\boldsymbol{q}}(t) = \boldsymbol{g}$. For the initial conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}(t_0) = \boldsymbol{v}_0$ Algorithm 4.1 provides the velocity

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{v}_0 + \int_{t_0}^{t} \mathrm{d}t' \boldsymbol{g} = \boldsymbol{v}_0 + \boldsymbol{g} \cdot (t - t_0)$$

This equation can be integrated again, providing the position of the particle

$$\begin{aligned} \boldsymbol{q}(t) &= \boldsymbol{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \dot{\boldsymbol{q}}(t) = \boldsymbol{q}_0 + \int_{t_0}^{t} \mathrm{d}t' \left(\boldsymbol{v}_0 + \boldsymbol{g}\,(t - t_0)\right) \\ &= \boldsymbol{q}_0 + \boldsymbol{v}_0 \int_{t_0}^{t} \mathrm{d}t' + \boldsymbol{g} \int_{t_0}^{t} \mathrm{d}t'\ (t - t_0) \\ &= \boldsymbol{q}_0 + \boldsymbol{v}_0\,(t - t_0) + \boldsymbol{g} \int_{0}^{t-t_0} \mathrm{d}t''\, t'' \\ &= \boldsymbol{q}_0 + \boldsymbol{v}_0\,(t - t_0) + \frac{1}{2}\,\boldsymbol{g}\,(t - t_0)^2 \end{aligned}$$

When we introduce the components of the vectors $\boldsymbol{q}$ and $\boldsymbol{v} = \dot{\boldsymbol{q}}$ as $\boldsymbol{q} = (q_1, q_2, \dots)$ and $\boldsymbol{v} = (v_1, v_2, \dots)$, and choose the component direction anti-parallel to $\boldsymbol{g}$ as $z = q_1$, then

$$z(t) = q_1(t) = z(t_0) + v_1(t_0)\,(t - t_0) - \frac{g}{2}\,(t - t_0)^2 \tag{4.2.2a}$$

$$q_i(t) = q_i(t_0) + v_i(t_0)\,(t - t_0)\,, \quad \text{for } i > 1 \tag{4.2.2b}$$

It is illuminating to discuss these solutions form the perspective of non-dimensionalization and the evolution in phase space.

For $i > 1$ the EOM is $\ddot{q}_i = 0$. In phase space the direction field at (q_i, v_i) is then given by the vectors $(v_i(t_0), 0)$ pointing in horizontal direction, as shown in Figure 4.2(left). Moreover, for Equation (4.2.2b) we have $\dot{q}_i(t) = v_i(t_0) = \text{const}$ irrespective of $q_i(t)$. Therefore, the solutions take the form of horizontal lines. When introducing dimensionless units by adopting the velocity scale $v_i(t_0)$ one obtains

$$\hat{v}_i(t) = \frac{\dot{q}_i(t)}{\dot{q}_i(t_0)} = 1$$

For this problem all trajectories are identical up to a rescaling of the length and time units. By rescaling, all horizontal lines in the phase space can be mapped into the same dimensionless solution. From the point of view of the Buckingham-Pi Theorem 1.1 this is due to the fact that there are no dimensionless parameters in the solutions—not even due to the choice of initial conditions.

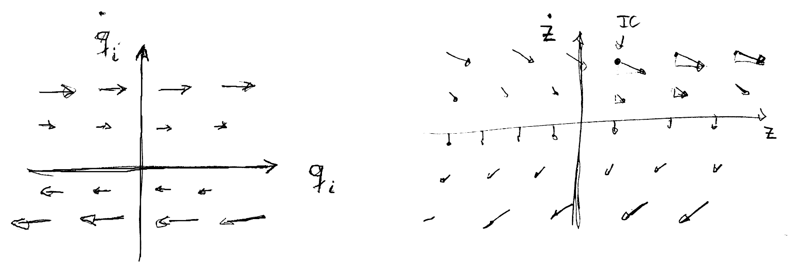


Figure 4.2: Phase-space flows for motion for free flight. (left) For direction perpendicular to g where there is no acceleration. The trajectories are horizontal lines. (right) For z anti parallel to g there is a constant acceleration $-g$. The trajectories take the form of parabola that are open to the left.

For Equation (4.2.2a) the arrows at position (z, v_z) in the phase space are directed to $(v, -g)$. For $v = 0$ they point straight down, for large v they point right and only a little bit down, and for large negative v they point left and only a little bit down, as marked in

Figure 4.2(right). The phase-space trajectories are found by observing that $\dot{z} = v_z(t_0) - g\,(t - t_0)$ implies $t - t_0 = \big[v_z(t_0) - v_z\big]/g$ such that

$$
\begin{aligned}
z &= z(t_0) + v_z(t_0)\,\frac{v_z(t_0) - v_z}{g} - \frac{\big[v_z(t_0) - v_z\big]^2}{2\,g} \\
&= z(t_0) + \frac{v_z(t_0) - v_z}{2\,g}\,\big[2\,v_z(t_0) - (v_z(t_0) - v_z)\big] \\
&= z(t_0) + \frac{v_z^2(t_0) - v_z^2}{2\,g}
\end{aligned}
\tag{4.2.3}
$$

As function of v_z these are parabola with a maximum at $v_z = 0$ and height $z_{\max} = z(t_0) + v_z^2(t_0)/2g$, as shown in Figure 4.2(right). In this case the EOM involves the constant g such that only one of the initial conditions can be absorbed into dimensionless units. For dimensionless units based on the velocity scale $v_z(t_0)$ and the length scale $v_z^2(t_0)/2\,g$ we have

$$\frac{z}{v_z^2(t_0)/2\,g} = I - \left(\frac{v_z}{v_z(t_0)}\right)^2 \quad \text{with} \quad I = 1 + \frac{2\,g\,z(t_0)}{v_z^2(t_0)}$$

The trajectories in this dimensionless representation are shown in Figure 4.3. They all have the shape of a normal parabola, but the parabolas are shifted by the dimensionless constant I that is formed by the gravitational acceleration g, and the initial conditions $z(t_0)$ and $v_z(t_0)$.

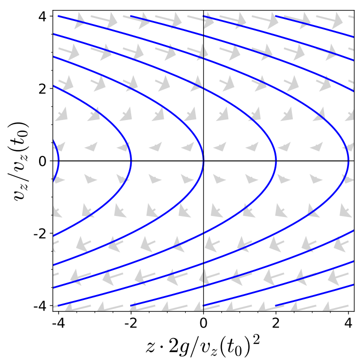


Figure 4.3: Dimensionless phase-space trajectories of a particle falling in the gravitational field without friction.

4.2.4 *Self Test*

Problem 4.1. **Estimating the depth of a pond**

You drop a stone into a pond and count n seconds till you hear it hit the water. How long a chord do you have to attach to your bucket to get up some water.

Problem 4.2. **Integrating the EOM for the flight of an Earth-bound rocket**

Integrate the EOM for rocket flight derived in Section 3.5,

$$
\begin{aligned}
\dot{V}_R(t) &= -g + \frac{a\,\rho\,v_f^2}{m + M_0 - a\,\rho\,v_f\,t} \\
\dot{z}(t) &= V_R(t)
\end{aligned}
$$

for a rocket that is launched with velocity v_0 at a height H_0, i. e. for the ICs

$$V_R(t_0) = v_0 \qquad \text{and} \qquad z(t_0) = H_0$$

How do the solutions Equations (3.5.1a) and (3.5.2a) change? Was there a way to anticipate the impact of the changing the initial height H_0?

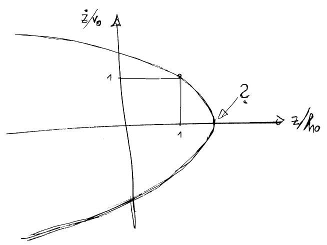


Figure 4.4: Sketch of the universal form of the free-flight trajectories in phase space, Equation (4.2.3).

Problem 4.3. **Alternative dimensionless units for trajectories with constant acceleration**

Discuss the shape of the trajectories that emerges when introducing dimensionless units based on the velocity scale $v_z(t_0)$ and the length scale $z(t_0)$ into Equation (4.2.3).

Hint: You will find parabolas as shown to in the margin. Discuss their shape and the position of their maximum.

4.3 *Separation of variables — Settling with Stokes drag*

The settling of a ball in a viscous medium can be described by the equations of motion

$$m\,\ddot{h}(t) = -m\,g - \mu\,\dot{h}(t)\,. \tag{4.3.1a}$$

Here $h(t)$ is the vertical position of the ball (height), g is the acceleration due to gravity, and the contribution $-\mu\,\dot{h}(t)$ describes Stokes friction, i. e. the viscous drag on the ball. It has the same form as the friction opposing the motion of the mine cart in Example 3.6.

The Stokes friction coefficient μ depends the viscosity of the fluid η and the geometry of the body. The viscosity $[\eta]$ of a fluid is measured in terms of Pa $=$ kg/m s. For air and water it takes values of about $\eta_{\text{air}} \simeq 2 \times 10^{-5}$ kg/m s, and $\eta_{\text{water}} \simeq 1 \times 10^{-3}$ kg/m s, respectively. The size of the ball will be given by its radius R. Hence, dimensional analysis implies that

$$\mu \propto R\,\eta$$

For a sphere of radius R the proportionality constant takes the value of 6π.

This problems involves the parameters g, μ and m that will absorbed into dimensionless units by introducing the dimensionless units for height $\hat{h} = h\,\mu^2/m^2\,g$, velocity $\hat{v} = \dot{h}\,\mu/m\,g$, and time $\tau = (t - t_0)\,\mu/m$. In these units the EOM takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\hat{h}(\tau) = -1 - \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{h}(\tau) \quad \Leftrightarrow \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{h} & = \hat{v} \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{v} & = -1 - \hat{v} \end{cases} \tag{4.3.1b}$$

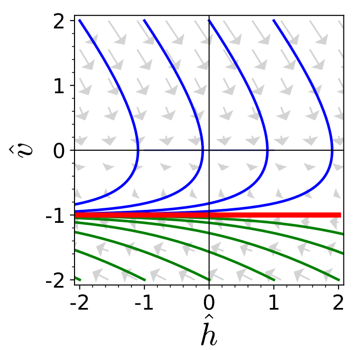


Figure 4.5: Dimensionless phase-space trajectories of a particle subjected to a constant acceleration g and Stokes drag.

The corresponding phase-space plot is shown in Figure 4.5. For positive (i. e. upwards) velocities the resulting direction field in phase space point to the lower right, and for $\hat{v}$ it points straight down. However, the arrows are steeper than for the case without friction, Figure 4.3. For $\hat{v} > 0$ the trajectories in the two cases look similar, but with friction they follow curves that are broader than the parabola for the frictionless fall. For downwards, the flows differ qualitatively: Trajectories started with zero velocity never cross the $\hat{v} = -1$ line, and trajectories that are started with a speed larger than 1 are no longer accelerated by gravity, but slowed down by friction until they also reach their terminal velocity -1 that is marked by a red line.

After having reached this qualitative insight into the dynamics, we will look now for the explicit solution of the EOM. Equation (4.3.1) can be integrated once, yielding an ODE for the settling velocity starting from a height $\hat{h}_0$ with velocity $\hat{v}_0$,

$$\dot{h}(\tau) = \hat{v}_0 - \tau - \big(\hat{h}(\tau) - \hat{h}_0\big)$$

This equation can not be solved by integration, employing Algorithm 4.1, because its right-hand side explicitly depends on the function $h(t)$ that must still be determined as solution of the ODE. It is a better strategy in this case to adopt another solution strategy.

4.3.1 *Solving ODEs by separation of variables*

In the case at hand the ODE (4.3.1a) can be interpreted as a first order ODE for the settling velocity $v = \dot{h}$ where $\dot{v}$ is provided as a function of only v. Such an ODE is best solved by separation of variables.

Algorithm 4.2: Separation of variables

A one-dimensional first-order ODE of the form

$$\dot{f}(t) = g(f(t))\, h(t)$$

can be solved by *separation of variables*. For the initial condition $f(t_0) = f_0$ one obtains then

$$\int_{t_0}^{t} \mathrm{d}t'\, h(t') = \int_{t_0}^{t} \mathrm{d}t'\, \frac{\dot{f}(t')}{g(f(t'))} = \int_{f_0}^{f(t)} \mathrm{d}f\, \frac{1}{g(f)}$$

which provides the solution in terms of two integrals.

Remark 4.4. Let us assume that we find the antiderivatives $H(t)$ with $\mathrm{d}H/\mathrm{d}t = h(t)$ as well as $A(f)$ with $\mathrm{d}A/\mathrm{d}f = 1/g(f)$ and inverse $I(f)$, i. e. $I(A(f)) = f$. Then separation of variables provides the explicit solution

$$\begin{aligned} H(t) - H(t_0) &= A(f(t)) - A(f(t_0)) \\ \Rightarrow \quad f(t) &= I\big(H(t) - H(t_0) + A(f(t_0))\big) \end{aligned}$$

We will see an example of this type when we resume the discussion of Stokes drag in Section 4.3.2.

Remark 4.5. Often the integrals can be performed but the inverse $I(f)$ can not be given in a closed form. If one can find the inverse of $H(t)$, i. e. a function $J(H)$ with $J(H(t)) = t$ then the solution can still be given in the (rather unusual) explicit form

$$t = J\big(A(f) - A(f(t_0)) + H(t_0)\big)$$

This is always possible for autonomous ODEs, i. e. in particular for Equation (4.3.1a) with $f(t) = \dot{h}(t)$.

Example 4.2: Separation of variable

The solution of the differential equation

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{\mathrm{e}^{y^2(x)}}{1-2\,y^2(x)} \qquad \text{with} \quad y(0)=0$$

obeys

$$x = \int_0^x \mathrm{d}x' = \int_0^{y(x)} \mathrm{d}y\,(1+2\,y^2)\,\mathrm{e}^{y^2} = y(x)\;\mathrm{e}^{y^2(x)}$$

One can not solve this equation to specify $y(x)$, but it is easy to plot $y\;\mathrm{e}^{y^2}$ and swap the axes (see Figure 4.6).

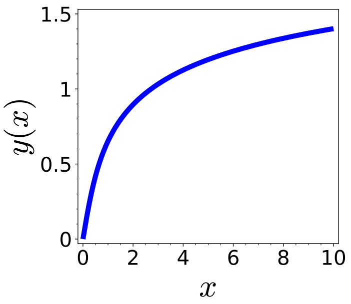


Figure 4.6: Solution of the ODE discussed in Example 4.2.

Remark 4.6. When neither of the inverse functions are known, then the solution can only be stated as an implicit equation

$$H(t) - A(f) = H(t_0) - A(f(t_0)) = \text{const}$$

Hence, the solutions amount to the contour lines of the function $G(t,f) = H(t) - A(f)$ that is plotted to this end as function of the two variables (t,f).[2]

[2] Plotting of contour lines is supported by all scientific plot programs. In Gnuplot it is facilitated via the "set implicit" option for a 2d plot command "plot", or by using "set contour" together with a 3d plot called by "splot". In Sage there are the commands 'plot_implicit()' and 'contour_plot()'. In Python with Matplotlib there is 'matplotlib.pyplot.contour()'.

Example 4.3: Separation of variable

The solution of the differential equation

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{y(x)\;(1+\cos x)}{1+y(x)} \qquad \text{with} \quad y(0)=1$$

obeys

$$\begin{aligned} x + \sin x - 1 &= \int_0^x \mathrm{d}x'\,(1+\cos x') \\ &= \int_1^{y(x)} \mathrm{d}y\,\left(\frac{1}{y}+1\right) = \ln y + y - 1 \end{aligned}$$

One can not solve this equation to specify $y(x)$ or $x(y)$. Hence, the solution is given as implicit equation

$$\Leftrightarrow \quad G(x,y) = y + \ln y - x - \sin x = 0$$

whose solution is plotted in Figure 4.7.

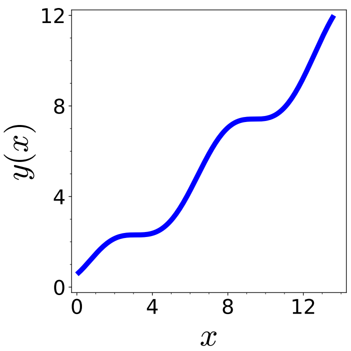


Figure 4.7: Solution of the ODE discussed in Example 4.3.

4.3.2 *Solving the EOM for settling with Stokes drag*

For Equation (4.3.1a) we will now derive the velocity $v(t) = \dot{h}(t)$ for an initial velocity v_0 by applying Algorithm 4.2. In order to simplify notations we perform the derivation in dimensionless units, Equation (4.3.1b), and introducing the physical variables in the end. Separation of variables provides that for a particle with initial velocity $\hat{v}_0$

$$\tau = \int_0^\tau \mathrm{d}\tau' = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w \, \frac{1}{1+w} = -\ln \frac{1+\hat{v}(\tau)}{1+\hat{v}_0}$$

$$\Leftrightarrow \quad \hat{v}(\tau) = -1 + \left(\hat{v}_0 + 1\right) \mathrm{e}^{-\tau} \tag{4.3.2}$$

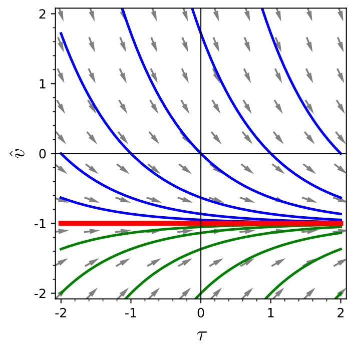


Figure 4.8: Sketch of $w(\tau)$ as obtained in Equation (4.3.2).

The solutions are shown in Figure 4.8. Stokes drag entails that for large times, $\tau \gg 1$, the ball is sinking with the constant Stokes velocity that takes the value -1 in our dimensionless units. Due to $-1 = \hat{v}_\infty = \mu v_\infty / m\, g$ this implies $v_\infty = -m\, g/\mu$ in terms of the physical units.

The position of the sphere can be obtained by integrating Equation (4.3.2) for $\hat{v}(\tau) = \mathrm{d}\hat{h}/\mathrm{d}\tau$ with initial condition $\hat{h}_0$,

$$\hat{h}(\tau) = \hat{h}_0 + \int_0^\tau \mathrm{d}\tau \, \frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = \hat{h}_0 + \int_0^\tau \mathrm{d}\tau \left(-1 + \left(\hat{v}_0 + 1\right) \mathrm{e}^{-\tau}\right)$$
$$= \hat{h}_0 - \tau + \left(\hat{v}_0 + 1\right) \left(1 - \mathrm{e}^{-\tau}\right) \tag{4.3.3a}$$

or in terms of physical units

$$h(t) = h_0 - v_\infty \,(t - t_0) + \frac{m}{\mu} \,(v_0 - v_\infty) \left[1 - \exp\left(-\frac{\mu}{m}(t - t_0)\right)\right] \tag{4.3.3b}$$

4.3.3 *Relation to free fall*

It is instructive to explore how the evolution with Stokes friction is related to the free flight $h_f(t) = h_0 + v_0(t - t_0) - g\,(t - t_0)^2$ obtained in Section 4.2. This can most effectively be done by Taylor expansion of Equation (4.3.3) for small τ, and subsequently expressing the result in physical units.

Definition 4.4: Taylor expansion

The *Taylor expansion* to order N provides an approximation of a function $f(x)$ at a position x_0. It is obtained by matching the first N derivatives of the function and of a polynomial of order N that represents the Taylor approximation (or *Taylor approximation*),

$$f(x) \simeq \sum_{n=0}^{N} \left. \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} \right|_{x=x_0} \frac{(x - x_0)^n}{n!}$$

Remark 4.7 (Leading-order Taylor expansion). The first-order, or leading-order Taylor expansion is a linear function $t(x) = t_0 + t_1 x$ with coefficients $t_0 = f(x_0)$ and $t_1 = f'(x_0)$. Hence, we have $t(x_0) = t_0 = f(x_0)$ and $t'(x) = t_1 = f'(x_0)$. This is a tangent to the function $f(x)$ that approximates f at x_0 by having the same functional value and slope. Examples for the sine function are shown in Figure 4.9.

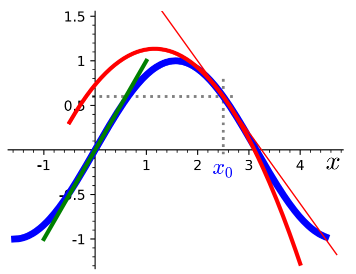


Figure 4.9: The leading order and second order Taylor approximations of the sine function at the origin (green) and at the position $x_0 = 2.5$.

Remark 4.8 (Second-order Taylor expansion). The second-order Taylor expansion is a quadratic function $t(x) = t_0 + t_1 x + t_2 x^2$ with coefficients $t_0 = f(x_0)$, $t_1 = f'(x_0)$, and $t_2 = f''(x_0)/2$. As for the first-order approximation, we have $t(x_0) = t_0 = f(x_0)$ and $t'(x) = t_1 = f'(x_0)$. Moreover, in this case we also have $t''(x) = 2\,t_2 = f'(x_0)$. Examples for the sine function are shown in Figure 4.9.

Example 4.4: Taylor approximations of the sine function

For $\sin x$ the even derivatives vanish at the origin, and the odd $2n-1$ derivative amounts to -1^n. Hence, the first few terms of the Taylor expansion at the origin are given by

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

At the origin the first and second order Taylor approximation agree, as shown by the green line in Figure 4.9.
For the sine-function the expansion at a position x_0 is given by

$$\sin x \simeq \sin(x_0)\left[1 - \frac{(x-x_0)^2}{2!} + \frac{(x-x_0)^4}{4!} - \frac{(x-x_0)^6}{6!} + \cdots\right]$$
$$+ \cos(x_0)\left[(x-x_0) - \frac{(x-x_0)^3}{3!} + \frac{(x-x_0)^5}{5!} - \cdots\right]$$

The red lines in Figure 4.9 show the first-order (thin red line) and the second order (thick red line) approximation for $x_0 = 2.5$. The second order approximation remains closer to the sine-function for a bit longer than the linear first-order approximation.

Example 4.5: Taylor approximations of the exponential function

The derivatives of the exponential function $f(x) = \exp(ax)$ amount to $f^{(n)}(x) = a^n f(x)$ such that its expansion at a position x_0 is given by

$$e^{a\,x} \simeq e^{a\,x_0}\left[1 + a\,(x-x_0) + \frac{a^2\,(x-x_0)^2}{2} + \cdots \right.$$
$$\left. \cdots + \frac{a^n\,(x-x_0)^n}{n!} + \cdots\right]$$

For $x_0 = 0$ this simplifies to

$$e^{a\,x} = \sum_{n=0}^{\infty} \frac{(a\,x)^n}{n!} \simeq 1 + a\,x + \frac{(a\,x)^2}{2} + \cdots$$

Based on the Taylor expansion of the exponential function $e^{-\tau} = \sum_{n=0}^{\infty}(-\tau)^n/n!$ we find for Equation (4.3.3)

$$\hat{h}(\tau) = \hat{h}_0 - \tau + (\hat{v}_0 + 1)\left(\tau - \frac{\tau^2}{2} + \frac{\tau^3}{6} - \cdots\right)$$
$$= \hat{h}_0 + \hat{v}_0\,\tau - (\hat{v}_0 + 1)\,\frac{\tau^2}{2}\left(1 - \frac{\tau}{3} + \cdots\right)$$

The solution with physical units is obtained by substituting $\hat{h} = \mu^2 h/m^2 g$, $\hat{h}_0 = \mu^2 h_0/m^2 g$, $\tau = \mu (t - t_0)/m$, and $\hat{v}_0 = \mu \dot{h}(t_0)/m g$. Hence,

$$h(t) = h_0 + v_0 (t - t_0) - \frac{\mu}{m} \left(v_0 + \frac{m g}{\mu} \right) \frac{(t - t_0)^2}{2} \left(1 - \frac{\mu (t - t_0)}{3 m} + \cdots \right)$$
$$= h_0 + v_0 (t - t_0) - \frac{g}{2} (t - t_0)^2 \left(1 - \frac{v_0}{v_\infty} \right) \left(1 - \frac{\mu (t - t_0)}{3 m} + \cdots \right)$$

This implies that Stokes friction provides a small corrections to the free flight if the initial velocity is small as compared to the asymptotic velocity of free flight, $|v_0| \ll v_\infty = m g/\mu$. Further, one must restrict the attention to times that are small as compared to the time scale m/μ where the asymptotic velocity is reached. Equation (4.3.2) implies that this amounts to situations where the velocity $|v(t)|$ is small as compared to the Stokes settling speed v_∞. This is discussed now for two concrete cases:

Example 4.6: Stokes friction for a steel ball

A steel ball with a diameter of 1 cm has a mass of about

$$m = \frac{4\pi}{3}\, 2 \times 10^3\,\text{kg/m}^3\, \frac{1 \times 10^{-6}\,\text{m}^3}{8} \simeq 1 \times 10^{-3}\,\text{kg}$$

In air it will reach a terminal velocity of about

$$v_{\text{air}} = \frac{m g}{\mu_{\text{air}}} = \frac{3 m g}{2 \eta_{\text{air}} R} = \frac{3 \times 1 \times 10^{-3}\,\text{kg}\, 10\,\text{m/s}^2}{2 \times 2 \times 10^{-5}\,\text{kg/m s}\; 1 \times 10^{-2}\,\text{m}}$$
$$\simeq 7.5 \times 10^4\,\text{m/s}$$

Saturation to this velocity occurs on time scales

$$t_{\text{air}} = \frac{m}{\mu_{\text{air}}} = \frac{m}{\eta_{\text{air}} R} = \frac{1 \times 10^{-3}\,\text{kg}}{2 \times 10^{-5}\,\text{kg/m s}\; 1 \times 10^{-2}\,\text{m}} = 5 \times 10^3\,\text{s}$$

and this time the bullet will have dropped by a distance $g\, t_c^2/2 = 2.5 \times 10^7$ m which is much more than the thickness of the atmosphere. We conclude that Stokes friction is not relevant for the motion of a bullet in air.
Even in water, where the viscosity is larger by a factor of 50, we will have

$$v_{\text{water}} = \frac{3 m g}{2 \eta_{\text{water}} R} = \frac{3 \times 1 \times 10^{-3}\,\text{kg}\, 10\,\text{m/s}^2}{2 \times 1 \times 10^{-3}\,\text{kg/m s}\; 1 \times 10^{-2}\,\text{m}}$$
$$\simeq 1.5 \times 10^3\,\text{m/s}$$

Saturation to this velocity occurs on time scales

$$t_{\text{water}} = \frac{m}{\eta_{\text{water}} R} = \frac{1 \times 10^{-3}\,\text{kg}}{1 \times 10^{-3}\,\text{kg/m s}\; 1 \times 10^{-2}\,\text{m}} = 100\,\text{s}$$

and this time the bullet will have dropped by a distance $g\, t_{\text{water}}^2/2 = 5 \times 10^4$ m which is deeper than the deepest point in our Oceans.[3]

[3] In Section 4.4 we will see what goes wrong here.

Example 4.7: Stokes friction for sperms

Sperms are cells equipped with cilia that allow them to swim towards the egg for fertilization. They have a characteristic size L of a few micrometers and they swim in an environment that is approximated here as water. Their mass is of the order of $m_{\text{sperms}} = \rho_{\text{water}} L^3$. In this case their asymptotic speed is reached at a time scale

$$t_{\text{spermium}} = \frac{m_{\text{spermium}}}{\mu_{\text{spermium}}} = \frac{\rho_{\text{water}} L^2}{\eta_{\text{water}}}$$
$$= \frac{1 \times 10^3\,\text{kg/m}^3\; 1 \times 10^{-12}\,\text{m}^2}{1 \times 10^{-3}\,\text{kg/m s}} = 1 \times 10^{-6}\,\text{s}$$

Stokes friction plays a major role for their swimming. See Purcell (1977) for more details.

4.3.4 *Self Test*

Problem 4.4. **Solving ODEs by separation of variables**

Determine the solutions of the following ODEs

a) $\dfrac{dy}{dx} = \dfrac{\cos^2 y}{\sin^2 x}$ such that $y(\pi/4) = 0$

b) $\dfrac{dy}{dx} = \dfrac{3\,x^2\,y}{2y^2 + 1}$ such that $y(0) = 1$

c) $\dfrac{dy}{dx} = -\dfrac{1 + y^3}{x\,y^2\,(1 + x^2)}$ such that $y(1) = 2$

Problem 4.5. **Taylor approximations of the Cosine function**

Find the Taylor approximation for the cosine function

a) analogously to the discussion in Example 4.4, and

b) based on Euler's equation $e^{ix} = \cos x + i \sin x$.
Hint: Insert $a = i$ into the expansion provided in Example 4.5, and collect real and imaginary parts.

Problem 4.6. **Taylor series**

Find the Taylor approximation for the sine function close to

a) $x = \pi/4$ b) $x = \pi/2$ c) $x = 3\,\pi/2$

Hint: Make use of the result of Problem 4.5 and the symmetries of trigonometric functions.

Problem 4.7. **Stopping distance of a yacht**

A yacht of mass 750 kg is sailing on the sailing into the harbor with a speed of 6 m/s. At this moment it is experiencing a friction force of 900 N. At time $t = 0$ the skipper switches off the motor such that only the friction is acting on the boat. Let the water resistance be proportional to the speed.

a) How long will it take till the yacht has come to rest?

b) How long will if take till the speed has been reduced to 1.5 m/s and which distance has the yacht traversed till that time?

Problem 4.8. **Free fall with viscous friction**

In Equation (4.3.3) we derived the time evolution of the height $h(t)$ of a ball that is falling a gravitational field and subjected to Stokes drag.

a) Make a plot of $\hat{h}(\tau)$ as function of τ, where you compare the evolution of trajectories that start with different initial velocity $\hat{v}_0$ from the same height $\hat{h}_0 = 0$.

b) Make a plot of $h(t)$ as function of t, where you compare the evolution of trajectories that start with the same initial velocity v_0 from the same height h_0, but are subjected to a different drag μ (for instance because they have different radius).

4.4 *Worked example: Free flight with turbulent friction*

In Example 4.6 we reached the puzzling conclusion that — for all physically relevant parameters — Stokes friction plays no role for the motion of a steel ball in air and water. On the other hand, we know from experience that friction arises to the very least for large velocities, like for gun shots. This apparent contradiction is resolved by observing that the drag is not due to Stokes drag. Rather for most settings in our daily live friction arises because the motion of the fluid around the considered object goes turbulent, as anticipated in Problem 3.3. A ball of mass m, radius R, and mass density $\rho_{\text{ball}} = 3m/4\pi R^3$ that is moving with speed v through a fluid of mass density ρ_{fluid} will experience a turbulent drag force of modulus

$$F_D = m \, \frac{\rho_{\text{fluid}} \, C_D}{8 \, \rho_{\text{ball}} \, R} \, v^2 = m \, \kappa \, v^2 \tag{4.4.1}$$

Here, C_D is a dimensionless number that typically takes values between 0.5 and 1. A very beautiful description of the physics of this equation has been provided in an instruction video of the NASA (click here to check it out).

To address motion affected by turbulent drag we measure time in units of $(\kappa g)^{-1/2}$ and velocity in units of $(g/\kappa)^{1/2}$. The dimen-

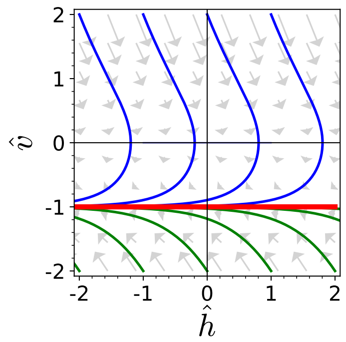


Figure 4.10: Dimensionless phase-space trajectories of a particle subjected to a constant acceleration g and turbulent drag.

sionless velocity $\hat{v}(\tau)$ will then obey the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{h}(\tau) = \hat{v} \tag{4.4.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,\hat{v}(\tau) = -1 - \hat{v}^2(\tau)\,\mathrm{sign}\big(\hat{v}(\tau)\big) \tag{4.4.2b}$$

The requirement that the friction force always acts in the direction opposing the motion has been incorporated here by the sign function

$$\mathrm{sign}(x) = \begin{cases} -1 & \text{for } \; x < 0 \\ \;\;0 & \text{for } \; x = 0 \\ \;\;1 & \text{for } \; x > 0 \end{cases}$$

The phase-space flow for this EOM is shown in Figure 4.10. It looks similar to the one for Stokes drag, with the important difference that the change of velocity grows much faster for large $|\hat{v}|$. For $\hat{v} > 0$ this gives even rise to an inflection point where the curvature of the trajectories indicated by blue lines crosses from convex to concave.

The equation for the velocity can again be solved by separation of variables,

$$\tau = \int_0^\tau \mathrm{d}\tau' = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w\, \frac{1}{1 + w^2\,\mathrm{sign}(w)}$$

In order to deal with the sign-function, we deal with the integral separately for four types of initial conditions in either of the intervals $\{(-\infty, -1), [-1, -1], (-1, 0], [0, \infty)\}$.

First we consider the initial condition where $\hat{v}_0 = -1$. In that case $\frac{\mathrm{d}}{\mathrm{d}\tau} w(\tau) = 0$ such that

$$\hat{v}(\tau) = -1 \qquad \text{for} \quad \hat{v}_0 = -1 \tag{4.4.3a}$$

Next we consider initial conditions where $\hat{v}_0 < 0$, but $\hat{v}_0 \neq 0$. In this case

$$\tau = \int_{\hat{v}_0}^{v(\tau)} \frac{\mathrm{d}w}{1 - w^2} = \frac{1}{2}\,\ln\left(\frac{1 - \hat{v}(\tau)}{1 + \hat{v}(\tau)} \cdot \frac{1 + \hat{v}_0}{1 - \hat{v}_0}\right)$$

$$\Leftrightarrow \quad \hat{v}(\tau) = \begin{cases} -\tanh\big(\tau - \mathrm{atanh}\,\hat{v}_0\big) & \text{for} \quad -1 < \hat{v}_0 \le 0 \\ -\mathrm{cotanh}\big(\tau - \mathrm{acotanh}\,\hat{v}_0\big) & \text{for} \quad -1 > \hat{v}_0 \end{cases} \tag{4.4.3b}$$

Finally, we consider the case $\hat{v}_0 > 0$. We expect in that case that the particle moves up, $\hat{v}(\tau) > 0$, till some time τ_c, and then it start falling due to the action of gravity. However, in that case its velocity heads down with $-1 < \hat{v}_0 \le 0$ such that it must follow the solution $\hat{v}(\tau) = -\tanh(\tau - \tau_c)$ obtained in Equation (4.4.3b). For $\tau < \tau_c$ we find

$$\tau = -\int_{\hat{v}_0}^{\hat{v}(\tau)} \mathrm{d}w\, \frac{1}{1 + w^2} = -\arctan(\hat{v}(\tau)) + \arctan(\hat{v}_0)$$

$$\Leftrightarrow \quad \hat{v}(\tau) = -\tan\big(\tau - \arctan(\hat{v}_0)\big)$$

such that

$$\hat{v}(\tau) = \begin{cases} -\tan(\tau - \tau_c) & \text{for} \quad \hat{v}_0 > 0 \wedge \tau < \tau_c = \arctan(\hat{v}_0) \\ -\tanh(\tau - \tau_c) & \text{for} \quad \hat{v}_0 > 0 \wedge \tau \geq \tau_c = \arctan(\hat{v}_0) \end{cases} \tag{4.4.3c}$$

A solution Equation (4.4.3) that passes through the origin and another one through $(0, -2)$ are shown in Figure 4.11.

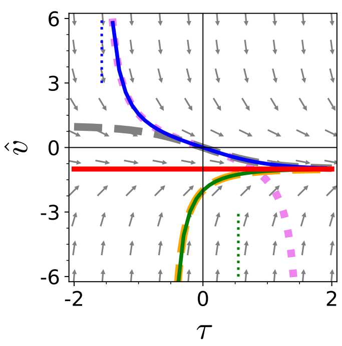


Figure 4.11: Solutions Equation (4.4.3) of Equation (4.4.2a).

4.4.1 *Range of applicability*

Turbulent friction applies whenever

$$\mu\,|v| \lesssim m\,\kappa\,v^2 \quad \Leftrightarrow \quad |v| \gtrsim v_c = \frac{\mu}{m\kappa} \simeq \frac{\eta_{\text{fluid}}}{\rho_{\text{fluid}}\,R}$$

For the 1 cm steel ball considered in Example 4.6 the cross-over velocity v_c yields

$$v_c = \begin{cases} \dfrac{2\times 10^{-5}\,\text{kg/m s}}{1\,\text{kg/m}^3 \times 1\times 10^{-2}\,\text{m}} = 2\,\text{mm/s} & \text{for air} \\ \dfrac{1\times 10^{-3}\,\text{kg/m s}}{1\times 10^{3}\,\text{kg/m}^3 \times 1\times 10^{-2}\,\text{m}} = 0.1\,\text{mm/s} & \text{for water} \end{cases}$$

Moreover, the characteristic time for turbulent drag is

$$t_c = (\kappa g)^{-1/2} = \sqrt{\frac{\rho_{\text{ball}}\,R}{\rho_{\text{fluid}}\,g}}$$
$$= \begin{cases} \sqrt{\dfrac{2\times 10^{3}\,\text{kg/m}^3 \times 1\times 10^{-2}\,\text{m}}{1\,\text{kg/m}^3 \times 10\,\text{m/s}^2}} \simeq 1.4\,\text{s} & \text{for air} \\ \sqrt{\dfrac{2\times 10^{3}\,\text{kg/m}^3 \times 1\times 10^{-2}\,\text{m}}{1\times 10^{3}\,\text{kg/m}^3 \times 10\,\text{m/s}^2}} \simeq 0.04\,\text{s} & \text{for water} \end{cases}$$

As a consequence, one may safely assume that Stokes friction is always negligible for the steel ball. Either friction may be neglected or turbulent friction must be considered.

4.4.2 *Self Test*

Problem 4.9. **Turbulent friction**

Assume that the Earth atmosphere gives rise to the same turbulent drag, irrespective of height.

a) What is the maximum time after which a steel ball that is shot up with vertical velocity v_0 will hit the ground?

b) Does it make a noticeable difference when you require that v_0 must not surpass the speed of light $c = 3 \times 10^8\,\text{m/s}$?

Problem 4.10. **Free fall with turbulent friction**

In Equation (4.4.3) we derived the velocity of a ball that is accelerated by gravity and slowed down by turbulent drag.

a) How will the height $\hat{h}$ of the trajectories evolve for large times τ?

b) Determine the full time dependence of the dimensionless height $\hat{h}$ by solving the ODE $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{h} = \hat{v}$.
Hint: Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \ln \cos \tau = - \tan \tau \,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau} \ln \cosh \tau = \tanh \tau \,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau} \ln \sinh \tau = \text{cotanh}\tau$$

c) Make a plot of $\hat{h}(\tau)$ as function of τ, where you compare the evolution of trajectories that start with different initial velocity $\hat{v}_0$ from the same height $\hat{h}_0 = 0$.

d) Insert the definitions of the dimensionless units in order to find the solutions for the velocity $v(t)$ and height $h(t)$ in physical units.

Problem 4.11. **Stopping length of a yacht**

For moderate speeds[4] a yacht experiences a turbulent drag force of the form given in Equation (4.4.1). In the following we assume that $\kappa^{-1} \simeq 10\,\mathrm{m}$.

[4] Recall Problem 1.7 for a discussion of large speeds.

a) The sails of the yacht are shortened at a speed of $v_0 = 10\,\mathrm{m\,s^{-1}}$ and some position x_0. Subsequently, it is running straight ahead. Determine the position $x(t)$ of the yacht.

b) You will find that the yacht never comes to rest. Is that in line with your physical intuition? What might be the origin of this finding?

c) What would happen when the yacht is rather subjected to Stokes drag, Equation (4.3.1a), with a friction coefficient of the order of $\mu \simeq R\,\eta = 3\,\mathrm{m}\,1 \times 10^{-3}\,\mathrm{kg/m\,s} = 3 \times 10^{-3}\,\mathrm{kg/s}$.

d) How does the result of c) refer to that of a). Does the comparison help to solve the issue raised in b)?

4.5 *Linear ODEs — Particle suspended from a spring*

There are two forces acting on a particle is suspended from a spring: the gravitational force $-m\,g$ and the spring force $-k\,z(t)$ where $z(t)$ measures the displacement of the spring from its rest position. Hence, the EOM of the particle takes the form

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) \tag{4.5.1}$$

This equation can neither be integrated directly, because its right hand side depends on $z(t)$, nor can it be solved by separation of variables, because its right hand side depends on $z(t)$ rather than only on $\dot{z}(t)$. It falls into the very important class of *linear ODEs*.

Definition 4.5: Linear ODEs

An ODE is called a *linear ODE* when $z(t)$ and its derivatives only appear as linear terms in the ODE. Hence, an N^{th} order linear ODEs for $z(t)$ takes the general form

$$I(t) = z^{(N)}(t) + c_{N-1}(t)\, z^{(N-1)}(t) + \cdots + c_0(t)\, z(t)$$

The functions $I(t)$, $c_\nu(t)$, $\nu = 0 \cdots N-1$, are called the coefficients of the linear ODE. When they do not depend on time we speak of a linear ODE with *constant coefficients*. In particular, $I(t)$ is called *inhomogeneity*; when it vanishes the ODE is called *homogeneous*.

Example 4.8: Particle suspended from a spring

Equation (4.5.1) is an inhomogeneous second-order linear ODE with the constant coefficients $f_0 = k$, $f_1 = 0$, and inhomogeneity $I = m\, g$.

Remark 4.9. An N^{th}-order linear ODE where the coefficient in front of the N^{th} derivative takes the value $c_N \neq 1$ can be stated in the form given in Definition 4.5 by division with c_N.

Example 4.9: Damped harmonic oscillator

The harmonic oscillator with damping γ and spring constant k

$$m\, \ddot{x}(t) = -m\, \gamma\, \dot{x}(t) - k\, x(t)$$

is described by a homogeneous second order, linear ODE with the constant coefficients $k_1 = \gamma$ and $k_0 = k/m$.

4.5.1 *Solving linear ODEs with constant coefficients*

Linear ODEs with constant coefficients are solved as follows

Algorithm 4.3: Linear ODEs with constant coefficients

An N^{th}-order linear ODE with constant coefficients,

$$I = \sum_{\nu=0}^{N} c_\nu\, f^{(\nu)}(t)$$

can be recast into a homogeneous ODE by considering $h(t) = f(t) - I/c_0$, which is a solution of the corresponding homogeneous, linear ODE

$$0 = \sum_{\nu=0}^{N} c_\nu\, h^{(\nu)}(t)$$

Its solutions can be written as

$$h(t) = \sum_{k=1}^{N} A_k \, \mathrm{e}^{\lambda_k \, t}$$

where the numbers λ_k, $k = 1 \cdots N$ are the N distinct roots of the characteristic polynomial

$$0 = \sum_{\nu=0}^{N} c_\nu \, \lambda^\nu$$

and the amplitudes A_k, $k = 1 \dots N$ must be chosen such that $f(t) = I + c_0 \, h(t)$ obeys the initial conditions

$$\begin{aligned} f(t_0) &= \frac{I}{c_0} + \sum_{k=1}^{N} A_k \, \mathrm{e}^{\lambda_k \, t_0} \\ f^{(1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k \, \mathrm{e}^{\lambda_k \, t_0} \\ \vdots &= \vdots \\ f^{(N-1)}(t_0) &= \sum_{k=1}^{N} A_k \, \lambda_k^{N-1} \, \mathrm{e}^{\lambda_k \, t_0} \end{aligned}$$

The idea underlying this algorithm is founded on three insights:

- the solutions of a homogeneous linear ODE form a vector space,
- $\exp(\lambda t)$ is a solution of the ODE iff it is a root of the characteristic polynomial, and
- the functions $\{\exp(\lambda_i t), i = 1, \dots, N\}$ form a basis of the vector space.

The proof will be provided in Problem 4.15.

Remark 4.10. When the polynomial only has $M < N$ distinct roots the set of functions $\{\exp(\lambda_i \, t), i = 1, \dots, M\}$ is missing $N - M$ elements to form a basis for the space of solutions. The set is augmented then by functions of the form $t \, \exp(\kappa \, t)$ for double roots, $t^2 \, \exp(\kappa \, t)$ for triple roots, etc. In this course we only deal with second order ODEs, where at most double roots arise. The solution strategy for that case will be discussed in Section 4.5.3. 🕮

4.5.2 *Solving the ODE for the mass suspended from a spring*

For Equation (4.5.1) this implies that $h(t) = z(t) + mg/k$ with

$$0 = \ddot{h}(t) + \frac{k}{m} \, h(t) \tag{4.5.2}$$

such that we obtain

$$\lambda_\pm = \pm \sqrt{\frac{k}{m}} = \pm \omega \qquad \text{as solution of} \quad 0 = \lambda^2 + \frac{k}{m}$$

Consequently, the motion of the spring is described by

$$z(t) = -\frac{mg}{k} + A_+ \, e^{\omega\,(t-t_0)} + A_- \, e^{-\omega\,(t-t_0)}$$

This is a real-valued function if and only if A_+ and A_- are canonically conjugated complex numbers, such that we can write $A_\pm = A \, \exp(\pm i\,\varphi)/2$ with $A \in \mathbb{R}$. As a consequence of $\cos x = (e^{ix} + e^{-ix})/2$ we then obtain

$$z(t) = -\frac{mg}{k} + A \; \cos\big(\varphi + \omega\,(t - t_0)\big) \tag{4.5.3a}$$

where A and φ must be fixed based on the initial conditions

$$\begin{aligned} z(t_0) &= -\frac{mg}{k} + A \; \cos(\varphi) \\ \dot{z}(t_0) &= \qquad -\omega\, A \; \sin(\varphi) \end{aligned}$$

or

$$A^2 = \left(z(t_0) + \frac{mg}{k}\right)^2 + \frac{\dot{z}^2(t_0)}{\omega^2} \qquad \text{and} \qquad \varphi = \arcsin\left(\frac{\dot{z}(t_0)}{\omega\, A}\right) \tag{4.5.3b}$$

4.5.3 *Solution for the damped harmonic oscillator*

The damped harmonic oscillator is described by the linear EOM

$$0 = \ddot{x}(t) + \gamma\,\dot{x}(t) + \frac{k}{m}\,x(t) \qquad \text{with} \quad \gamma,\, k,\, m \in \mathbb{R}_+ \,. \tag{4.5.4}$$

It characteristic polynomial

$$0 = \lambda^2 + \gamma\,\lambda + \frac{k}{m}$$

has the solutions

$$\lambda_\pm = -\frac{1}{2}\,\left(\gamma \pm \sqrt{\gamma^2 - 4\,k/m}\right)$$

Here λ_+ and λ_- can either be both real, a pair of complex conjugated numbers, or we have to deal with the case $\gamma^2 = 4\,k/m$ where there only is a single root. We treat the cases one after the other.

1. Two real roots

In this case $\gamma^2 < 4\,k/m$ such that $\lambda_\pm \in \mathbb{R}_-$. The motion of the oscillator is described by

$$x(t) = A_+ \, e^{\lambda_+\,(t-t_0)} + A_- \, e^{\lambda_-\,(t-t_0)}$$

which is a real-valued function for amplitudes $A_\pm \in \mathbb{R}$. The solution for the initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$ is then found by solving the equations

$$\left.\begin{aligned} x_0 &= A_+ + A_- \\ v_0 &= A_+\,\lambda_+ + A_-\,\lambda_- \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{cases} A_+ = m\,(x_0\,\lambda_- - v_0)/\sqrt{\gamma^2 - 4\,k/m} \\ A_- = -m\,(x_0\,\lambda_+ - v_0)/\sqrt{\gamma^2 - 4\,k/m} \end{cases}$$

Problem 4.13 instructs the reader to plot these solutions for different combinations of A_+ and A_-.

2. Two complex roots

This discussion is analogous to the one provided in Section 4.5.2. One obtains

$$x(t) = A\, e^{-\gamma\,(t-t_0)/2}\, \cos\big(\varphi + \omega_\gamma\,(t-t_0)\big) \tag{4.5.5}$$

where A and φ must be fixed based on the initial conditions

$$x(t_0) = A\, \cos(\varphi)$$
$$\dot{x}(t_0) = -\omega_\gamma\, A\, \sin(\varphi)$$

or $\quad A^2 = z^2(t_0) + \dfrac{\dot{z}^2(t_0)}{\omega_\gamma^2} \qquad$ and $\qquad \varphi = \arcsin\left(\dfrac{\dot{z}(t_0)}{\omega_\gamma\, A}\right)$

In Problem 4.14 the reader is advised to fill in the details of this derivation.

3. A single double root

For $\gamma^2 = 4\,k/m$ the characteristic polynomial has a single root $\lambda = -\gamma/2$ such that we only find a single solution $\exp(\lambda t)$ of the ODE. The ODE is solved then as follows:

Algorithm 4.4: Linear 2nd order ODEs: the degenerate case

A 2^{nd}-order linear homogeneous ODE whose characteristic polynomial has a double root at $\lambda = c$ takes the form

$$0 = \ddot{h}(t) - 2\,c\,\dot{h}(t) + c^2\,h(t) \quad \text{with} \quad c \in \mathbb{C}$$

This ODE has two independent solutions $\exp(\lambda\, t)$ and $t\, \exp(\lambda\, t)$ such that its general solutions can be written as

$$h(t) = (A + B\,(t-t_0))\, e^{c\,(t-t_0)}$$

Here the amplitudes A and B must be chosen such that the solution obeys the initial condition

$$\left.\begin{aligned} h(t_0) &= h_0 = A \\ \dot{h}(t_0) &= v_0 = c\,A + B \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} A &= h_0 \\ B &= v_0 - c\,h_0 \end{aligned}\right.$$

Remark 4.11. The function $t\,\exp(ct)$ is a solution of the ODE iff the characteristic polynomial has a double root:

Proof.

$$\begin{aligned} 0 &= \frac{d^2}{dt^2}\big(t\,e^{c\,t}\big) + a\,\frac{d}{dt}\big(t\,e^{c\,t}\big) + b\,\big(t\,e^{c\,t}\big) \\ &= e^{c\,t}\,\Big[(2\,c+a) + \big(c^2 + a\,c + b\big)\,t\Big] \\ \Rightarrow \quad & (2\,c+a) = 0 \quad \wedge \quad \big(c^2 + a\,c + b\big) = 0 \end{aligned}$$

The first equations holds iff $a = -2\,c$ and the second condition implies then that $b = c^2$. □

For the damped harmonic oscillator we have $c = -\gamma/2$ such that

$$x(t) = \left[x_0 + \left(v_0 + \frac{\gamma\, x_0}{2}\right)\,(t - t_0)\right]\; e^{-\gamma\, t/2}$$

is the solution with $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$.

4.5.4 *Self Test*

Problem 4.12. **Alternative solution for the mass suspended from a spring**

In Equation (4.5.3) we provided the solution of the EOM (4.5.2) of a mass suspended from a spring. Occasionally one also finds the solution given in the form

$$z(t) = -\frac{mg}{k} + A_1\;\cos\bigl(\omega\,(t - t_0)\bigr) + A_2\;\cos\bigl(\omega\,(t - t_0)\bigr)$$

What is the relation between these two solutions? How does one find one from the other?

Hint: Start from Equation (4.5.3) and use trigonometric relations.

Problem 4.13. **Overdamped solutions of the damped harmonic oscillator: time dependence and in phase-space portrait**

In this exercise we discuss the form of the overdamped solutions of the damped harmonic oscillator.

a) Consider first ICs where A_+ and A_- are positive. Verify that there is a time t_c where the two contributions to $x(t)$ are equal. Plot $x(t)\;\exp(-\lambda_+)\,(t_c - t_0)/A_+$ as function of $t - t_0 - t_c$, choosing a log-scale for the ordinate axis. You should observe linear behavior for large negative and positive values on the mantissa $t - t_0 - t_c$. What are the slopes of these lines? What is the value where the function intersects with $t - t_0 - t_c = 0$?

b) Consider first ICs where $A_+ > 0 > A_-$ and plot $x(t)$ as function of $t - t_0$. Add the functions that describe the asymptotics of $x(t)$ for very small and for large times. You will find that this function has a root and a maximum. Find the time where this happens, and the function value at the maximum.

c) Sketch the motion in phase space! Make use to this end of the special points that you evaluated in b).

Problem 4.14. **Damped oscillations of the damped harmonic oscillator: derivation and phase-space portrait**

For $\gamma^2 > 4\,k/m$ the damped harmonic oscillator shows damped oscillations as given in Equation (4.5.5).

a) Why should the amplitudes $A_\pm$ of the two solutions be complex conjugate?

b) Choose the ansatz $A_\pm = A\ \exp(\pm i\varphi)/2$ and derive the result provided in Equation (4.5.5).

c) Note that there is no explicit γ dependence of IC.
Why does it drop out?

d) How does this motion look like in phase space?

Problem 4.15. **The solutions of a homogeneous linear ODE form a vector space**

The set of solutions S_N of a homogeneous N^{th}-order homogeneous linear ODE,

$$0 = \sum_{\nu=0}^{N} c_\nu(t)\, f^{(\nu)}(t)\,, \tag{4.5.6}$$

forms a vector space (cf. the Definition 2.9). Proof to this this end that

a) $(\mathsf{S}, +)$ is a commutative group. The non-trivial statement that must be checked to this end is that

$$\forall s_1(t), s_2(t) \in \mathsf{S} : \quad s_1(t) + s_2(t) \in \mathsf{S}$$

b) Verify that

$$\forall \alpha \in \mathbb{C}\,,\ \ s(t) \in \mathsf{S} : \quad \alpha\, s(t) \in \mathsf{S}$$

and show that the other properties of a vector space follow trivially from the properties of real functions.

c) Show that the vector space S_N has dimension N.

d) Show that the functions $\exp(\lambda t)$ are a solution of Equation (4.5.6) iff λ is a root of the characteristic polynomial.

e) In Algorithm 4.3 we wrote the solutions as $h(t) = \sum_k A_k\ \exp(\lambda_k\, t)$.
Show that this can be interpreted as a representation of the vector $h(t)$ as a linear combination with coordinates A_k with respect to a basis $\{\exp(\lambda_k\, t), k = 1, \ldots, N\}$.
Why is it important to this end that the characteristic polynomial has distinct roots?

f) What about inhomogeneous, linear ODEs? Do their solutions form a vector space, too? If yes: proof it! If no: provide counterexamples for all properties that are violated.

add: variation of constants: fish pond, bells

4.6 *Employing constants of motion — the center of mass (CM) inertial frame*

One of the most important objectives of physics is the description of the motion of interacting particles. As a first step in this direction we discuss how to employ constants of motion to determine the motion of two point particles that interact with a conservative force depending only on the scalar distance between the particles, the interaction most commonly encountered in physical systems. The impact of spatial extension will be the topic of Chapter 5.

Definition 4.6: Point Particles

A *point particle* is an idealization of a physical object where its mass is considered to be concentrated in a single point in space $\boldsymbol{x}$. Point particles can not collide. However, their motion can be subjected to forces that depend on their position $\boldsymbol{x}$.

Example 4.10: Kepler Problem

The Kepler problem addresses the motion of a planet of mass m that orbits around a sun of mass M. The sun and the planet are so far apart that it is justified to consider their masses as concentrated in the positions $\boldsymbol{q}_P$ and $\boldsymbol{q}_S$, and to approximate their interaction as arising from the potential

$$\Phi(R) = -\frac{mMG}{R}$$

where $G = 6.672\,59 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$ is the constant of gravitation and $R = |\boldsymbol{q}_P - \boldsymbol{q}_S|$ is the distance between planet and sun. Planet and sun are considered as point particles.

Remark 4.12. The approximation of point particles has been introduced by Newton upon providing the first mathematical model for the Kepler problem. Subsequently, it has extremely successfully been applied in celestial mechanics. *Celestial Mechanics* addresses the problem of discussing the motion of all planets and their moons based on pair interactions deriving from the potential provided in Example 4.10. How to the tiny interactions between the planets impact their motion over long times? Is our solar system stable, or will–at some time in the far future–some planet or moon borrow energy from the other bodies and escape into outer space? 🕮

Remark 4.13. A straightforward application of the Kepler problem is the discussion of the motion of the Moon around Earth where the predictions have been tested extremely accurately based on satellite data and the return time of light signals send to Moon and reflected by mirrors on its surface that have been left there by space missions. The measurements clearly reveal the limitations of the model: Most noticeably, the Moon gives rise to tidal forces on Earth that induce a tiny amount of dissipation. Even in celestial mechanics there are small dissipative corrections to conservative interaction.

add references and exercises: Earth/Moon dissipation

In Section 3.4 we learned that conservation laws impose constraints on the motion of bodies that can be used to simplify the description of their motion. We consider the motion of N particles of masses m_i, $i = 1, \dots, N$ at the positions $\boldsymbol{q}_i$, $i = 1, \dots, N$ that are subjected to forces $\boldsymbol{F}_{ij}$ acting between every pair (i, j) of particles. There is not self-interaction $\boldsymbol{F}_{ii} = \boldsymbol{0}$, and the forces obey Newtons 3rd law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moreover, they are conservative, and depend only on the distance of the particles, $\boldsymbol{F}_{ij} = \nabla\Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|)$. Here the indices ij indicate that the force may depend on additional scalar parameters such as the mass or charge of the particles.

4.6.1 Center of mass motion and relative motion

We first determine the evolution of the position of the center of mass $\boldsymbol{Q}$ of the system

$$\boldsymbol{Q} = \frac{1}{M} \sum_i m_i \, \boldsymbol{q}_i \quad \text{with total mass} \quad M = \sum_i m_i \tag{4.6.1}$$

Its evolution is not subjected to external forces

$$\ddot{\boldsymbol{Q}} = \frac{1}{M} \sum_i m_i \, \ddot{\boldsymbol{q}}_i = \frac{1}{M} \sum_i \sum_j \boldsymbol{F}_{ij} = \boldsymbol{0} \tag{4.6.2}$$

due to Newtons 3rd law.

Hence, we find for an initial position $\boldsymbol{Q}_0$ and initial velocity $\boldsymbol{V}_0$ at an initial time t_0 that

$$\boldsymbol{Q}(t) = \boldsymbol{Q}_0 + \boldsymbol{V}_0 \, (t - t_0) \tag{4.6.3}$$

Now we introduce the coordinates relative to the center of mass $\boldsymbol{r}_i = \boldsymbol{q}_i - \boldsymbol{Q}$ and we observe that

$$\begin{aligned} m_i \, \ddot{\boldsymbol{r}}_i &= m_i \, \ddot{\boldsymbol{q}}_i - m_i \, \ddot{\boldsymbol{Q}} = m_i \, \ddot{\boldsymbol{q}}_i \\ &= \sum_j \boldsymbol{F}_{ij} = -\nabla_{\boldsymbol{q}_i} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) = -\frac{\boldsymbol{q}_i - \boldsymbol{q}_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|} \, \Phi'_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|) \\ &= -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \, \Phi'_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) \end{aligned} \tag{4.6.4}$$

where $\Phi'_{ij}(x)$ denotes the derivative of $\Phi_{ij}(x)$ with respect to its scalar argument x.

Hence, the EOMs for $\boldsymbol{Q}$ and for the positions $\boldsymbol{r}_i$ relative to the center of mass can be solved separately of each other, and the EOM for the CM has a trivial solution, Equation (4.6.3). We may therefore always address the motion of the particles in a setting where their center of mass is fixed at the origin of the coordinate system.

4.6.2 Angular momentum in celestial mechanics

The total angular momentum is conserved for systems where all forces are due to pairwise interactions between particles pairs of particles ij that obey Newtons 3rd law $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$ with forces acting along the line connecting particle i and j, i.e. in particular for the forces of the form given in Equation (4.6.4). After all,

$$\begin{aligned}
\boldsymbol{L} &= \sum_i \boldsymbol{q}_i \times m_i\, \dot{\boldsymbol{q}}_i = \sum_i (\boldsymbol{Q} + \boldsymbol{r}_i) \times m_i\, (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{r}}_i) \\
&= \sum_i (m_i\, \boldsymbol{Q} \times \boldsymbol{Q} + \boldsymbol{Q} \times m_i\, \dot{\boldsymbol{r}}_i + m_i\, \boldsymbol{r}_i \times \dot{\boldsymbol{Q}} + \boldsymbol{r}_i \times m_i\, \dot{\boldsymbol{r}}_i) \\
&= M\boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times m_i\, \dot{\boldsymbol{r}}_i
\end{aligned}$$

where the terms that contain only a single factor of $\boldsymbol{r}_i$ or $\dot{\boldsymbol{r}}_i$ vanish because $\sum_i m_i\, \boldsymbol{r}_i = \sum_i m_i\, (\boldsymbol{q}_i - \boldsymbol{Q}) = \boldsymbol{Q} - \boldsymbol{Q} = \boldsymbol{0}$. Now we have

$$M\, \ddot{\boldsymbol{Q}} = \sum_i m_i\, \ddot{\boldsymbol{q}}_i = \sum_i \sum_j \boldsymbol{F}_{ij} = \sum_{i<j} \boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = \boldsymbol{0}$$

such that we obtain for the time derivative

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} &= M\, \boldsymbol{Q} \times \ddot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times m_i\, \ddot{r}_i = \boldsymbol{Q} \times M\, \ddot{\boldsymbol{Q}} + \sum_i \boldsymbol{r}_i \times \sum_j \boldsymbol{F}_{ij} \\
&= \frac{1}{2} \left(\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} \right) \\
&= \frac{1}{2} \left(\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \sum_{ij} \boldsymbol{r}_j \times \boldsymbol{F}_{ij} \right) \\
&= \frac{1}{2} \sum_{ij} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = \boldsymbol{0}
\end{aligned}$$

Upon moving to the second line we used that $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$, and the antisymmetry of the forces $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moving to the third line we swapped the names of the summation indices i and j. In the last line, we collected terms and used that $\boldsymbol{F}_{ij}$ is parallel to $\boldsymbol{r}_i - \boldsymbol{r}_j$. We summarize this important finding the following

Theorem 4.1: Angular momentum conservation

The relative angular momentum is conserved for systems with pairwise interaction forces acting parallel to the distance between particles. The total angular momentum is conserved when external forces vanish or when they give rise to a center-of-mass forces $M\ddot{\boldsymbol{Q}}$ aligned parallel to $\boldsymbol{Q}$.

Remark 4.14. An example for the latter case is a harmonic force $\boldsymbol{F}_i = c\, m_i\, \boldsymbol{q}_i$. The proof is provided as Problem 4.19c).

Conservation of the relative angular momentum implies important constraints on the motion. In celestial mechanics this is vividly displayed in the shape of galaxies, solar systems and planetary ring structures. All these systems emerge by the gravitational collapse of large stellar dust clouds. Let cloud be spherically symmetric and uniform initially, consisting of a huge number of small dust particles. By statistical fluctuations the cloud will have an angular momentum, of the order of $M\,D^2\,\omega$ where M is the total mass of the cloud, D is the diameter of the cloud and ω is a tiny number with the unit of a rotation frequency. For a solar system the cloud will collapse until virtually all of its mass is concentrated eventually in the sun in its very center. This involves a change of the diameter of the region holding the mass of about 10^4. For conserved angular momentum the frequency ω is growing by a factor of 10^8. In Problem 4.17 you will show that the initial angular momentum can not be coped by a spin of the central star. The competing constraints of the tendency of gravity to lump together the matter in the cloud and the need to conserve angular moment eventually form a solar system with a central very massive star or double star that is surrounded by planets moving around the star at a distance large as compare to the size of the star.

4.6.3 *Self Test*

Problem 4.16. **The CM of the solar system and the position of the sun**

Verify that the center of mass of Sun can lie more than a sun-diameter away from the center of mass of the solar system.

Hint: Relevant parameters are provided in Table A.1.

Problem 4.17. **Angular momentum of the solar system**

The solar system has a total angular momentum of about $L_{SoSy} = 3.3212 \times 10^{45}\,\mathrm{kgm^2s^{-1}}$.

a) Assume that the mass was initially distributed in a ball of a radius of about 40 AU. Estimate the corresponding effective frequency ω.

b) Assume that the mass in concentrated in two point particles that circulate around each other at a distance of about the sun diameter. Compare their rotation speed to the speed of light.

c) Verify that 98% of L_{SoSy} is accounted for by the orbital angular momenta of the planets.

d) How does this imply the disk-like structure of our solar system?

e) Speculate about other effects that contribute to the remaining 2% of the total angular momentum.

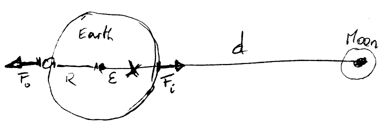


Figure 4.12: Distances adopted for the estimate of the forces inducing tidal forces F_o and F_i: The distance d between Earth and Moon, the Earth radius R, and the distance ϵ between the center of mass of Earth and the joint center of mass of Earth and Moon (indicated by a cross $\times$). Tides emerge at the side facing the Moon (inwards) and opposing the Moon (outwards).

Problem 4.18. **Tidal forces**

Gravitational forces of Moon and centripetal forces due to the rotation with frequency $\Omega = 1/\text{month}$ of Earth around the common center of mass of Earth and Moon (cross in Figure 4.12) give rise to tides. On the outwards facing side the resulting acceleration on a mass element on the Earth surface can be estimated as

$$a_0 = g - (\epsilon + R)\,\Omega^2 + \frac{G\,M_M}{(d+R)^2}$$

where M_M is the Moon mass.

a) Assume that Earth and Moon evolve on circular paths and employ the force balance for a stable motion in order to show that

$$a_0 = g - R\,\Omega^2 \left[1 - \frac{M_M}{M_E + M_M}\,\frac{R}{d}\,\left(1 + \mathcal{O}(R/d)\right)\right]$$

and determine the higher-order correction terms that are indicated here as $(1 + \mathcal{O}(R/d))$.

b) Determine also the change of the acceleration on the side towards the moon. How does it differ from a_i?

c) Determine the relative change of the gravitational acceleration due to the presence of moon, and the difference between a_i and a_o.

d) So far we only discussed the component of the acceleration along a line connecting Earth and Moon at the innermost and outermost points of the Earth surface. What about the other components of the gravitational acceleration:
when considering tides at mid latitudes?
at positions half-way between the two points (i. e. top and bottom sides of Earth in the figure).

e) What is the impact of the Earth rotation? How does it break the symmetry? What does this imply about the relative strength of the two tidal waves every day?

Problem 4.19. **Center of mass and constants of motion**

How do the expressions for the constants of motion discussed in Section 3.4 behave when separating the center of mass motion and the relative motion, $\boldsymbol{q}_i(t) = \boldsymbol{Q}(t) + \boldsymbol{r}_i(t)$.

a) Show that the kinetic energy $T = \sum_i m_i \dot{\boldsymbol{q}}_i^2$ takes the new value

$$T = \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 + \sum_i \frac{m_i}{2}\,\dot{\boldsymbol{r}}_i^2$$

b) Assume that the system is moving in a gravitational field, and that the other forces on the particle arise from pair-wise conservative interactions as discussed Equation (4.6.4). Show that the total energy can be written as

$$E = \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 - M\boldsymbol{g}\cdot\boldsymbol{Q} + \sum_i \frac{m_i}{2}\,\dot{\boldsymbol{r}}_i^2 + \sum_{i<j} \Phi_{ij}\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)$$

c) Show that the total angular momentum is conserved for a systems with the particles interactions given in Equation (4.6.4) and an additional external force

$$\boldsymbol{F}_i = c\,m_i\,\boldsymbol{q}_i$$

acting on each particle i.

4.7 Worked example: the Kepler problem

The first problem tackled in theoretical mechanics was the motion of two point particles with gravitational interaction. It is formulated in terms of three laws. The second law holds for all central forces, the 3rd law is a consequence of mechanical similarity, and the 1st law is based on a solution of the EOM. We first explore the general arguments, and then illustrate their application to the Kepler problem.

4.7.1 Conservation of angular momentum and Kepler's 2nd Law

Angular momentum conservation also has important consequences for the motion of two particles. The center of mass of the two particles takes the form

$$Q = \frac{m_1}{m_1 + m_2} \, (Q + \boldsymbol{r}_1) + \frac{m_2}{m_1 + m_2} \, (Q + \boldsymbol{r}_2) = Q + \frac{m_1 \, \boldsymbol{r}_1 + m_2 \, \boldsymbol{r}_2}{m_1 + m_2}$$

such that

$$m_1 \, \boldsymbol{r}_1 + m_2 \, \boldsymbol{r}_2 = \boldsymbol{0} \quad \text{and in particular} \quad \boldsymbol{p} = m_2 \, \dot{\boldsymbol{r}}_2 = -m_1 \, \dot{\boldsymbol{r}}_1 \, .$$

This has important consequences for the evolution of the conserved angular momentum of the relative motion

$$\boldsymbol{L} = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{p} \, .$$

In view of

$$\begin{aligned} \boldsymbol{p} = m_2 \, \dot{\boldsymbol{r}}_2 = m_2 \, \dot{\boldsymbol{q}}_2 - m_2 \, \dot{Q} &= \frac{m_1 m_2}{m_1 + m_2} \, \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{q}_2 - \boldsymbol{q}_1) \\ &= \mu \, \dot{\boldsymbol{R}} \quad \text{with} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad \boldsymbol{R} = \boldsymbol{q}_2 - \boldsymbol{q}_1 \end{aligned}$$

the angular momentum of the relative motion can be expressed in terms of the vector $\boldsymbol{R}$ connecting the two masses

$$\boldsymbol{L} = \boldsymbol{R} \times \mu \dot{\boldsymbol{R}}$$

It takes the then form the angular momentum of a single particle with mass μ, and this also applies for the relation between the acceleration and the force

$$\mu \ddot{\boldsymbol{R}} = m_2 \, \ddot{\boldsymbol{r}}_2 = \boldsymbol{F} \, .$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{L} = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{R} \times \mu \dot{\boldsymbol{R}} = \boldsymbol{R} \times \boldsymbol{F} = \boldsymbol{0}$$

for forces $\boldsymbol{F}$ acting along the line $\boldsymbol{R}$ connecting the two particles.

The conservation of angular momentum has two important consequences:

1. The direction of $\boldsymbol{L}$ is fixed. As a consequence the positions and the velocities of the planet and the sun always lie in a plane that is orthogonal to $\boldsymbol{L}$, and the force $\mu \ddot{\boldsymbol{R}}$ also lies in the plane because the

force is parallel to $\boldsymbol{R}$. Therefore, the motion is constrained to the plane for all times.

2. The absolute value of $\boldsymbol{L}$ is fixed, and this has a geometric interpretation that was first formulated in the context of planetary motion

Theorem 4.2: Kepler's second law

A segment joining the two particles, planet and sun in the Kepler problem, sweeps out equal areas Δa in equal time intervals Δt.

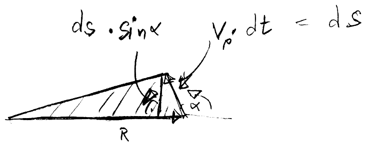


Figure 4.13: Area passed over by the trajectory.

Proof. For the time interval $[t_0, t_1]$ with length $\Delta t = t_1 - t_0$ one has

$$|\boldsymbol{L}|\,\Delta t = \int_{t_0}^{t_1} \mathrm{d}t\,|\boldsymbol{R} \times (m_2 \boldsymbol{v}_2)| = m_2 \int_{t_0}^{t_1} \mathrm{d}t\,|\boldsymbol{R}|\,|\boldsymbol{v}_2|\sin\alpha$$

where α is the angle between $\boldsymbol{R}$ and $\boldsymbol{v}_2$. Further, $\mathrm{d}s = v_2\,\mathrm{d}t$ is the path length that the trajectory traverses in a time unit $\mathrm{d}t$, such that $\mathrm{d}a = \mathrm{d}t\,|\boldsymbol{R}|\,|\boldsymbol{v}_2|\sin\alpha/2$ is the area swiped over in $\mathrm{d}t$ (see the sketch in Figure 4.13). Hence,

$$|\boldsymbol{L}|\,\Delta t = \frac{1}{2}\int_0^{\Delta a} \mathrm{d}a = \Delta a \qquad \Rightarrow \qquad \Delta a = \frac{2\,|\boldsymbol{L}|}{m_2}\,\Delta t$$

such that Δa is proportional to Δt. □

4.8 *Mechanical similarity — Kepler's 3rd Law*

Two solutions of a differential equations are called *similar* when they can be transformed into one another by a rescaling of the time-, length-, and mass-scales. We indicate the rescaled quantities by a prime, and denote the scale factors as τ, λ, and α, respectively,

$$t' = \tau t\,, \qquad q'_i = \lambda q_i\,, \qquad m'_i = \alpha m_i$$

We explore the consequences of this idea for the Kepler problem, i. e. for two point particles interacting by a gravitation force $\boldsymbol{F}$ deriving from the following potential

$$\Phi(|\boldsymbol{R}|) = \frac{m_1 m_2 G}{|\boldsymbol{R}|} \qquad \Rightarrow \qquad \boldsymbol{F} = -\nabla\Phi(|\boldsymbol{R}|) = \frac{m_1 m_2 G}{|\boldsymbol{R}|^3}\,\boldsymbol{R}$$

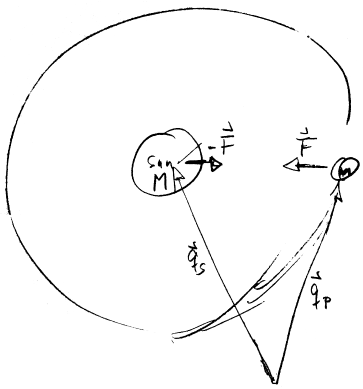


Figure 4.14: Setup of and notations for the motion of a planet around the sun. Here m_S and m_P are the mass of the sun and the planet, respectively, and $\boldsymbol{q}_S$ and $\boldsymbol{q}_P$ are their positions. The relative position is $\boldsymbol{R} = \boldsymbol{q}_P - \boldsymbol{q}_S$.

adapt Figure 4.14

The setup for a planet going around the sun is sketched in Figure 4.14. acting on the planet and pointing towards the sun. We only consider the relative motion and assume that there are no other forces acting on the sun and the planet.

Information about the period and the shape of the trajectory is obtained from the energy for the relative motion

$$E = \frac{\mu}{2}\,\dot{\boldsymbol{R}}^2 + \Phi(|\boldsymbol{R}|)$$

This energy is conserved because

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\,\dot{\boldsymbol{R}}^2 + \Phi(|\boldsymbol{R}|)\right) = \mu\,\dot{\boldsymbol{R}}\cdot\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}\cdot\nabla\Phi(|\boldsymbol{R}|)\\ &= \dot{\boldsymbol{R}}\cdot\left(\mu\,\ddot{\boldsymbol{R}} - \boldsymbol{F}\right) = \boldsymbol{0}\end{aligned}$$

Here, we used that $\boldsymbol{F} = \mu\, \ddot{\boldsymbol{R}}$.

In our planetary system the trajectories of the planets are all circular to a good approximation. They are therefore described by the *same* solution of the EOM up to a rescaling of the length scale and the time scale. The former accounts to their different distance to the Sun, and the latter to the different periods of their motion. We observe now that $\mu = m_1\, m_2/(m_1 + m_2)$ and that the Sun mass m_S is 1000 times larger than the mass of Jupiter, the largest planet. Therefore, for the motion of the planets we have $(m_1 + m_2)/\mu \simeq m_S$ and

$$\frac{E}{\mu} \simeq \frac{\dot{\boldsymbol{R}}^2}{2} + \frac{m_S\, G}{|R|}$$

We expect that different planets follow the same trajectory up to rescaling space and time units, and a different constant value of their energy. We hence explore the consequences of the scaling $\lambda \boldsymbol{R}(t)$ and τt

$$\frac{E}{\mu} \simeq \frac{\lambda^2}{\tau^2} \frac{\dot{\boldsymbol{R}}^2}{2} + \frac{1}{\lambda} \frac{m_S\, G}{|R|}$$
$$\Leftrightarrow \frac{E\, \tau^2}{\mu\, \lambda^2} \simeq \frac{\dot{\boldsymbol{R}}^2}{2} + \frac{\tau^2}{\lambda^3} \frac{m_S\, G}{|R|}$$

where the right-hand side remains invariant iff $\tau^2/\lambda^3 = \text{const}$. This entails

Theorem 4.3: Kepler's third law

The square of the period T of the planets in our planetary system are proportional to the third power of their distance D to the sun.

4.9 *Solving ODEs by coordinate transformations: Kepler's 1st law*

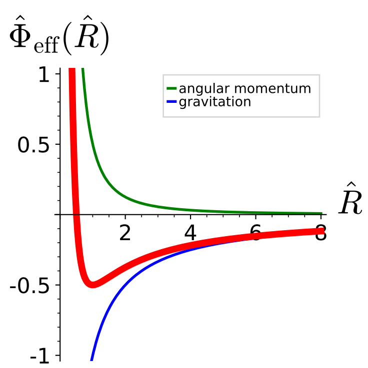


Figure 4.15: Effective potential $\hat{\Phi} = \Phi\, R_0/m_1\, m_2\, G$ for the Kepler problem as function of the dimensionless distance $\hat{R} = R/R_0$, where $R_0 = L^2/\mu\, m_1\, m_2\, G$.

In polar coordinates $\boldsymbol{R} = (R, \theta)$ the kinetic energy takes the form $\mu \dot{\boldsymbol{R}}^2/2 = \mu\left(\dot{R}^2 + (R\dot{\theta})^2\right)/2$ while the conservation of angular momentum implies $R\dot{\theta} = L/(\mu R)$ with $L = |\boldsymbol{L}|$. Consequently,

$$E = \frac{\mu}{2}\, \dot{R}^2(t) + \frac{L^2}{2\mu\, R^2(t)} - \frac{m_1 m_2 G}{R(t)} \tag{4.9.1}$$

which is equivalent to the motion of a particle of mass μ at position R in the one-dimensional effective potential (Figure 4.15)

$$\Phi_{\text{eff}}(R) = \frac{L^2}{2\mu\, R^2} - \frac{m_1 m_2 G}{R}\,.$$

The first, repulsive contribution to the effective potential arises from angular momentum conservation, and the second, attractive contribution is due to gravity.

There is no elementary way to determine the function $R(t)$. However, based on Equation (4.9.1) one can plot the trajectories in phase space, $\dot{R}(R)$ for different energies. This plot is provided in Figure 4.16. For negative energies there are bounded trajectories that oscillate in the minimum of the potential Φ_{eff}. For zero energy the trajectory reaches till $R = \infty$, and reaches infinity with zero speed. For a positive energy the trajectory reaches till $R = \infty$, and it will go there with speed $\dot{R} = \sqrt{2E/\mu}$.

However, one can determine the shape $R(\theta)$ of the trajectories by observing

$$\dot{R}(\theta) = \dot{\theta}\, \frac{\mathrm{d}R(\theta)}{\mathrm{d}\theta} = \frac{L}{\mu\, R^2}\, R'(\theta)$$

such that

$$E = \frac{L^2}{2\,\mu} \left(\frac{R'^2}{R^4} + \frac{1}{R^2} \right) - \frac{m_1\, m_2\, G}{R}$$

In terms of $w(\theta) = 1/R(\theta)$ this implies

$$\frac{\mu\, E}{L^2} = \frac{1}{2}\, \left(w'(\theta)\right)^2 + \frac{1}{2}\, w^2(\theta) - \frac{m_1\, m_2\, \mu\, G}{L^2}\, w(\theta) \tag{4.9.2}$$

and differentiating with respect to θ provides

$$0 = w'(\theta)\, \left[w''(\theta) + w(\theta) - \frac{m_1\, m_2\, \mu\, G}{L^2} \right] .$$

The expression in the square bracket is a second order linear ODE with solution

$$w(\theta) = \frac{\mu\, m_1\, m_2\, G}{L^2}\, \left[1 + \epsilon\, \cos\left(\theta - \theta_0\right)\right]$$

where ϵ and θ_0 are integration constants that must be determined from the initial conditions. Inserting $w(\theta)$ into Equation (4.9.2) yields

$$\frac{\mu\, E}{L^2} = \frac{\epsilon^2 - 1}{2}\, \left(\frac{m_1\, m_2\, \mu\, G}{L^2} \right)^2 \quad \Rightarrow \quad \epsilon^2 = 1 + \frac{2\, E\, L^2}{\mu\, \left(m_1\, m_2\, G\right)^2}$$

Hence, ϵ is fully determined by the parameters and the conservation laws of the Kepler problem, while θ_0 determines the orientation of the trajectory in the plane. Commonly, one chooses the coordinate frame where $\theta_0 = 0$.

For the motion of a planet around the sun this entails

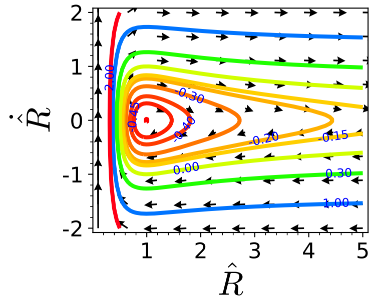


Figure 4.16: The phase-space flow for the EOM of $R(t)$ provided by Equation (4.9.1). The plot adopts dimensionless units with length scale R_0 introduced in Figure 4.15 and a time scale $t_0 = \sqrt{\mu\, R_0^3 / m_1\, m_2\, G}$. Solid lines refer to solutions for different dimensionless energy, with values marked on the contour lines.

Theorem 4.4: Kepler's first law

The trajectories of planets around the sun are described by sections of the cone with a plane,

$$R(\theta) = \frac{R_0}{1 + \epsilon\, \cos(\theta - \theta_0)} \tag{4.9.3}$$

where $R_0 = L^2 / m_1\, m_2\, \mu\, G$ sets the length scale of the trajectory and θ_0 the orientation in the plane. The parameter ϵ sets its shape: for $\epsilon = 0$ the shape amounts to a circle with radius R_0, for $0 < \epsilon < 1$ to an ellipse, for $\epsilon = 1$ to a parabola, and for $\epsilon > 1$ to a hyperbola.

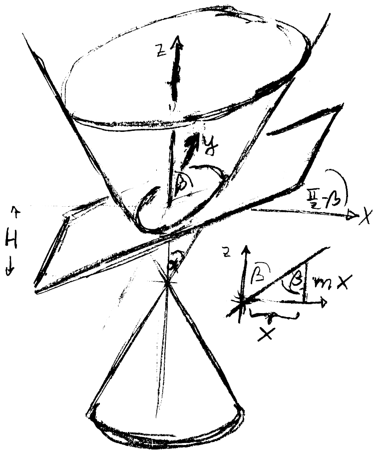


Figure 4.17: Section of a cone double and a plane. The axis are drawn here at the intersection point of the plane with the cone axis, in order to emphasize the the plane is tilted around the y axis. For calculations in the main text the vertex of the cone will be chosen as origin of the coordinate system.

Proof. We consider the section of a cone with opening angle α and its symmetry axis aligned along the z-axis, and a plane, as sketched in Figure 4.17. The origin of the coordinate system is a the vertex of the cone. The plane is tilted with respect to the y-axis such that it forms an angle β with the z-axis, and it intersects the z-axis at height H. The points in the plane have coordinates

$$\boldsymbol{q} = \begin{pmatrix} x \\ y \\ H + m\,H \end{pmatrix} \qquad \text{with} \quad m^{-1} = \tan\beta$$

The point $\boldsymbol{q}$ lies on the cone when $\boldsymbol{q} \cdot \hat{\boldsymbol{z}} = |\boldsymbol{q}| \cos\alpha$, which entails

$$(H + m\,x)^2 = \cos^2\alpha \, \left(x^2 + y^2 + (H + m\,x)^2 \right)$$

Henceforth, we adopt dimensionless coordinates $\hat{x} = x/H \tan\alpha$ and $\hat{y} = y/H \tan\alpha$, and we introduce the abbreviation $\epsilon = m \tan\alpha$. We will denote the distance from the origin in the (x, y)-plane as $R = \sqrt{\hat{x}^2 + \hat{y}^2}$, and introduce θ such that $\hat{x} = R \cos\theta$. This entails

$$(1 + \epsilon\, R \cos\theta)^2 = R^2$$

with solutions

$$\begin{aligned} R_\pm &= \frac{\epsilon \cos\theta}{1 - \epsilon^2 \cos^2\theta} \pm \frac{\left[\epsilon^2 \cos^2\theta + \left(1 - \epsilon^2 \cos^2\theta\right)\right]^{1/2}}{1 - \epsilon^2 \cos^2\theta} \\ &= \frac{\epsilon \cos\theta \pm 1}{1 - \epsilon^2 \cos^2\theta} = \frac{-1}{\pm 1 + \epsilon \cos\theta} \end{aligned}$$

Hence, Equation (4.9.3) describes a cone section with length scale $R_0 = H \tan\alpha$ and eccentricity $\epsilon = m \tan\alpha$.

The eccentricity amounts to the ratio of the slope m of the z coordinates of points in the plane as function of x, and the slope $1/\tan\alpha$ of the line obtained as intersection of the double cone and the (x, z) plane. This ratio determines the shape of the conic section (see Figure 4.18).

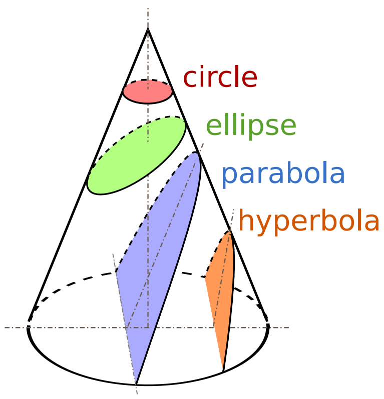


Figure 4.18: Shape of conic sections for parameters, $\epsilon = 0$ circle, $0 < \epsilon < 1$ ellipse, $\epsilon = 1$ parabola, and $\epsilon > 1$ hyperbola.

For $\epsilon = 0$ the shape is a circle $R^2 = 1$.

For $\epsilon = 1$ the shape is a parabola described by $1 + 2\,\epsilon\,\hat{x} = \hat{y}^2$.

For $0 < \epsilon < 1$ the shape is an ellipse described by

$$\frac{1 + \epsilon^2}{1 - \epsilon^2} = \hat{y}^2 + \left(1 - \epsilon^2\right) \left(\hat{x} - \frac{\epsilon}{1 - \epsilon^2} \right)^2$$

For $1 < \epsilon$ the shape is a hyperbola described by

$$\hat{y} = \pm \sqrt{ \frac{-1}{\epsilon^2 - 1} + (\epsilon^2 - 1) \left(\hat{x} + \frac{\epsilon}{\epsilon^2 - 1} \right)^2 }$$

□

add remark on physical interpretation of cone and plane

4.9.1 *Self Test*

Problem 4.20. **Keeping the Moon at a distance**

Something goes wrong at the farewell party for the settlers of the new Moon colony *Sleeping Beauty 1* such that an extremely annoyed evil fairy switches off gravity for the Moon. Luckily there also is a good fairy at the party. She cannot undo the curse but offers to strip all protons from all water-molecules in a bucket of water that you give to her, and hide them on Moon. The Coulomb attraction between electrons on Earth and protons on Moon can then undo the damage.

a) How much water would you give to her?

b) What will happen to the Earth-Moon system when you are off by 20%, by a factor of two, or even by an order of magnitude?
Hint: The idea is that you discuss the motion for an initial condition where Earth and Moon are at their present position and move with their present velocity, while the gravitational force is changed by a the specified factor.

Problem 4.21. **Mechanical similarity and dimensional analysis**

We discuss here the relation between dimensional analysis, introduced in Section 1.2, and mechanical similarity, adopting the notations introduced in the beginning of Section 4.8.

a) We consider a system with kinetic energy $T = \frac{1}{2} \sum_i m_i \, \dot{\boldsymbol{q}}_i^2$, and consider a potential that admits the following scaling

$$V' = \mu^\alpha \, \lambda^\beta \, V$$

Show that the EOM are then invariant when one rescales time as

$$\tau = \mu^{(1-\alpha)/2} \, \lambda^{(2-\beta)/2}$$

b) Consider now two pendulums, $V = mgz$ with different masses and length of the pendulum arms. Which factors τ, λ, and μ relate their trajectories? How will the periods of the pendulums thus be related to the ratio of the mass and the length of the arms? Which scaling do you expect based on a dimensional analysis?

c) What do you find for the according discussion of the periods of a mass subjected to a harmonic force, $V = k \, |\boldsymbol{q}|^2/2$?

d) Discuss the period of the trajectories in the Kepler problem, $V = mMG/|\boldsymbol{q}|$. In this case the dimensional analysis is tricky because the masses of the sun and of the planet appear in the problem. What does the similarity analysis reveal about the relevance of the mass of the planet for Kepler's third law?

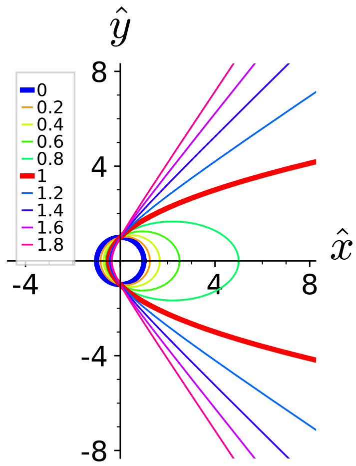


Figure 4.19: Conic sections for different eccentricity ϵ.

Problem 4.22. **Conic Sections**

In the margin we show the shape of conic sections for different eccentricity ϵ.

a) Show that all conic sections intersect the $\hat{y}$ axis at ± 1.

b) Show that the conic sections intersect the $\hat{x}$ axis at $-1/(\epsilon \pm 1)$. Where are these points located for different conic sections?

c) How does Equation (4.9.1) look like after introducing the dimensionless units adopted in Figures 4.15 and 4.16? Write down the solution of the EOM in dimensionless units.

d) Find an alternative choice of the length scale such that all trajectories intersect the x-axis at the potion -1, and prepare the corresponding plot of the trajectory shapes in the (x, y)-plane.

4.10 *Problems*

4.10.1 *Rehearsing Concepts*

Problem 4.23. **Maximum distance of flight**

There is a well-known rule that one should through a ball at an angle of roughly $\theta = \pi/4$ to achieve a maximum width.

a) Solve the equation of motion of the ball thrown in x direction with another velocity component in vertical z direction. Do not consider friction in this discussion, and verify that the ball will then proceeds on a parabolic trajectory in the (x, z) plane.

b) Well-trained shot put pushers push the put with an initial angle substantially smaller than $\pi/4$, i.e., they provide more forward than upward thrust. Verify that this is a good idea when the height H of the release point of the trajectory over the ground is noticeable as compared to the length L between the release point and touchdown, i.e. when H/L is not small.

What is the optimum choice of θ for the shot put?

c) Consider now friction:

- Is it relevant for the conclusions on throwing shot puts?
- Is it relevant for throwing a ball?
- How much does it impact the maximum distance that one can reach in a gun shot?

Problem 4.24. **Phase-space portraits for a scattering problem**

a) Sketch the potential $\Phi(x) = 1 - 1/\cosh x$ for $x \in \mathbb{R}$.

b) Sketch the direction field in the phase space for the EOM $\ddot{x} = -\partial_x \Phi(x)$.

c) Show that $E = \frac{1}{2}\,\dot{x}^2 + \Phi(x)$ is a constant of motion of the EOM.

d) Use energy conservation to determine the shape of the trajectories in phase space, and add a few trajectories to the plot started in b).

Add to the sketch a the phase portrait of the motion in this potential, i. e. the solutions of in the phase space $(x, \dot{x})$.

Problem 4.25. **Another linear ODEs with constant coefficients**

Consider the ODE

$$\ddot{x} = a\,x \qquad \text{with} \quad a \in \mathbb{R}_+$$

a) Sketch the direction field in phase space.

b) Find the solutions of $x(x)$.

c) Add the trajectories the are proceeding through the points $(x(t_0), \dot{x}(t_0) \in \{(1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1), (1,-1)\}$ to the plot started in a).
Hint: Only two cases must be solved explicitly. All other solutions can be inferred from symmetry arguments.

Problem 4.26. **Stokes drag**

The EOM for Stokes friction, Equation (4.3.1) is a linear differential equation. Adopt the strategy for solving linear differential equations, Algorithm 4.3, to find the solution Equation (4.3.3b).

4.10.2 *Practicing Concepts*

Problem 4.27. **Egyptian water clocks**

In ancient Egypt time was measured by following how water is running out of a container with a constant cross section A. At a water level h in the container, the water will then run out at a speed

$$v(t) = -c\,\sqrt{2g\,h(t)}$$

where the numerical constant c accounts for the viscosity of water and the geometry of the vessel. For Egyptian water clocks this constant takes values of the order of $c \simeq 0.6$.

a) How does the height $h(t)$ of the water in he container evolve after the plug is pulled?

For use as a clock it would be desirable to change the design of the clock such that $h(t)$ would decrease linearly in time. How can the construction of the water clock be amended to reach that aim?

Problem 4.28. **Damped oscillator**

Physical systems are subjected to friction. This can be taken into account by augmenting the EOM of a particle suspended from a spring, Equation (4.5.1), by a friction term

$$m\,\ddot{z}(t) = -m\,g - k\,z(t) - \mu\,\dot{z}(t)$$

a) How does friction affect the motion $z(t)$ of the particle? What is the condition that there are still oscillations, even though with a damping? For which parameters will they disappear, and how do the solutions look like in that case?

b) Sketch the evolution of the trajectories in phase space, for the two settings with and without oscillations.

c) For the borderline case the characteristic polynomial will only have a single root, λ. Verify that the general solution can then be written as

$$z(t) = z_0 + A_1\,\mathrm{e}^{\lambda\,(t-t_0)} + A_2\,t\,\mathrm{e}^{\lambda\,(t-t_0)}$$

d) Determine the solutions for a particle for the following initial conditions:
the particle is at rest and at a distance A from its equilibrium position,
the particle is at the equilibrium position, but it has an initial velocity v_0.
Indicate the form of these trajectories in the phase-space plots.

Problem 4.29. **One-dimensional collisions in the center-of-mass frame**

In Example 3.12 we discussed one-dimensional collisions for settings where the second particle is initially at rest. Now, we consider the situation where both particles are moving from the beginning. Specifically, we consider a setting with two particles of masses m_1 and m_2 with the initial conditions $\big(q_1(t_0), v_1\big)$ and $\big(q_2(t_0), v_2\big)$.

a) Show that the center of mass $Q(t) = \big(m_1\,x_1(t) + m_2\,x_2(t)\big)/M$ with $M = m_1 + m_2$ of the two particles evolves as

$$Q(t) = Q(t_0) + \dot{Q}(t_0)\,(t-t_0) \qquad \text{where} \qquad \dot{Q}(t_0) = a_1\,v_1 + a_2\,v_2$$

and determine the associated real constants a_1 and a_2.

b) We denote the relative coordinates as $x_i = q_i - Q$ and associate it with a momentum $m_i\,\dot{x}_i$. Show that the relative momenta add up to zero before and after the collision,

$$0 = m_1\,x_1 + m_2\,x_2 = m_1\,(q_1 - Q) + m_2\,(q_2 - Q)$$

and that they swap signs upon collision.
Hint: This is a consequence of energy conservation.

c) Determine the time evolution before and after the collision.

d) Verify the consistency of your result with the special case treated in Example 3.12.

Problem 4.30. **Motion in a harmonic central force field**

A particle of mass m and at position $\boldsymbol{r}(t)$ is moving under the influence of a central force field

$$\boldsymbol{F}(\boldsymbol{r}) = -k\,\boldsymbol{r}\,.$$

a) We want to use the force to build a particle trap,[5] i.e. to make sure that the particle trajectories $\boldsymbol{r}(t)$ are bounded: For all initial conditions there is a bound B such that $|\boldsymbol{r}(t)| < B$ for all times t. What is the requirement on k to achieve this aim?

[5] Particle traps with much more elaborate force fields, e.g. the Penning- and the Paul-trap, are used to fix particles in space for storage and use in high precision spectroscopy.

b) Determine the energy of the particle and show that the energy is conserved.

c) Demonstrate that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times m\,\dot{\boldsymbol{r}}$ of the particle is conserved, too. Is this also true when considering a different origin of the coordinate system?
Hint: The center of the force field is no longer coincide with the origin of the coordinate system in that case.

d) Let (x_1, x_2) be the coordinates in the plane that is singled out by the angular momentum conservation. Show that $m\ddot{x}_i(t) + k\,x_i(t) = 0$ for $i \in \{1,2\}$. Determine the solution of these equations. Sketch the trajectories in the phase space $(x_i, \dot{x}_i)$. What determines the shape of the trajectories?

e) Show that the trajectories in the configuration space (x_1, x_2) are ellipses. What determines the shape of these trajectories?

f) Discuss the relation between the amplitude and shape of the trajectory, as determined by the ratio and the geometric mean of the major axes of the ellipse in configuration space, and the period of the trajectory.

4.10.3 *Mathematical Foundation*

Problem 4.31. **Differential equations and functional dependencies**

Determine ODEs whose general solutions are of the form

a) $y(x) = C\,x^2 - x$

b) $y^2(x) = A\,x + B$

Here, A, B, and C are real constants that will be determined by the IC of the ODE.

Problem 4.32. **Separation of variables for a non-autonomous ODE**

We consider the ODE

$$y'(x) = \frac{x}{y}$$

a) How many degrees of freedom does this system have? What is its space? State it as a first order ODE in terms of the phase-space variables.

b) Sketch the direction field in phase space.

c) Find the solution of the ODE for ICs (x_0, y_0) with $y_0 \neq 0$ and

   i. $x_0 < 0$ and $x_0 < y_0 < -x_0$

   ii. $x_0 > 0$ and $x_0 > y_0 > -x_0$

   iii. other ICs with $|x_0| \neq |y_0|$

   iv. $|x_0| = |y_0|$

d) Determine the largest interval of values $x \in \mathbb{R}$ where the solutions $y(x)$ obtained in b) are defined.

e) Is the function $y(x) = |x|$ a solution of the ODE? If in doubt: Where do you see problems for this solution?

Problem 4.33. **Effective potentials and phase-space portraits**

We consider ODEs of the form

$$\ddot{x}(t) = -\frac{\mathrm{d}}{\mathrm{d}x} V_{\text{eff}}(x)$$

Sketch the solutions for trajectories in the following potentials in the phase space $(x, \dot{x})$.

a) $V_{\text{eff}} = x \sin x$ b) $V_{\text{eff}} = x \cos x$
c) $V_{\text{eff}} = x - \sin x$ d) $V_{\text{eff}} = x - \cos x$
e) $V_{\text{eff}} = \mathrm{e}^{x} \sin x$ f) $V_{\text{eff}} = \mathrm{e}^{-x} \sin x$

Problem 4.34. **Central forces conserve angular momentum**

Consider a system of N particles at the positions $\boldsymbol{q}_i$ with masses m_i where each pair (ij) interacts by a force $\boldsymbol{F}_{ij}(|\boldsymbol{d}_{ij}|)$ acting parallel to the displacement vector $\boldsymbol{d}_{ij} = \boldsymbol{q}_j - \boldsymbol{q}_i$ from particle i to j. Proof the following statements:

a) The evolution of the center of mass of the system

$$\boldsymbol{Q} = \frac{1}{M} \sum_{i=0}^{N} m_i \boldsymbol{q}_i \quad \text{with} \quad M = \sum_{i=0}^{N} m_i$$

is force free, i. e. $\ddot{\boldsymbol{Q}} = \boldsymbol{0}$.

b) The total angular momentum can be written as

$$\boldsymbol{L}_{\text{tot}} = M\,\boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \sum_{i<j} \mu_{ij}\, \boldsymbol{d}_{ij} \times \dot{\boldsymbol{d}}_{ij}$$

Determine the factors μ_{ij}.

c) The two contributions to the angular momentum, $M\,\boldsymbol{Q} \times \dot{\boldsymbol{Q}}$ and the sum $\sum_{i<j} \mu_{ij}\, \boldsymbol{d}_{ij} \times \dot{\boldsymbol{d}}_{ij}$ are both conserved.

Problem 4.35. **Impact of translations on conservation laws**

We consider a coordinate transformation where the origin of the coordinate systems is moved to a new time-dependent position $\boldsymbol{x}(t)$,

$$\boldsymbol{q}_i(t) = \boldsymbol{x}(t) + \boldsymbol{r}_i(t)$$

a) Show that the expressions for the kinetic energy are related by

$$T = \sum_i \frac{m_i}{2}\, \dot{\boldsymbol{q}}_i^2 = \frac{M}{2}\, \dot{\boldsymbol{x}}^2 + M\, \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{Q}} + \sum_i \frac{m_i}{2}\, \dot{\boldsymbol{r}}_i^2$$

Here, $M = \sum_i m_i$ and $\boldsymbol{Q} = M^{-1} \sum_i m_i \boldsymbol{q}_i$ are the total mass and the center of mass, respectively.

b) Show that the expressions for the total energy for motion in an external field are related by

$$\begin{aligned} E &= T - M\,\boldsymbol{g} \cdot \boldsymbol{Q} + \sum_{i<j} \Phi_{ij}\big(|\boldsymbol{q}_i - \boldsymbol{q}_j|\big) \\ &= T - M\,\boldsymbol{g} \cdot \boldsymbol{Q} + \sum_{i<j} \Phi_{ij}\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) - M\,\boldsymbol{g} \cdot \boldsymbol{x} \end{aligned}$$

c) Show that the angular momentum transforms as follows

$$L = \sum_i m_i\, \boldsymbol{q}_i \times \dot{\boldsymbol{q}}_i = M\,\boldsymbol{x} \times \dot{\boldsymbol{Q}} + M\,(\boldsymbol{x} + \boldsymbol{Q}) \times \dot{\boldsymbol{x}} + \sum_i m_i\, \boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i$$

d) Show that conservation laws are mapped to conservation laws iff we consider a Galilei transformation, i. e. a transformation where $\dot{x}$ =const.

4.10.4 *Transfer and Bonus Problems, Riddles*

Problem 4.36. **Light intensity at single-slit diffraction**

Monochromatic light of wave length λ that is passion through a slit will produce an diffraction pattern on a screen where the intensity follows (cf. Figure 4.20, top panel)

$$I(x) = I_{\max} \left(\frac{\sin x}{x} \right)^2$$

Here the light intensity $I(x)$ is the power per unit area that is observed at a distance x to the side from the direction straight ahead

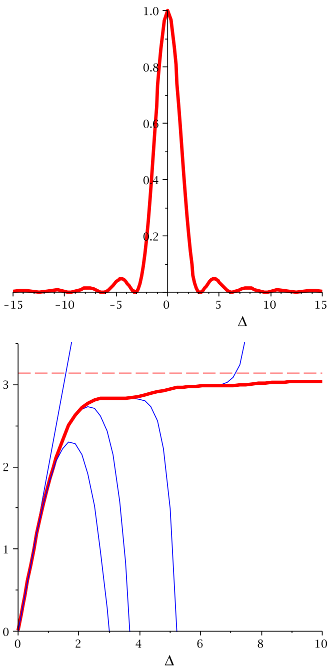


Figure 4.20: The upper panel shows the light intensity $I(x)/I_{\max}$, and the lower panel the fraction of light in the center region of width Δ, i. e. the power $P(\Delta) = \left[\int_{-\Delta}^{\Delta} I(x) \mathrm{d}x\right] / I_{\max}$. The red dotted value marks the asymptotic value π and the blue line the approximations obtained by a Taylor approximation up to order 2, 4, 8, 16, and 32. according to the Taylor series evaluated in Problem 4.36.b).

from the light source through the slit to the screen. We are interested in the total power $P(\Delta)$ that falls into a region of width $|x| < \Delta$. Since there is no antiderivative for $I(x)$ we will find approximate solutions by considering Taylor approximations of $I(x)$ that can be integrated without effort.

a) Show that $\sin^2 x = (1 - \cos 2x)/2$, and use the Taylor expansion of the cosine-function to show that

$$\frac{\sin^2 x}{x^2} = \frac{1 - \cos 2x}{2x^2} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} \, (2x)^{2n}$$

b) Determine the Taylor approximations for $P(\Delta)$ by integrating the expression found in a).

c) Write a program that is numerically determines $P(\Delta)$ and compares it to Taylor approximations of different order, as shown in the lower panel of Figure 4.20.

Problem 4.37. **Tricky issues in a classical population model**

The Lotka-Volterra model is considered the first model addressing the evolution of populations in theoretical biology. It predicts oscillations of populations, and still today it is cited in the context of data of Lynx and Hare that were collected in Canada in the late 19th century (cf. Figure 4.21).

Let $H(t)$ be the population of prey animals (Hare) and $L(t)$ be the population of its predator (Lynx). When there are no predators the population of prey grows exponentially with a rate a, and this rate is reduced by $-bL(t)$, when prey is consumed by predators. In absence of food the predators die at a rate d, and this rate is reduced by $-cH(t)$, when they find food.

$$\dot{H}(t) = H(t) \; [a - b\, L(t)]$$

$$\dot{L}(t) = L(t) \; [c\, H(t) - d]$$

a) Let $u(\tau) \propto H(t)$, $v(\tau) \propto L(t)$, and $\tau \propto t$. Find suitable proportionality constants and a dimensionless parameter Π such that

$$\dot{u}(\tau) = \quad u(\tau) \; [1 - v(t)]$$

$$\dot{v}(\tau) = \Pi^2 \, v(\tau) \; [u(\tau) - 1]$$

b) Show that the EOM for this biological system has fixed points at $(0,0)$ and $(1,1)$. How does the population model behave close to these fixed points?

c) Sketch the evolution of the solutions in the (u,v)-plane, and compare your result with the data reported on the lynx and hare that are shown in Figure 4.21. Can you find the qualitative difference of the data and behavior predicted by the model?
Hint: Look at the orientation of the flow in phase space. Who would be eating whom?

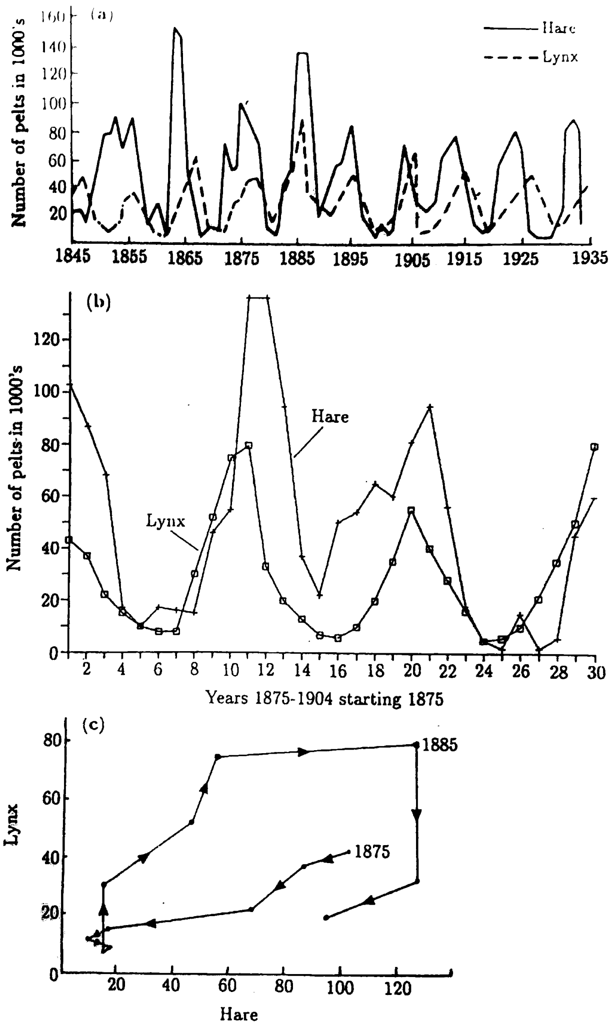


Figure 4.21: (a) Annual oscillations of the skins of hare and lynx offered to the Hudson Bay company. (b) Data with higher time resolution for the 30 years between 1875 and 1904. (c) Presentation of the data presented in (b) as a phase-space plot. [reproduced from Fig. 3.3. of Murray (2002). The book provides a thorough discussion of populations models, their assumptions and artifacts for a range of different populations models.]

d) One can infer the form of the trajectories in phase space by observing that

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\dot{v}}{\dot{u}} = \pi^2 \, \frac{v\,(u-1)}{u\,(1-v)} \,.$$

Why does this hold?

e) Find the solution of the ODE by separation of variables and show that the result implies the following constant of motion

$$\Phi(u,v) = \ln\left(v\,u^{\alpha}\right) - v - \alpha\,u\,, \qquad \text{with a suitably chosen } \alpha > 0.$$

Verify this result by also determining the time derivative of $\Phi(u(\tau), v(\tau))$. Here $(u(\tau), v(\tau))$ is a solution of the EOM.

Remark: The presence of a conservation law should be considered an artifact of the model whenever there is no model-immanent (i. e. required by the biological problem in this cases) reason for it to exist.

grav nut cracker

4.11 Further reading

fix reference!

An excellent mathematical treatise of the theory of ODEs that is well-accessible for physicists is given by Arnol'd (1992).

5
Impact of Spatial Extension

In Chapter 4 we discussed the motion of point particles. However, in our environment the spatial extension of particles in crucial. Physical objects always keep a minimum distance due to their spatial extension. When they had zero extension, one could neither blow up water droplets by impact with a laser (Figure 5.1), nor work clackers (Figure 5.2) or hit a ball with a tennis racket (Figure 5.3). Even giving spin to a ball only works due to the distance between the surface of the racket and the center of the ball.

At the end of this chapter we will be able to discuss the evolution of balls with spin, and their reflections from flat surfaces. Why is spin of so much importance in table tennis? How can a wing man score a goal in Handball, even when the goal keeper is fully blocking he direct path to the goal?[1]

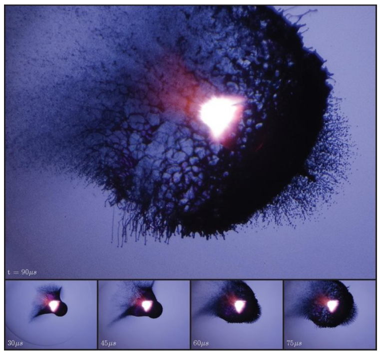


Figure 5.1: Impact of a laser pulse on a microdrop of opaque liquid that is thus blown up; cf. Klein, et al, Phys. Rev. Appl. 3 (2015) 044018

Figure 5.2: Girl playing with clackers.

Figure 5.3: Man running to return a tennis ball.

[1] See Wikipedia's description of Handball formations and this picture of an attack by Bertrand Gille, the IHF World Player of the Year 2002.

What is the magic of Beckham's banana kicks?

5.1 *Motivation and outline: How do particles collide?*

In order to get a first impression about this idea we consider the case of two particles at the positions $\boldsymbol{q}_i$, $i \in \{1,2\}$ that interact by a repulsive Coulomb force that derives from a potential $\Phi_C(|\boldsymbol{R}|)$ with $\boldsymbol{R} = \boldsymbol{q}_2 - \boldsymbol{q}_1$,

$$\Phi_C(|\boldsymbol{R}|) = \frac{C}{|\boldsymbol{R}|} \quad \Rightarrow \quad \boldsymbol{F}_c(\boldsymbol{q}_i) = -\nabla_{\boldsymbol{q}_i}\Phi_C(|\boldsymbol{q}_i - \boldsymbol{q}_{2-i}|) = \frac{C\;(\boldsymbol{q}_i - \boldsymbol{q}_{2-i})}{|\boldsymbol{q}_i - \boldsymbol{q}_{2-i}|^3}$$

Here, $2-i$ is the index of the other particle (1 for $i = 2$ and 2 for $i = 1$), and the constant C is the product of the permittivity of the vacuum and the particle charges. For charges of opposite signs this force has agrees with the gravitational force when one substitutes $C \to -m_1\, m_2\, G$. This results in the same dimensionless equations of motion as obtained for the Kepler problem, with the important difference that the length and time units adopted to defined the dimensionless units take vastly different values.

When the two particles carry charges of equal signs the force is repulsive, giving rise to the EOM

$$0 = w''(\theta) + w(\theta) + \frac{\mu\, C}{L^2}\, w(\theta)$$

such that

$$R(\theta) = \frac{1}{W(\theta)} = \frac{R_0}{-1 + \epsilon\, \cos(\theta - \theta_0)} \quad \text{where} \quad R_0 = \frac{L^2}{\mu\, C}$$

agrees with Equation (4.9.3) up to a change of the sign of the one in the denominator and the length unit R_0.

Remark 5.1. It is illuminating to adopt a different perspective on the origin of the minus sign in front of the one. Let us write the force on particle 1 as $\boldsymbol{F}_1 = F_1\, \hat{\boldsymbol{e}}(\theta)$ where $\hat{\boldsymbol{e}}(\theta)$ is the vector pointing from particle 1 to particle 2. The strength of the scalar force F_1 will be positive for an attractive force and negative for a repulsive force. In the dimensionless force $F t_0^2 / \mu\, R_0$ the change of sign is taken into account by the sign of C in $R_0 = L^2/\mu\, C$ and the solution takes the form of Equation (4.9.3). In order to obtain a positive length scale $|R_0| = \pm R_0$ we multiply the numerator and denominator of the solution by the ± 1 and absorb the factor in front of ϵ in a rotation of the angle by π such that the polar coordinates are always aligned with the direction of the force. Hence, one finds

$$R(\theta) = \frac{1}{W(\theta)} = \frac{|R_0|}{\pm 1 + \epsilon\, \cos(\theta - \theta_0)} \quad \text{where} \quad \pm 1 = \operatorname{sign}(C)$$

At this point dimensionless units play out their strength. We obtain the solution of the nontrivial EOM by an analysis of the ODE and mapping of parameters to a known problem, rather than going again through the involved analysis. ◻

The phase-space portrait and the shape of the orbits for repulsive interactions are plotted in Figure 5.4. We observe that the trajectory

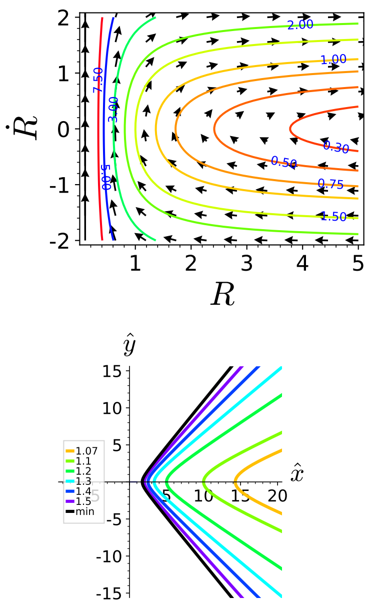


Figure 5.4: Phase-space flow and the shape of trajectories for scattering with a repulsive Coulomb potential.

shape describes the approach of the other particle from a perspective of an observer that sits on a particle located in the origin. When the observer sits on a particle that has a much larger mass than the approaching particle, then an outside observer will see virtually no motion of the mass-rich particle and the lines in Figure 5.4 describe the lines of the trajectories of the light particle in a plane selected by the initial angular momentum of the scattering problem. In general, two particles of masses m_1 and m_2 will be at opposite sides of the center of mass. In a coordinate system with its origin at the center of mass the lines in Figure 5.4 describe the particle trajectories up to factors $m_1/(m_1+m_2)$ and $-m_2/(m_1+m_2)$ for the first and second particle, respectively. A pair of trajectories for $m_1 = 0.3\,(m_1+m_2)$ and $\epsilon = 1.2$ is shown in Figure 5.5. The approximation as point particles is well justified when the sum of the particle radii is much smaller than their closest approach $R_0/(\epsilon-1)$.

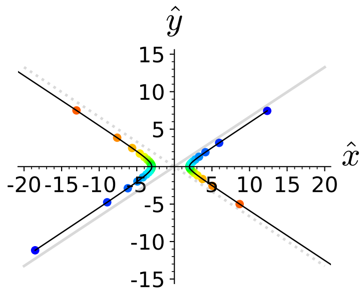


Figure 5.5: The two black lines show the scattering trajectories of two particles with $\epsilon = 1.2$ and relative mass $m_1 = 0.3\,(m_1+m_2)$. They approach each other along the solid gray line and separate along the dotted line. Particle 1 is initially at the top right. Corresponding positions are marked by dots of matching color.

Outline

In Section 5.2 we study the collision of spherical hard-ball particles that only interact by a force kick vertical to the surfaces at their contact point when they touch. Then we compare the Coulomb case and the force-kick case in order to explore which features of the outgoing trajectories are provided by conservation laws, irrespective of the type of interaction. In Section 5.3 we discuss the forces of an extended object (Earth) on a point particle moving without further interactions in its gravitational field. In Section 5.4 we further explore the impact spatial extension of solid particles: How does their shape matter? How are particles set into spinning motion, and how does the spin evolve? Section 5.5 addresses the motion of particles with internal degrees of freedom. Finally, in Section 5.6 we wrap up the findings of this section by discussing the reflections of balls: How do balls pick up spin in collisions? What happens upon multiple collisions in a channel with parallel walls? How should one return a ping-pong ball arriving with severe spin? How much energy is dissipated into vibrations of the ball?

check and update upon finalizing Chapter

5.1.1 Self Test

Problem 5.1. **Scattering angle for the Coulomb potential**

For the choice of coordinates adopted in Figures 5.4 and 5.5 the trajectories have an asymptotic angle θ with the $\hat{x}$-axis when they approach each other and they separate with an asymptotic angle $-\theta$.

a) Show that

$$\tan^2\theta = \frac{2\,E\,L^2}{\mu\,C^2} \tag{5.1.1}$$

b) The parameter dependence of the scattering angle θ is shown in Figure 5.6. What happens to the line for very large values of $2\,E\,L^2/\mu\,C^2$?

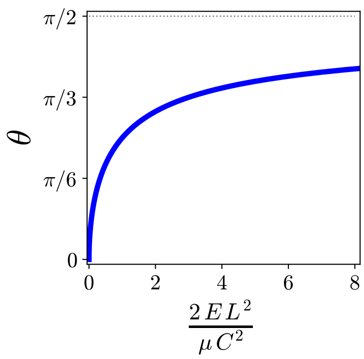


Figure 5.6: Scattering angle θ for a collision of two particles that interact by a repulsive Coulomb potential..

c) How would the scattering trajectories in Figure 5.5 look like for $\theta = \pi/2$? Does this comply with your finding in b)?

5.2 Collisions of hard-ball particles

We consider two spherical particles and denote their radii and masses as R_i and m_i with $i \in \{1, 2\}$, respectively. At the initial time $t = t_0$ the particles motion is not (yet) subjected to a force such that

$$\boldsymbol{q}_i(t) = \boldsymbol{q}_i(t_0) + v_i\,(t - t_0)\,, \quad \text{for } \; i \in \{1, 2\}$$

5.2.1 Center of mass motion

Analogous to the treatment of the Kepler problem, we decompose the motion of the particles into a center-of-mass motion $\boldsymbol{Q}(t)$ and a relative motion $\boldsymbol{r}(t)$. Introducing the notion $M = m_1 + m_2$ the former amounts to

$$M\,\boldsymbol{Q}(t) = m_1\,\boldsymbol{q}_1(t) + m_2\,\boldsymbol{q}_2(t) = M\,\boldsymbol{Q}(t_0) + \dot{\boldsymbol{Q}}(t_0)\,(t - t_0) \tag{5.2.1}$$

Since there are not external forces the total momentum $M\,\dot{\boldsymbol{Q}}(t)$ is conserved (cf. Theorem 3.5) such that Equation (5.2.1) applies for all times – even when the particles collide. A collision will therefore only impact the evolution relative to the center of mass. Equation (5.2.1) holds for all times.

5.2.2 Condition for collisions

To explore the relative motion we write $\boldsymbol{q}_i = \boldsymbol{Q} + \boldsymbol{x}_i$, and we introduce the momentum $\boldsymbol{p} = m_1\,\dot{\boldsymbol{x}}_1 = -m_2\,\dot{\boldsymbol{x}}_2$ and the distance coordinate $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. With these notations the angular momentum of the relative motion reads $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, and it is conserved when the collision force is acting along the line connecting the centers of the particles (cf. Theorem 3.6 and the discussion of Kepler's problem in Section 4.7). Moreover, $\boldsymbol{r}(t)$ is the only time-dependent quantity in this equation because $\boldsymbol{L}$ and $\boldsymbol{p}$ are preserved. Let us first assume that the particles do not collide, and that the closest approach occurs at some time t_c to a distance $r_c = |\boldsymbol{r}(t_c)|$. Then the vectors $\boldsymbol{r}(t_c)$ and $\boldsymbol{p}$ will be orthogonal, and $|\boldsymbol{L}| = r_c\,|\boldsymbol{p}|$. By the properties of the vector product the distance of the closest encounter will always be

$$r_c = \frac{|\boldsymbol{L}|}{|\boldsymbol{p}|} = \frac{\left|m_1\boldsymbol{q}_1(t_0) \times \dot{\boldsymbol{q}}_1(t_0) + m_2\boldsymbol{q}_2(t_0) \times \dot{\boldsymbol{q}}_2(t_0) - M\boldsymbol{Q}(t_0) \times \dot{\boldsymbol{Q}}(t_0)\right|}{m_1\left|\dot{\boldsymbol{q}}_1(t_0) - \dot{\boldsymbol{Q}}\right|}$$

and there will be no collision if $r_c > R_1 + R_2$.

5.2.3 The collision

Conservation of angular momentum implies that the relative motion of the particles proceeds in a plane. When they collide they approach until, at time t_c, they reach a position $\boldsymbol{r}(t_c)$ where their distance is $|\boldsymbol{r}(t_c)| = R_1 + R_2$. We denote the direction of $\boldsymbol{r}$ at this time as $\hat{\boldsymbol{\beta}}$ and augment it by an orthogonal direction $\hat{\boldsymbol{\alpha}}$ such that $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{L}} = \boldsymbol{L}/|\boldsymbol{L}|)$ form an orthonormal basis. We select the origin of the associated coordinate system such that

$$\boldsymbol{p} = (\boldsymbol{p} \cdot \hat{\boldsymbol{\alpha}})\, \hat{\boldsymbol{\alpha}} + (\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})\, \hat{\boldsymbol{\beta}}$$

At the collision there is a force $\boldsymbol{F} = F\, \hat{\boldsymbol{\beta}}$ acting on the particles, that acts in the direction of the line $\boldsymbol{r}(t_c)$ connecting the particles. Hence,

1. the momentum component in the $\hat{\boldsymbol{\alpha}}$ direction is preserved during the collision because there is no force acting in this direction
2. the collision in $\hat{\boldsymbol{\beta}}$ direction proceeds like a one-dimensional collision, Example 3.12, with the exception that one must retrace the argument using the center-of-mass frame, as discussed in Problem 4.29.

Consequently, we obtain the following momentum $\boldsymbol{p}'$ after the collision

$$\boldsymbol{p}' = (\boldsymbol{p} \cdot \hat{\boldsymbol{\alpha}})\, \hat{\boldsymbol{\alpha}} - (\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})\, \hat{\boldsymbol{\beta}} = \boldsymbol{p} - 2\, (\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})\, \hat{\boldsymbol{\beta}}$$

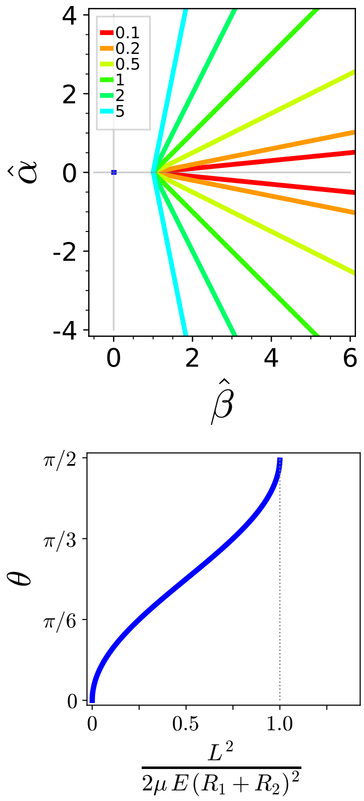


Figure 5.7: Collision of two hard-ball particles with radii R_1 and R_2: (top) Trajectory shape. The labels denote the ratios $(\boldsymbol{p} \cdot \hat{\boldsymbol{\alpha}})/(\boldsymbol{p} \cdot \hat{\boldsymbol{\beta}})$. (bottom) Scattering angle θ.

5.2.4 *Self Test*

Problem 5.2. **Scattering angle for hard-ball particles**

In Figure 5.7 we show shows the trajectory shape and the scattering angle for hard-ball scattering.

a) What is the dimensionless length scale adopted to plot the trajectory shapes?

b) What is the impact of the angular momentum on the trajectory shape?
What is the impact of the energy?

c) Verify that

$$\sin^2\theta = \frac{L^2}{2\mu\, E\, (R_1 + R_2)^2} \tag{5.2.2}$$

and that this dependence is plotted in the lower panel of Figure 5.7.

d) What happens when $L^2 > 2\mu\, E\, (R_1 + R_2)^2$?
Which angle θ will one observe in that case?

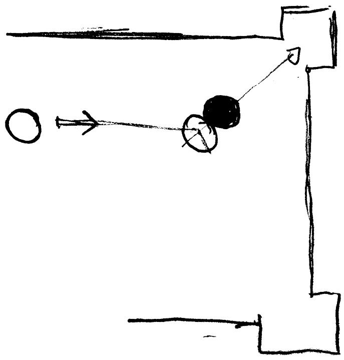
e) Show that Equations (5.1.1) and (5.2.2) agree when one identifies the length scale $R_1 + R_2$ of the hard-ball system with the distance R_{eff} of symmetry point of the cone section from the origin, i. e. with the mean value of the two intersection points with the $\hat{x}$-axis

$$R_{\text{eff}} = \frac{1}{2}\left(\frac{R_0}{1+\epsilon} + \frac{R_0}{1-\epsilon}\right) = \frac{\epsilon\, R_0}{1-\epsilon^2}$$

Can you provide a physical argument why that must be true?

Problem 5.3. **Reflection from a wall**

Show that a particle reflected at a flat wall follows the same trajectory as a particle that collides with a particle of the same mass and at a position obtained as mirror image of the particle.



Problem 5.4. **Collisions on a billiard table**

The sketch to the right shows a billiard table. The white ball should be kicked (i.e. set into motion with velocity $\boldsymbol{v}$), and hit the black ball such that it ends up in pocket to the top right.

What is tricky about the sketched track?

What might be a better alternative?

5.3 *Volume integrals — A professor falling through Earth*

The center of mass of a set of particles was defined in Equation (4.6.1) as a weighted sum of their positions. Now we consider an extended object that is characterized by a mass distribution $\rho(\boldsymbol{q})$. We will always assume that the distribution varies slowly in space in side the object. Outside it vanishes. The weighted sum over the particle positions will then be generalized to become a volume integral.

5.3.1 *Determine volume and mass by volume integrals*

In Section 3.4.2 we introduced line integrals by dividing the integration path into small steps $\{s_i\}$, and approximating the integral as a sum over the contributions of the individual pieces. The definition of a volume integrals proceeds analogously. Now, we integrate over a region $R \subset \mathbb{R}^D$, and we start by partitioning this region into small *volume elements* ΔV_i.

Definition 5.1: Partition of Space

A set $\{\Delta V_i\,,\ i \in \mathsf{I}\}$ is a *partition* of a region $R \subset \mathbb{R}^D$ iff

a) $\forall i \in \mathsf{I} \, : \, \Delta V_i \subset R$,

b) $\forall i, j \in \mathsf{I} \, : \, i \neq j \Rightarrow \Delta V_i \bigcap \Delta V_j = \emptyset$,

c) $\forall x \in R \; \exists i \in \mathsf{I} \, : \, x \in \Delta V_i$.

Definition 5.1 entails that the union of the elements of the partition amounts to the region R,

$$R = \bigcup_{i \in \mathsf{I}} \Delta V_i$$

Let now $V = \|R\|$ denote the volume of the region R. For every partition it can be written as

$$V = \|R\| = \sum_{i \in \mathsf{I}} \|\Delta V_i\|$$

In the limit of small volume elements we write this sum as a

Definition 5.2: Volume Integral

The *volume integral* F of a function $f(\boldsymbol{q})$ over a region $R \subset \mathbb{R}^D$ is defined as follows as limit of a sum over the elements of a partition,[2] $\{\Delta V_i\,,\ i \in \mathsf{I}\}$ of R and points $\boldsymbol{q}_i \in \Delta V_i$,

$$F = \int_R \mathrm{d}^D q\; f(\boldsymbol{q}) = \lim_{\|\Delta V_i\| \to 0} \sum_{i \in \mathsf{I}} \|\Delta V_i\|\, f(\boldsymbol{q}_i)$$

For Cartesian coordinates $(q_1, q_2, \dots, q_D)$ the integration volume element is $\mathrm{d}^D q = \mathrm{d}q_1 \cdots \mathrm{d}q_D$ and the integral amounts to

$$F = \int_{I_1} \mathrm{d}q_1 \int_{I_2(q_1)} \mathrm{d}q_2 \cdots \int_{I(q_1, \cdots, q_{D-1}} \mathrm{d}q_D\; f(q_1, \cdots, q_D)$$

where the boundaries of the integrals must be chosen such that $(q_1, \cdots, q_D) \in R$.

Remark 5.2. For the function $f(\boldsymbol{q}) = 1$ the volume integral provides the D-dimensional volume of the region R. ⎅

The mass $m(V)$ contained in a volume $\mathcal{V}$ can be expressed as a volume integral

$$\begin{aligned} m(V) &= \int_{\mathcal{V}} \mathrm{d}^3 q\; \rho(\boldsymbol{q}) = \int_{\mathcal{V}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\; \rho(x,y,z) \\ &= \int_{x_{\min}}^{x_{\max}} \mathrm{d}x \left[\int_{y_{\min}(x)}^{y_{\max}(x)} \mathrm{d}y \left(\int_{z_{\min}(x,y)}^{z_{\max}(x,y)} \mathrm{d}z\; \rho(x,y,z) \right) \right] \end{aligned}$$

where the integration runs over all $\boldsymbol{q} \in \mathcal{V}$, a volume with smallest x-value $x_{\min}$ and largest x-value $x_{\max}$, its y-values between $y_{\min}(x)$ and $y_{\max}(x)$ for a given x, and z-values between $z_{\min}(x,y)$ and $y_{\max}(x,y)$ for given x and y.

Remark 5.3. We adopt the convention that the mass density is zero outside an object. As a consequence its total M mass is obtained as

$$M = \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})$$

The boundaries of the integral that define the shape of the body have been absorbed into the definition of the density. ⎅

We illustrate the steps taken to evaluate a volume integral by calculating the area and volume of some simple geometric shapes:

Example 5.1: Surface areas of rectangles and circles

a) The surface area of the rectangle $R \subset \mathbb{R}^2$ with $(x,y) \in R$ iff $0 \le x \le a$ and $-b < y < b$ is

$$\|R\| = \int_R \mathrm{d}^2 q = \int_0^a \mathrm{d}x \int_{-b}^{b} \mathrm{d}y = 2ab$$

b) The surface area of the circle C with center at the origin

[2] Considerable care is taken in calculus courses to explore under which conditions the limit exists and is well-defined. Here, we assume that the function f varies smoothly inside the region. In other words, we assume that for all partition elements the difference $|f(\boldsymbol{q}) - f(\boldsymbol{q}_i)| \lll |f(\boldsymbol{q}_i)|$ for all points $\boldsymbol{q} \in \Delta V_i$.

and radius R is

$$\|C\| = \int_C \mathrm{d}^2 q = \int_{-R}^{R} \mathrm{d}x \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}y = 2 \int_{-R}^{R} \mathrm{d}x \sqrt{R^2 - x^2}$$
$$= 2\,R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos\theta \, \sqrt{1 - \sin^2\theta} = 2\,R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos^2\theta$$
$$= R^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \left(\cos^2\theta + \sin^2\theta\right) = \pi\,R^2$$

The choice of the integration boundaries is illustrated in Figure 5.8. Upon moving to the second line of this equation we substituted $x = R\,\sin\theta$, and in the step to the third line we made use of the π-periodicity of $\cos^2\theta$.

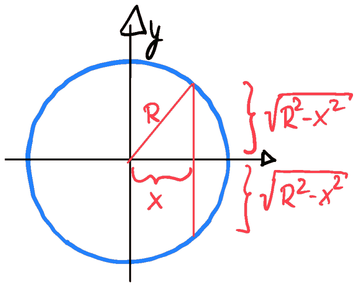


Figure 5.8: Notations adopted in the surface integral performed in Example 5.1b).

Example 5.2: Volume of a sphere

The volume of a three-dimensional sphere S with center at the origin and radius R is

$$\|S\| = \int_S \mathrm{d}^2 q = \int_{-R}^{R} \mathrm{d}x \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}y \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \mathrm{d}z$$
$$= \int_{-R}^{R} \mathrm{d}x \, \pi \left(\sqrt{R^2 - x^2}\right)^2 = \pi \int_{-R}^{R} \mathrm{d}x \, \left(R^2 - x^2\right)$$
$$= \pi \left(2\,R^3 - \frac{2}{3}\,R^3\right) = \frac{4\,\pi}{3} R^3$$

Upon moving to the second line we observed that the y and z integrals agreed with the ones performed to evaluate the are of a circle, cf. Example 5.1b).

5.3.2 *Change of variables*

The shape of a circle with center at the origin and radius R can much easier be described by polar coordinates rather than Cartesian coordinates:[3] $\{(\rho, \theta) \in \mathbb{R}^+ \times [0, 2\pi) \; : \; \rho < R\}$. To take advantage of this simplification we have to introduce a transformation of the integration coordinates from Cartesian to polar coordinates. A heuristic guess based on Figure 5.9 suggests that a volume element $\mathrm{d}x\,\mathrm{d}y$ at the position $(x, y) = (\rho\,\cos\theta, \rho\,\sin\theta)$ should be replaced by $\rho\,\mathrm{d}\theta\,\mathrm{d}\rho$. One readily verifies that this is a reasonable choice by working out the area of the circle with radius R:

[3] In order to avoid confusion with the radius of the circle the radial coordinate of the polar coordinates is here denoted as ρ.

$$\|C\| = \int_C \mathrm{d}^2 q = \int_0^R \mathrm{d}R \int_0^{2\pi} \mathrm{d}\theta \, R = \pi \int_0^R \mathrm{d}R \, 2\,R = \pi\,R^2$$

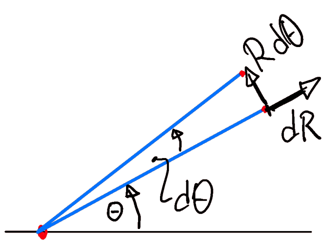


Figure 5.9: Integration volume for polar coordinates.

with a much easier calculation than in Example 5.1b).

Formally the change of the integration volume is determined by generalizing the substitution rule for integrals, as illustrated in Figure 3.13 for one dimensional integrals. In this rule the derivative $f'(x)$ account for the change of the width of the rectangles that are summed to approximate the integral. In order to generalize this idea we recall from the discussion of line integrals that

$$\frac{d\boldsymbol{q}}{dR}\, dR = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} dR \quad \text{and} \quad \frac{d\boldsymbol{q}}{d\theta}\, d\theta = \begin{pmatrix} -R\, \sin\theta \\ R\, \cos\theta \end{pmatrix} dR$$

In general the derivatives involved in the definition of the length elements do not have unit length and they need not be orthogonal. Their length reflects the change of the length unit that we also encounter in the one-dimensional case. The angle between the vectors indicates that in two-dimensions one can also partition space by using parallelograms rather than rectangle. The unit area for the integration will always be the area spanned by the two vectors.

In D dimensions the integration volume is defined by the volume spanned by the D derivative vectors of the position vector $\boldsymbol{q}$ with respect to the new coordinates. It is commonly expressed in terms of the Jacobi determinant. We first introduce the notion of a

Definition 5.3: Determinant

The *determinant* of a matrix amounts to the volume spanned by its column vectors. For a matrix A it is denoted as $\det \mathsf{A}$.

Remark 5.4. The determinant of 2×2 and 3×3 matrices takes the form of the (sum of) products along the diagonals from left to right minus the (sum of) products of the diagonals from right to left,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} \\ - a_{11}\, a_{23}\, a_{32} + a_{22}\, a_{31}\, a_{13} + a_{33}\, a_{12}\, a_{21}$$

These expressions are entailed by the geometric interpretation of the cross product in Section 2.9.2.

Without proof we provide the following general rule for calculating determinants

provide a reference

Theorem 5.1: Recursive calculation of determinants

Let A be a $D \times D$ matrix with $D \in \mathbb{N}$ and entries a_{ij} where $i, j \in \{1, \cdots, D\}$. For $D = 1$ we define $\det \mathsf{A} = a_{11}$. For $D > 1$ we introduce the $(D-1) \times (D-1)$ submatrices A_{ij} that are obtained from A by dropping its ith row and jth column. The determinant of A can then be calculated by a recursion that either works along a row j or a column k of A,

$$\det \mathsf{A} = \sum_{j=1}^{D} (-1)^{j+k} \det \mathsf{A}_{jk} = \sum_{k=1}^{D} (-1)^{j+k} \det \mathsf{A}_{jk}$$

Altogether this allows us to identify the factor involved in a change of the integration variables as the Jacobi determinant.

Theorem 5.2: Jacobi matrix and determinant

We consider a change of integration variables from the coordinates $\boldsymbol{x} = (x_1, x_2, \cdots, x_D)$ to $(y_1, y_2, \cdots, y_D)$ that is defined by the functions $x_1(\boldsymbol{y})$, $x_2(\boldsymbol{y})$, $\cdots$, $x_D(\boldsymbol{y})$. Then the integration volume changes as

$$\mathrm{d}x_1 \cdots \mathrm{d}x_D = \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_D} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_D} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_D}{\partial y_1} & \frac{\partial x_D}{\partial y_2} & \cdots & \frac{\partial x_D}{\partial y_D} \end{pmatrix} \mathrm{d}y_1 \cdots \mathrm{d}y_D$$

The matrix involved in this transition is called the *Jacobi matrix* of the transition, and the determinant is called the *Jacobi determinant*.

Example 5.3: Integration volumes

a) *polar coordinates* $(x, y) = \rho\,(\cos\theta,\, \sin\theta)$

transform as

$$\mathrm{d}x\,\mathrm{d}y = \det \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \mathrm{d}\rho\,\mathrm{d}\theta = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

b) *cylindrical coordinates* $(x, y, z) = (\rho\cos\theta,\, \rho\sin\theta, z)$

transform as

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \det \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z$$

c) *spherical coordinates* $(x, y, z) = \rho\,(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$

transform as

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\,\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi = \rho^2\,\mathrm{d}\rho\,\mathrm{d}\cos\theta\,\mathrm{d}\phi$$

5.3.3 *The force of an extended object (Earth) on a point particle (professor)*

As a first step towards discussing extended objects we consider the force exerted by an extended object on a point particle. The force is obtained by integrating the forces originating from the mass elements of the body,

$$\boldsymbol{F}_{\text{tot}} = \int_{\mathbb{R}^3} \mathrm{d}^3q\, \boldsymbol{F}(\boldsymbol{q})$$

where $\boldsymbol{q}$ is the vector from the position of the point particle to the mass element that is exerting the force. This expression involves the vector-valued generalization of volume integrals. It should be interpreted component-wise, stating the that the components $\boldsymbol{F}_{\text{tot},i} = \hat{\boldsymbol{e}}_i \cdot \boldsymbol{F}_{\text{tot}}$ of the total force in some orthonormal base $\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z$ amount to

$$\hat{\boldsymbol{e}}_i \cdot \boldsymbol{F}_{\text{tot}} = \int_{\mathbb{R}^3} \mathrm{d}^3 q \, (\hat{\boldsymbol{e}}_i \cdot \boldsymbol{F}(\boldsymbol{q}))$$

or explicitly
$$\begin{aligned} \boldsymbol{F}_{\text{tot},x} &= \int_{\mathbb{R}^3} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, F_x(x,y,z) \\ \boldsymbol{F}_{\text{tot},y} &= \int_{\mathbb{R}^3} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, F_y(x,y,z) \\ \boldsymbol{F}_{\text{tot},z} &= \int_{\mathbb{R}^3} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, F_z(x,y,z) \end{aligned}$$

The consequences can nicely be explored when an evil witch switches off electromagnetic interactions between a physics professor and its environment. In the absence of interaction with other matter the professor will freely fall towards the center of Earth, accelerated by a force that arises as sum of the mass elements constituting Earth (see Figure 5.10). For the professor of mass m at position $\boldsymbol{q}_P$ and the mass element at position $\boldsymbol{q}_e$ this force amounts to $\boldsymbol{F}(\boldsymbol{q}_P, \boldsymbol{q}_e) = -\nabla(m\,\rho(\boldsymbol{q}_e)\,G) / \,|\boldsymbol{q}_P - \boldsymbol{q}_e|$. For simplicity we assume that Earth is spherical and that its mass density takes a uniform value $\rho = 3\,M_E/4\pi\,R^3$. Then, the force on the professor takes the form

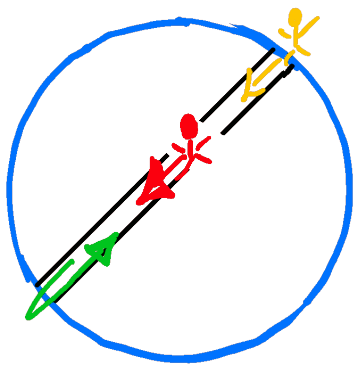


Figure 5.10: Initially positioned at the upper right (yellow), the professor will fall down (red), and eventually pop out at the other side and return (green).

$$\begin{aligned} \boldsymbol{F}_{\text{tot}} &= -\int_{\mathbb{R}^3} \mathrm{d}^3 q \, \nabla \frac{m\,\rho(\boldsymbol{q}_e)\,G}{|\boldsymbol{q}_P - \boldsymbol{q}_e|} && (5.3.1) \\ &= -m\,\rho\,G\,\nabla \int_{\text{Earth}} \mathrm{d}^3 q \, \frac{1}{\sqrt{q_P^2 + q_e^2 - 2\,q_P\,q_e\,\cos\theta}} && (5.3.2) \end{aligned}$$

where θ is the angle between the two vectors $|\boldsymbol{q}_P|$ and $|\boldsymbol{q}_e|$, while q_P and q_e denote their respective length.

The integral is best evaluated by adopting a spherical coordinates (r, θ, ϕ) for the integration where r runs from zero to the Earth radius R, the angle θ from zero to π, and ϕ all around from zero to 2π,

$$\begin{aligned} \boldsymbol{F}_{\text{tot}} &= -m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, r^2 \int_{-1} 1 \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi \, \frac{1}{\sqrt{q_P^2 + r^2 - 2\,q_P\,r\,\cos\theta}} \\ &= -2\pi\,m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, r^2 \left[\frac{-1}{q_P\,r} \sqrt{q_P^2 + r^2 - 2\,q_P\,r\,\cos\theta} \right]_{\cos\theta=-1}^{\cos\theta=1} \\ &= -2\pi\,m\,\rho\,G\,\nabla \int_0^R \mathrm{d}r\, \frac{r}{q_P} \left(|q_P + r| - |q_P - r| \right) \\ &= -4\pi\,m\,\rho\,G\,\nabla \left[\frac{1}{q_P} \int_0^{q_P} \mathrm{d}r\, r^2 + \int_{q_P}^R \mathrm{d}r\, r \right] \\ &= m\,\rho\,G\,\nabla \left[2\pi R^2 - \frac{2\pi}{3} \boldsymbol{q}_P \cdot \boldsymbol{q}_P \right] = -\frac{m\,g}{R}\,\boldsymbol{q}_P \end{aligned}$$

In the last step we used that the acceleration on the Earth surface is $g = MG/R = 4\pi\,\rho\,R^2\,G/3$. The professor moves under the influence of a *harmonic* central force, as studied in Problems 4.19, 4.21 and 4.30! After a while (cf. Problem 5.12) he reappears at the very same spot where he started, except that Earth moved on while he was under way.

5.3.4 Self Test

Problem 5.5. **Area of a parallelogram**

Determine the area of the parallelogram defined by the points $(0,0), (1,3), (4,4), (2,1)$ by

a) performing the volume integral,

b) determining the area spanned by the two vectors that define the sides starting at the corner $(0,0)$.

Problem 5.6. **Volume of a solid of revolution**

A solid of revolution is obtained by rotating some function $f(x)$ around the x axis. For instance, the function $\sqrt{R^2 - x^2}$ with $-R \leq x \leq R$ describes a sphere of radius R. The volume V of a solid of revolution are given by the integral

$$V = \pi \int \mathrm{d}x \; (f(x))^2 \tag{5.3.3}$$

a) Sketch the function $f(x) = \sqrt{R^2 - x^2}$ and verify that the solid of revolution is indeed a sphere.

b) Determine the volume of the sphere based on the given equation. Compare you calculation and the result to the calculation given in Example 5.2.

c) Show that the volume integral for a solid of revolution provides Equation (5.3.3) when one adopts cylindrical coordinates.

Problem 5.7. **Volume of a cone**

Determine the volume of a cone with symmetry axis along the z-axis, that stands on the (x, y)-plane where it traces a circle of radius R, while its vertex is at $(0, 0, H)$.

a) Perform the volume integral with Cartesian coordinates.

b) Perform the volume integral with cylindrical coordinates.

Problem 5.8. **Coordinate transformation to cylindrical coordinates**

Determine the Jacobi matrix and its determinant for the transformation from Cartesian to spherical coordinates, cf. Example 5.3c).

5.4 *Center of mass and spin of extended objects*

We consider a setting where there are only long distance force like gravity and no collisions between objects. The explicit calculation for the case of gravity in the previous section entails that in such a setting the force exerted by a planet on a point particle is identical to the one exerted by a mass point of identical mass that is located at the center of the planet (see also Problem 5.12e). In the present section we therefore explore which effects the force of a point particle exerts on an extended body.

5.4.1 *Evolution of the center of mass*

The force on the body is described by an integral that takes exactly the same form as Equation (5.3.1), where now $\boldsymbol{q}$ is a vector from the point particle to a volume element of the body.

The integral is best evaluated by introducing a coordinate frame $\hat{\boldsymbol{e}}_1(t), \ldots, \hat{\boldsymbol{e}}_3(t)$ with orientation fixed in the rotating body and origin in the body's center of mass $\boldsymbol{Q} = (Q_x, Q_y, Q_z)$. In immediate generalization of Equation (4.6.1) it is located at

$$\boldsymbol{Q} = \frac{1}{M} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, \boldsymbol{q} \quad \Leftrightarrow \quad \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \begin{pmatrix} M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_x \\ M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_y \\ M^{-1} \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, q_z \end{pmatrix}$$

A given mass element will always have the same coordinates (r_1, r_2, r_3) with respect to the body-fixed basis, and in a stationary coordinate frame this position can be specified as

$$\boldsymbol{q}(t) = \boldsymbol{Q}(t) + \sum_{i=1}^{3} r_i\, \hat{\boldsymbol{e}}_i(t)$$

Remark 5.5. The vector $\boldsymbol{r}$ describes the position $(r_1, \ldots, r_3)$ in the body with respect to its center of mass. When the body rotates $\boldsymbol{r}$ will evolve in time. However, the coordinates $(r_1, \ldots, r_3)$ are constant numbers describing the shape of the body when they are calculated in a coordinate system with base vectors $\{\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_3\}$ fixed in the body and origin in its center of mass. Hence,

$$\boldsymbol{r} = \sum_{i=1}^{3} r_i\, \hat{\boldsymbol{e}}_i(t) \quad \text{and} \quad \dot{\boldsymbol{r}} = \sum_{i=1}^{3} r_i\, \dot{\hat{\boldsymbol{e}}}_i(t) \tag{5.4.2}$$

(5.4.1)

We note that this choice of coordinates entails

$$\begin{aligned} M\, \boldsymbol{Q} &= \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, \boldsymbol{q} = \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, \Big(\boldsymbol{Q}(t) + \sum_{i=1}^{3} r_i\, \hat{\boldsymbol{e}}_i(t)\Big) \\ &= \boldsymbol{Q}(t) \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q}) + \sum_{i=1}^{3} \hat{\boldsymbol{e}}_i(t) \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, r_i \\ \Rightarrow \quad 0 &= \int_{\mathbb{R}^3} \mathrm{d}^3 q\, \rho(\boldsymbol{q})\, r_i = \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(\boldsymbol{r})\, r_i \end{aligned} \tag{5.4.3}$$

The latter equality holds because a shift of the origin by $\boldsymbol{Q}$ and rotation of the coordinate axes do not affect the integration volume (i. e. the Jacobi determinant of the transformation is one).

The acceleration $\ddot{\boldsymbol{q}}(t)$ takes the form

$$\ddot{\boldsymbol{q}}(t) = \ddot{\boldsymbol{Q}}(t) + \sum_{i=1}^{3} r_i\, \ddot{\hat{\boldsymbol{e}}}_i(t)$$

and the force on the spatially extended body results in

$$\begin{aligned}
\boldsymbol{F}_{\text{tot}} &= \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; \ddot{\boldsymbol{q}}(t) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; \left(\ddot{\boldsymbol{Q}}(t) + \sum_{i=1}^{3} r_i\, \ddot{\hat{\boldsymbol{e}}}_i(t) \right) \\
&= M\, \ddot{\boldsymbol{Q}} + \sum_{i=1}^{3} \ddot{\hat{\boldsymbol{e}}}_i(t) \int_{\mathbb{R}^3} \mathrm{d}^3 r\; \rho(\boldsymbol{r})\; r_i = M\, \ddot{\boldsymbol{Q}} \qquad (5.4.4)
\end{aligned}$$

The overall force $\boldsymbol{F}_{\text{tot}}$ results in an acceleration of the center of mass that behaves exactly as for a point particle described in the previous chapter. Thus, we have justified the assumption of point particles adopted in Chapter 4.

5.4.2 *Angular momentum and particle spin*

Let us now explore the angular momentum of a spatially extended particles. To this end we introduce the decomposition $\boldsymbol{q} = \boldsymbol{Q} + \boldsymbol{r}$ into the definition

$$\begin{aligned}
\boldsymbol{L}_{\text{tot}} &= \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; (\boldsymbol{q} \times \dot{\boldsymbol{q}}) = \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; ((\boldsymbol{Q} + \boldsymbol{R}) \times (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{r}})) \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; (\boldsymbol{Q} \times \dot{\boldsymbol{Q}}) + \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; (\boldsymbol{Q} \times \dot{\boldsymbol{r}}) \\
&+ \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; (\boldsymbol{r} \times \dot{\boldsymbol{Q}}) + \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \\
&= M\, \boldsymbol{Q} \times \dot{\boldsymbol{Q}} + \boldsymbol{Q} \times \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q}) \boldsymbol{r}(t) - \dot{\boldsymbol{Q}} \times \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q}) \boldsymbol{r}(t) \\
&+ \int_{\mathbb{R}^3} \mathrm{d}^3 q\; \rho(\boldsymbol{q})\; (\boldsymbol{r} \times \dot{\boldsymbol{r}})
\end{aligned}$$

The first summand amounts to the angular momentum of the center of mass, $\boldsymbol{L}_{CM} = M\, \boldsymbol{Q} \times \dot{\boldsymbol{Q}}$. The second and the third term vanish due to Equation (5.4.3). The forth term can be simplified by performing the integration in the comoving coordinate frame. The coordinate transformation involves a translation by $\boldsymbol{Q}$ and rotation. Hence, the Jacobi determinant is one, and the term only depends on the local coordinates $\boldsymbol{r}$. It is denoted as particle spin.

Definition 5.4: Particle Spin

The *total angular momentum* $\boldsymbol{L}_{\text{tot}}$ of a particle can be decomposed into the angular momentum $\boldsymbol{L}_{CM}$ of its center-of-mass motion, and its *spin* $\boldsymbol{S}$ around the center of mass, $\boldsymbol{Q}$,

$$\boldsymbol{L}_{\text{tot}} = \boldsymbol{L}_{CM} + \boldsymbol{S} \qquad (5.4.5a)$$

$$\text{with} \quad \boldsymbol{L}_{CM} = M\, \boldsymbol{Q} \times \dot{\boldsymbol{Q}} \qquad (5.4.5b)$$

$$\boldsymbol{S} = \int_{\mathbb{R}^3} \mathrm{d}^3 r\; \rho(r_1.r_2.r_3)\; \boldsymbol{r} \times \dot{\boldsymbol{r}} \qquad (5.4.5c)$$

Remark 5.6. The decomposition of the total angular momentum has important consequences in collisions. For spatially extended objects the conservation of angular momentum only implies that the sum of the spin and the angular momentum of the center-of-mass motion are conserved. As a consequence, the incoming and outgoing angle can differ for a reflection at a wall, and the center of mass of the particle can even move in different planes before and after the collision. This will be demonstrated in the worked example in Section 5.6.

The discussion of particle spin can further be simplified by expressing the rotation of the body by the vector $\boldsymbol{\Omega}$ that indicates the rotation axis and angular speed $|\boldsymbol{\Omega}|$, and exploring that the relative positions of the mass elements in the body do not change upon rotation. Due to Equation (5.4.1) we have

$$\begin{aligned} \boldsymbol{S} &= \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(\boldsymbol{r})\, \boldsymbol{r} \times \dot{\boldsymbol{r}} \\ &= \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_i \times \dot{\hat{\boldsymbol{r}}}_j \int \mathrm{d}^3 r\, r_i\, r_j\, \rho(\boldsymbol{r}) = \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_i \times \dot{\hat{\boldsymbol{r}}}_j\, t_{ij} \\ \text{with} \quad t_{ij} &= \int_{\mathbb{R}^3} \mathrm{d}r_1\, \mathrm{d}r_2\, \mathrm{d}r_3\, r_i\, r_j\, \rho(r_1, r_2, r_3) \end{aligned}$$

Note that the coefficients t_{ij} are properties of the body. They characterize the mass distribution of the body, and do not depend on the motion. The situation simplifies further when one observes that the velocities $\dot{\hat{\boldsymbol{r}}}_k$ are unit vectors that must be orthogonal to $\hat{\boldsymbol{r}}_k$ and to Ω.[4] Hence, the velocities can be expressed as

$$\dot{\hat{\boldsymbol{r}}}_k = \hat{\boldsymbol{r}}_k \times \boldsymbol{\Omega}$$

[4] Recall that $\hat{\boldsymbol{r}}_k \cdot \hat{\boldsymbol{r}}_k = 1$ such that $2\, \hat{\boldsymbol{r}}_k \cdot \dot{\hat{\boldsymbol{r}}}_k = 0$, and by construction the motion of mass elements is orthogonal to the axis of rotation.

With this notations the kth component of $\boldsymbol{S}$ can be expressed as

check signs!

$$\begin{aligned} S_k = \hat{\boldsymbol{r}}_k \cdot \boldsymbol{S} &= \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_k \cdot \left(\hat{\boldsymbol{r}}_i \times (\hat{\boldsymbol{r}}_j \times \boldsymbol{\Omega})\right) t_{ij} \\ &= \sum_{ij=1}^{3} \hat{\boldsymbol{r}}_k \cdot \left(\hat{\boldsymbol{r}}_j\, (\boldsymbol{\Omega} \cdot \hat{\boldsymbol{r}}_i) - \boldsymbol{\Omega}\, (\hat{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{r}}_j)\right) t_{ij} \\ &= \sum_{ij=1}^{3} \left(\delta_{jk}\, \Omega_i - \Omega_k\, \delta_{ij}\right) t_{ij} \\ &= \sum_{i=1}^{3} \Omega_i \sum_{j=1}^{3} \delta_{jk}\, t_{ij} - \Omega_k \sum_{ij=1}^{3} \delta_{ij} t_{ij} \\ &= \sum_{i=1}^{3} \Omega_i \left(t_{ik} - \delta_{ik} \sum_j t_{jj} \right) \end{aligned}$$

This amounts to a multiplication of the vector $\boldsymbol{\Omega}$ written in terms of its components Ω_i. We summarize this observation in the following definition

Definition 5.5: Tensor of Inertia

The rotation of a solid body with a fixed mass distribution $\rho(\boldsymbol{r})$ can be described by a vector Ω that defines the rotation axis and speed. It is related to the spin $\boldsymbol{S}$ of the body by multiplication with the *tensor or inertia*

$$\boldsymbol{S} = \Theta\, \boldsymbol{\Omega}\,,$$

i. e. a symmetric matrix with components

$$\Theta_{ij} = \int_{\mathbb{R}^3} \mathrm{d}r_1\, \mathrm{d}r_2\, \mathrm{d}r_3 \left(r_i\, r_j - \sum_{k=1}^{3} r_k\, r_k \right) \rho(r_1, r_2, r_3)$$

Example 5.4: Intertial tensor for a solid ball

For a ball of radius R with uniform mass density ρ the tensor of inertia has the following entries for its diagonal elements

$$\Theta_{ii} = \int_{\mathbb{R}^3} \mathrm{d}r_1\, \mathrm{d}r_2\, \mathrm{d}r_3 \left(r_i\, r_i - \sum_{k=1}^{3} r_k\, r_k \right) \rho(r_1, r_2, r_3)$$

We evaluate the integral in spherical coordinates with $r = |\boldsymbol{r}|$ and θ denoting the angle with respect to the i-axis, which we denote as z-axis in the following. Hence,

$$(r_x, r_y, r_z) = r\,(\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta)\,,$$

and

$$\begin{aligned}
\Theta_{ii} &= \rho \int_0^R \mathrm{d}r\, r^2 \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi\, (r_x^2 + r_y^2) \\
&= 2\,\pi\,\rho \int_0^R \mathrm{d}\rho\, r^2 \int_{-1}^{1} \mathrm{d}\cos\theta\, r^2\, \sin^2\theta \\
&= 2\,\pi\,\rho \left(\int_0^R \mathrm{d}\rho\, r^4 \right) \left(\int_{-1}^{1} \mathrm{d}\cos\theta\, \left(1 - \cos^2\theta\right) \right) \\
&= 2\,\pi\,\rho\, \frac{R^5}{5} \left(2 - \frac{2}{3} \right) = \frac{2}{5}\, M\, R^2
\end{aligned}$$

Moreover, for the off-diagonal element θ_{ik} we align the k-axis with $\phi = 0$ and find

$$\begin{aligned}
\Theta_{ik} &= \rho \int_0^R \mathrm{d}r\, r^2 \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi\, r_x\, r_z \\
&= \rho \int_0^R \mathrm{d}r\, r^2 \int_0^{\pi} \mathrm{d}\theta\, \sin\theta \int_0^{2\pi} \mathrm{d}\phi\, r^2\, \sin\theta\, \cos\theta \\
&= 2\pi\,\rho\, \frac{R^5}{5} \int_0^{\pi} \mathrm{d}\theta\, \sin^2\theta\, \cos\theta = 0
\end{aligned}$$

since $\sin^2\theta\, \cos\theta$ is antisymmetric with respect to $\pi/2$.

The finding that the off-diagonal elements of the tensor of inertia vanish is no coincidence. In **??** we will show that this happens whenever the mass distribution features a symmetry in the *ik* plane. Moreover, the ...theorem of linear algebra states that one can always choose coordinates where all off-diagonal elements of the tensor of inertia vanish.[5] The particular axes where this happens are called the axis of inertia of a body.

fill in name and reference

[5] For a general matrix this is not true. It is a consequence of the fact that Θ is symmetric, i. e. $\Theta_{ij} = \Theta_{ji}$ for all its entries.

check wording

Definition 5.6: Axis of inertia

For each solid body there is a choice of internal coordinate axes $\hat{\boldsymbol{r}}_i$, $i = 1, \cdots, 3$ where the tensor of inertia takes a diagonal form. The directions selected by the axis are called *axes of inertia*, and the related diagonal entry of the matrix of inertial is denoted as *moment of inertia*.

Remark 5.7. If the mass distribution of the body obeys reflection or rotation symmetry, the axes of inertia are invariant under the symmetry transformation.

5.4.3 *Time evolution of angular momentum and particle spin*

The angular momentum $\boldsymbol{L}_{CM}$ of its center-of-mass motion behaves in exactly the same way as for point particles.

The spin changes in time according to the differential equation

$$\begin{aligned}\dot{\boldsymbol{S}} &= \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \times \ddot{\boldsymbol{r}} \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \times \ddot{\boldsymbol{q}} = \int_{\mathbb{R}^3} \mathrm{d}^3 r\, \boldsymbol{r} \times \boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{r})\end{aligned}$$

In order to arrive at the second line, we noted that $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{q}} - \ddot{\boldsymbol{Q}}$, and that the integral for the $\ddot{\boldsymbol{Q}}$ contribution vanishes because $\int_{\mathbb{R}^3} \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} = 0$. Moreover, it is understood that the force $\boldsymbol{F}$ is zero for coordinates $\boldsymbol{r}$ outside the body.

Definition 5.7: Particle Torque

When the part $\boldsymbol{r}$ of a body is subjected to force $\boldsymbol{F}$ then its spin $\boldsymbol{S}$ is changing due to a torque $\boldsymbol{M}$

$$\dot{\boldsymbol{S}} = \boldsymbol{M} = \int_{\text{body}} \mathrm{d}^3 r\, \boldsymbol{r} \times \boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{r}) \tag{5.4.6}$$

Remark 5.8. Note that the torque is denoted by the letter capital M that is also frequently used for the mass. Nevertheless, there is no immediate risk that they are mixed up: The torque, $\boldsymbol{M}$, is a vector, while the mass, M, is a scalar. To further reduce the risk we will denote masses by a small letter m, when mass and torque appear in a problem.

In general the force $\boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{r})$ can only be evaluated after the CM motion has been determined. From the point of view of the rotating body it is a time-dependent force. This renders the motion of a particle in an inhomogeneous force field to be a very complex problem. However, the gravitational force on small spatial distances, where the gravitational acceleration $\boldsymbol{g}$ takes a constant value, forms a noticeable exception.

Theorem 5.3: Spinning motion and gravity

When an extended body moves subject to a spatially uniform acceleration $\boldsymbol{g}$, then its center of mass follows a free-flight parabola and its spin is preserved.

Proof. The statement about the center-of-mass motion follows from Equation (5.4.4).

Conservation of the spin is due to

$$\begin{aligned} \int \mathrm{d}^3 r\, \boldsymbol{r} \times (\rho(r_1, r_2, r_3)\, \boldsymbol{g}) &= \left(\int \mathrm{d}^3 r\, \rho(r_1, r_2, r_3)\, \boldsymbol{r} \right) \times \boldsymbol{g} \\ &= \boldsymbol{0} \times \boldsymbol{g} = \boldsymbol{0} \end{aligned}$$ □

Rather than in the free flight of a body, one also often encounters a spinning body that is fixed at some point. Besides gravity there is one additional force acting on the body that is constraining its motion. When this force acts *only* on the center of mass, then it has no effect on the spin and only changes the evolution of the center of mass. When it acts on another point on the body, then the total angular momentum is no longer conserved. This happens for instance for a spinning top where one fixes a point on its axis.

add:
discussion of motion with additional reference point
Euler angles

5.4.4 *Self Test*

add problems:
moments of inertia
ruler pendulum
suspension bridge
torque on triangle/tetraeder

5.5 *Bodies with internal degrees of freedom: Revisiting mobiles*

In Section 2.10 we worked out the positions of masses for a mobile where all masses are the same and where all sticks are straight. It is worth while to revisit this problem from a more advanced mathematical perspective.

5.5.1 *Mobile at rest*

The mobile is at rest when its center of mass does not move, $\dot{\boldsymbol{Q}} = \boldsymbol{0}$, and when it has not spin. It remains at rest, when it does not experience a total force that will induce a motion of the center of mass, and no torque that induces spin.

According to Equation (5.4.4) the center of mass can remain at rest when the total force $\boldsymbol{F}_{\text{tot}}$ vanishes. This implies that the force $\boldsymbol{F}_s$ at the suspension point of the mobile must balance the total weight of the mobile $M\,\boldsymbol{g}$,

$$\mathbf{0} = \boldsymbol{F}_{\text{tot}} = \boldsymbol{F}_s + M\,\boldsymbol{g} \quad \Rightarrow \quad \boldsymbol{F}_s = -M\,\boldsymbol{g}$$

According to Equation (5.4.6) the mobile will not topple (i. e. pick up or change its spin) when $\boldsymbol{M} = \mathbf{0}$, and in Section 5.4.3 we pointed out that the gravitational force does can not change the spin. Hence, we are left with the force $\boldsymbol{F}_s$ at the suspension point $\boldsymbol{q}_s = \boldsymbol{Q} + \boldsymbol{r}_s$,

$$\boldsymbol{M} = \boldsymbol{r}_s \times \boldsymbol{F}_s$$

It vanishes iff the force $\boldsymbol{F}_s$ acts parallel to the direction $\boldsymbol{r}_s$ from the position of the center of mass to the suspension point. Since $\boldsymbol{F}_s$ acts antiparallel to gravity this entails that the center of mass of the mobile must either be located directly below or above the suspension point, irrespective of the shape of the arms or distribution of the masses.

The mobile is not a stiff body. Rather its arms can move with respect to each other. We assume again that the mass of the arms may be neglected. The mass of the mobile is concentrated in its N weights that reside at the positions $\boldsymbol{q}_\nu$, $\nu = 1, \cdots, N$. The position of the suspension will be denoted as $\boldsymbol{q}_0$. Let us attach the mobile to a spring so that we can explicitly measure the suspension force. Clearly the forces on the mobile are conservative, such that there is a potential $\Phi(\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_N)$. The force $\boldsymbol{F}^{(\nu)}$ acting on particle ν (or on the suspension) can then be calculated by taking the derivatives with respect to the coordinates $\boldsymbol{q}_\nu = (q_{\nu,x}, q_{\nu,y}, q_{\nu,z})$, of the particle

$$\boldsymbol{F}^{(\nu)} = -\nabla_{\boldsymbol{q}_\nu} \Phi = \begin{pmatrix} -\partial q_{\nu,x} \Phi \\ -\partial q_{\nu,y} \Phi \\ -\partial q_{\nu,z} \Phi \end{pmatrix}$$

When the coordinates are collected into a single vector $\boldsymbol{q} = (\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_N)$ then the mobile is in equilibrium when the $\boldsymbol{q}$-gradient of $\Phi(\boldsymbol{q})$ vanishes, $\mathbf{0} = \nabla_{\boldsymbol{q}} \Phi(\boldsymbol{q})$. However, when taking the partial derivatives one has to keep in mind that one must not fix the values of the other coordinates but rather keep in mind the constraints of motion of the mobile (recall Example 3.7). Alternatively, one can account for the elasiticity of the cords and bars in the mobile, and the resulting restoring forces to pulling and bending. When also all these forces are accounted for the stationary point can be found by a variation principle.

move theory of variations from Chap 6.2 to this point

An more elegant way to deal with this problem will be presented in Chapter 6. Here we already note that the condition on Φ can be interpreted as a multi-dimensional requirement for a stationary point. The force will be zero, even when $\Phi(\boldsymbol{q})$ takes a maximum. However, in that case small fluctuations will induce forces that drive the system away from the stationary point. The mobile will stay put when $\Phi(\boldsymbol{q})$ takes a minimum. Small perturbations will then only lead to some wiggling close to the minimum. The mobile can slowly move because there are perturbations to its shape where all masses stay exactly at the same height. In terms of the potential this amounts to neutral directions where the potential is flat. In order to formally underpin this intuition we introduce multidimensional Taylor expansions.

5.5.2 *Multidimensional Taylor expansions*

We consider a scalar function $\Phi : \mathbb{R}^D \to \mathbb{R}$ that assigns a real value to its arguments $\boldsymbol{x} \in R^D$. For instance this may be the potential energy assigned to a configuration of masses characterized by a state vector $\boldsymbol{x}$. We select a reference point $\boldsymbol{x}_0$ and explore how $\Phi(\boldsymbol{x})$ deviates from $\Phi(\boldsymbol{x})$ for a small changes of the configuration, $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\epsilon}$, i. e. for a small change $\epsilon \in \mathbb{R}^D$ of the configuration. The multidimensional Taylor expansions states that

$$\Phi(\boldsymbol{x}) = \Phi(\boldsymbol{x}_0) + (\epsilon_i\,\partial_i)\Phi(\boldsymbol{x}_0) + \frac{1}{2}\,(\epsilon_i\,\partial_i)\,(\epsilon_j\,\partial_j)\Phi(\boldsymbol{x}_0) \\ + \frac{1}{3!}\,(\epsilon_i\,\partial_i)\,(\epsilon_j\,\partial_j)\,(\epsilon_k\,\partial_k)\Phi(\boldsymbol{x}_0) + \dots$$

Here, ϵ_i denotes the i-component of the vector $\boldsymbol{\varepsilon}$ with respect to an orthonormal basis $\hat{\boldsymbol{e}}_i$, and ∂_i is the partial derivative with respect to the according coordinate x_i of $\boldsymbol{x}$. Moreover, we use the Einstein convention that requires summation over repeated indices, i. e. $\epsilon_i\,\partial_i$ is an abbreviation for $\epsilon_i\,\partial_i = \sum_i \epsilon_i\,\partial_i$ where i runs of the set of indices labeling the base vectors, and analogous statement hold for $(\epsilon_j\,\partial_j)$ and $(\epsilon_k\,\partial_k)$.

Remark 5.9. $\partial_j\Phi(\boldsymbol{x}_0)$ should be interpreted as

$$\partial_j\Phi(\boldsymbol{x}_0) = \frac{\partial}{\partial x_j}\Phi(x_1,\dots,x_j,\dots)\bigg|_{\boldsymbol{x}=\boldsymbol{x}_0} .$$

For scalar arguments $x \in \mathbb{R}$ the expression for the multidimensional Taylor expansion reduces to the one for real functions that we have discussed before.

Proof. For a one-dimensional function $f(x)$ the Taylor expansion around x_0 with $x = x_0 + \epsilon$ the expression $(\epsilon_j\,\partial_j)$ reduces to $\epsilon\,\frac{\mathrm{d}}{\mathrm{d}x}$.

Consequently,

$$
\begin{aligned}
f(x) &= f(x_0) + \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x_0) + \frac{1}{2} \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}x}\right) \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x_0) \\
&\quad + \frac{1}{3!} \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}x}\right) \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}x}\right) \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x_0) + \dots \\
&= f(x_0) + \epsilon f'(x_0) + \frac{1}{2} \epsilon^2 f''(x_0) + \frac{1}{3!} \epsilon^3 f'''(x_0) + \dots
\end{aligned}
$$

These are the first terms of the 1D Taylor expansion. □

The first terms of the Taylor expansion can also be written in the form

$$\Phi(\boldsymbol{x}) = \Phi(\boldsymbol{x}_0) + (\boldsymbol{\varepsilon} \cdot \nabla)\Phi(\boldsymbol{x}_0) + \frac{1}{2} \boldsymbol{\varepsilon}^T \, \mathsf{C}(\boldsymbol{x}_0) \, \boldsymbol{\varepsilon} + \dots$$

where the matrix $\mathsf{C}(\boldsymbol{x}_0)$ has the components $c_{ij}(\boldsymbol{x}_0) = \partial_i \partial_j \Phi(\boldsymbol{x}_0)$.

Proof. For the first-order term we have

$$(\boldsymbol{\varepsilon} \cdot \nabla)\Phi(\boldsymbol{x}_0) = \left(\sum_j \epsilon_j \, \partial_j\right) \Phi(\boldsymbol{x}_0) = \left(\epsilon_j \, \partial_j\right) \Phi(\boldsymbol{x}_0)$$

where the second equality amounts to the simplification of notation achieved by the Einstein convention.

For the second-order term we have

$$
\begin{aligned}
\boldsymbol{\varepsilon}^T \, \mathsf{C}(\boldsymbol{x}_0) \, \boldsymbol{\varepsilon} &= \left(\epsilon_1 \, , \epsilon_2 \, , \epsilon_3 \, , \dots , \right) \begin{pmatrix} \partial_1^2 \Phi(x_0) & \partial_1 \, \partial_2 \Phi(x_0) & \partial_1 \, \partial_3 \Phi(x_0) & \dots \\ \partial_2 \, \partial_1 \Phi(x_0) & \partial_2^2 \Phi(x_0) & \partial_2 \, \partial_3 \Phi(x_0) & \dots \\ \partial_3 \, \partial_1 \Phi(x_0) & \partial_3 \, \partial_2 \Phi(x_0) & \partial_3^2 \Phi(x_0) & \dots \\ \vdots & \ddots & & \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \end{pmatrix} \\
&= \sum_{jk} \epsilon_j \, \partial_j \, \partial_k \Phi(x_0) \, \epsilon_k = \sum_{jk} (\epsilon_j \, \partial_j) \, (\epsilon_k \, \partial_k) \, \Phi(x_0) \\
&= \left(\sum_j \epsilon_j \, \partial_j\right) \left(\sum_k \epsilon_k \, \partial_k\right) \Phi(x_0) = \left(\epsilon_j \, \partial_j\right) \, \left(\epsilon_k \, \partial_k\right) \, \Phi(x_0)
\end{aligned}
$$

where the last equality amounts to the simplification of notation achieved by the Einstein convention. □

For scalar arguments the condition that $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$ amount to the requirement that the slope vanishes at an extremum.

When $\Phi(\boldsymbol{x})$ is a potential then the requirement $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$ amounts to the requirement that the force $\boldsymbol{F}(\boldsymbol{x})$ vanishes at the position x_0,

$$\boldsymbol{F}(\boldsymbol{x}_0) = -\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$$

Hence, we say that the function $\Phi(\boldsymbol{x})$ has a stationary point at $\boldsymbol{x}_0$ when $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$. This underpins the heuristic discussion of the potential energy of the mobile that we gave above.

In particular $\Phi(\boldsymbol{x})$ has a minimum at $\boldsymbol{x}_0$ iff

- $\nabla\Phi(\boldsymbol{x}_0) = \mathbf{0}$, and
- all eigenvalues of $\mathsf{C}(\boldsymbol{x}_0)$ are positive.

Proof. We explore how $\Phi(\boldsymbol{x}_0)$ changes when one considers a point $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\varepsilon}$ in the vicinity of $\boldsymbol{x}_0$, where we express the deviation in the orthonormal basis spanned by the eigenvectors $\hat{\boldsymbol{e}}_i$ of C. Adopting Einstein notation we have

$$\begin{aligned} \boldsymbol{\varepsilon} &= \epsilon_i\, \hat{\boldsymbol{e}}_i \\ \Rightarrow \quad \boldsymbol{\varepsilon}^T\, \mathsf{C}\, \boldsymbol{\varepsilon} &= \epsilon_i\, \hat{\boldsymbol{e}}_i \cdot \left(\mathsf{C}\epsilon_k\, \hat{\boldsymbol{e}}_k \right) = \epsilon_i\, \hat{\boldsymbol{e}}_i \cdot \left(\lambda_k \epsilon_k\, \hat{\boldsymbol{e}}_k \right) \\ &= \lambda_k \epsilon_k\, \epsilon_i\, \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_k = \lambda_k \epsilon_k\, \epsilon_i\, \delta_{ik} = \lambda_k \epsilon_k\, \epsilon_k \end{aligned}$$

such that

$$\Phi(\boldsymbol{x}_0 + \boldsymbol{\epsilon}) = \Phi(\boldsymbol{x}_0) + \epsilon_k \cdot \partial_k \Phi(\boldsymbol{x}_0) + \frac{1}{2} \lambda_k \epsilon_k\, \epsilon_k$$

1. Assume that $\partial_k \Phi(\boldsymbol{x}_0) \neq 0$ for some coordinate k. We will then choose the orientation of the associated unit vector such that $\partial_k \Phi(\boldsymbol{x}_0) = m > 0$ and consider a displacement $\boldsymbol{\varepsilon} = \epsilon\, \hat{\boldsymbol{e}}_k$. The change of the value of $\Phi(\boldsymbol{x}_0)$ amounts then to

$$\Phi(\boldsymbol{x}_0 + \epsilon\, \hat{\boldsymbol{e}}_k) - \Phi(\boldsymbol{x}_0) = m\, \epsilon + \frac{\lambda_k}{2} \epsilon^2 + \cdots = \epsilon\, \left(m + \frac{\lambda_k}{2} \epsilon + \dots \right)$$

For $|\epsilon| < 2m/|\lambda_k|$ the expression in the bracket takes a positive value, such that $\Phi(\boldsymbol{x}_0 + \epsilon\, \hat{\boldsymbol{e}}_k) < \Phi(\boldsymbol{x}_0)$ for small negative values of ϵ. Consequently, $\Phi(\boldsymbol{x}_0)$ can only be a minimum when $\nabla \Phi(\boldsymbol{x}_0) = \mathbf{0}$.

2. Assume that $\nabla \Phi(\boldsymbol{x}_0) = \mathbf{0}$ and that all eigenvalue $\lambda_k > 0$. For small $\boldsymbol{\varepsilon}$ the change of the value of $\Phi(\boldsymbol{x}_0)$ amounts then to

$$\Phi(\boldsymbol{x}_0 + \epsilon\, \hat{\boldsymbol{e}}_k) - \Phi(\boldsymbol{x}_0) \simeq \frac{1}{2}\, \lambda_k\, \epsilon_k^2 > 0$$

such that that the function takes values larger than $\Phi(\boldsymbol{x}_0)$ for all positions $\boldsymbol{x}$ in the vicinity of $\boldsymbol{x}_0$. □

Analogously, to the discussion of the minimum one shows that $\Phi(\boldsymbol{x}_0)$ takes a maximum when the gradient vanishes, $\nabla \Phi(\boldsymbol{x}_0) = \mathbf{0}$ and when all eigenvalue λ_k take negative values.

The function Φ takes a saddle at $\boldsymbol{x}_0$ when there are positive and negative eigenvalues and when $\nabla \Phi(\boldsymbol{x}_0) = \mathbf{0}$.

When some eigenvalues vanish and all others are positive (negative), then higher-order contributions of the Taylor expansion must be considered to determine if Φ takes a minimum (maximum).

5.5.3 *Self Test*

Problem 5.9. **Symmetry properties of the second-order contributions**

Verify that the left and the right eigenvectors of C are identical, up to transposition.

Why does this imply that the normalized eigenvectors span an orthonormal basis?

Problem 5.10. **Equipotential lines for a 2D potential**

Consider a potential $\Phi(\boldsymbol{x})$ with $\boldsymbol{x} \in \mathbb{R}^2$. Sketch the contour lines of the potential for the following situations

- $\nabla\Phi(\boldsymbol{x}) = (1, 1)$ and $\mathsf{C}(\boldsymbol{x}) = 0$ for all positions $\boldsymbol{x}$.
- $\nabla\Phi(1, 2) = \mathbf{0}$ and $\mathsf{C}(1, 2) = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ with
  1. $b > 1$,
  2. $1 > b > -1$,
  3. $b < -1$,
  4. $b = 1$.

5.6 *Worked example: Reflection of balls*

turn from question to worked example

We consider the reflection of a ball from the ground, the lower side of a table, and back. The ball is considered to be a sphere with radius R, mass m, and moments of inertia $m\alpha R^2$ (by symmetry they all agree). Its velocity at time t_0 will be denoted as $\dot{\boldsymbol{z}}_0$. It has no spin initially. $\boldsymbol{\omega}_0 = \mathbf{0}$. The velocity and the spin after the n^{th} collision will be denoted as $\dot{\boldsymbol{z}}_n$ and $\boldsymbol{\omega}_n$. We will disregard gravity such that the ball travels on a straight path in between collisions.

a) Sketch the setup, and the parameters adopted for the first collision: The positive x axis will be parallel to the floor and the origin will be put into the location of the collision. Its direction will be chosen such that the ball moves in the x-z plane. Take note of all quantities needed to discuss the angular momentum with respect to the origin.

b) Upon collision there is a force normal to the floor, $\boldsymbol{F}_\perp$, and a force tangential to the floor, $\boldsymbol{F}_\parallel$. The spin of the ball will *only* change due to the tangential force. The normal force $\boldsymbol{F}_\perp$ acts in the same way as for point particles. The velocity in vertical direction reverses direction and preserved its modulus. Denote the velocity component in horizontal direction as $v_n = \hat{x} \cdot \dot{\boldsymbol{z}}$, and demonstrate that conservation of energy and angular momentum imply that

$$v_n^2 + \alpha R^2 \omega_n^2 = v_{n+1}^2 + \alpha R^2 \omega_{n+1}^2$$

$$v_n - \alpha R\, \omega_n = v_{n+1} - \alpha R\, \omega_{n+1} \,.$$

Show that the tangential velocity component will therefore also reverse its direction and preserves the modulus,

$$v_n + R\, \omega_n = -(v_{n+1} + R\, \omega_{n+1}) \,.$$

c) Determine $v_1(v_0, \omega_0)$ and $\omega_1(v_0, \omega_0)$ for the initial conditions specified above. Now, we determine $v_2(v_1, \omega_1)$ and $\omega_2(v_1, \omega_1)$ by shifting the origin of the coordinate systems to the point where the next collision will arise, and we rotate by π to account for the

fact that we collide at the lower side of the table. What does this imply for v_1 and ω_1? Continue the iteration, and plot v_1, v_2 and v_3 as function of α. Discuss the result for a sphere with uniform mass distribution (what does this imply for ω?), and a sphere with $\omega = 1/3$.
Hint: For the plot one conveniently implements the recursion, rather than explicitly calculating v_3.

d) What changes in this discussion when the ball has a spin initially?

add:
tennis racket theorem?

5.7 Problems

5.7.1 Practicing Concepts

Problem 5.11. **Determining the volume, the mass, and the center of mass**

Determine the mass M, the area or volume V, and the center of mass $\boldsymbol{Q}$ of bodies with the following mass density and shape.

a) A triangle in two dimensions with constant mass density $\rho = 1\,\mathrm{kg/m^2}$ and side length 6 cm, 8 cm, and 10 cm.
Hint: Determine first the angles at the corners of the triangle. Decide then about a convenient choice of the coordinate system (position of the origin and direction of the coordinate axes).

b) A circle in two dimensions with center at position (a, b), radius $R = 5\,\mathrm{cm}$, and constant mass density $\rho = 1\,\mathrm{kg/m^2}$.
Hint: How do M, V and $\boldsymbol{Q}$ depend on the choice of the origin of the coordinate system?

c) A rectangle in two dimensions, parameterized by coordinates $0 \le x \le W$ and $0 \le y \le B$, and a mass density $\rho(x, y) = \alpha\, x$.
What is the dimension of α in this case?

d) A three-dimensional wedge with constant mass density $\rho = 1\,\mathrm{kg/m^3}$ that is parameterized by $0 \le x \le W$, $0 \le y \le B$, and $0 \le z \le H - Hx/W$.
Discuss the relation to the result of part b).

e) A cube with edge length L. When its axes are aligned parallel to the axes $\hat{x}$, $\hat{y}$, $\hat{z}$, it density takes the form $\rho(x, y, z) = \beta\, z$.
What is the dimension of β in this case?

Problem 5.12. **Return time and position of the professor**

a) How long will the professor take to arrive in down-under, and when will he reappear for the first time close to home?

b) How far will Earth have moved in that time? When this happens to him in Leipzig, where will he reappear, and when will he see land again for the next time?

c) Adopt an orthonormal coordinate system (x, y, z) that is co-rotating with Earth, with origin in the Earth center, z-axis oriented towards the North pole, and x-direction towards the latitude of Leipzig. Sketch the trajectory of the professor in the (x, y)-plane when he was at rest initially.

d) Observe that the professor is initially standing on the surface of Earth. What does this imply for his initial velocity? How does the trajectory change?

e) Let him how start with zero velocity from the Moon surface. What does this imply for the force law? How does the trajectory change?

5.7.2 *Proofs*

5.7.3 *Transfer and Bonus Problems, Riddles*

6
Integrable Dynamics

In Chapter 5 we considered objects that consist of a mass points with fixed relative positions, like a flying and spinning ping-pong ball. Rather than providing a description of each individual mass element, we established equations of motion for their center of mass and the orientation of the body in space. From the perspective of theoretical mechanics the fixing of relative positions is a constraint to their motion, just as the ropes of a swing enforces a motion on a one-dimensional circular track, rather than in two dimensions. The deflection angle θ of the pendulum, and the center of mass and orientation of the ball are examples of generalized coordinates that automatically take into account the constraints.

Marguerite Martyn, 1914
wikimedia/public domain
Figure 6.1: The point-particle idealization of a girl on a swing is the mathematical pendulum of Figures 1.2 and 1.3.

In this chapter we discuss how to set up generalized coordinates and how to find the associated equations of motion. The discussion will be driven by examples. The examples will be derived from the realm of integrable dynamics. These are systems where conservation laws can be used to break down the dynamics into separate problems that can be interpreted as motion with a single degree of freedom.

At the end of the chapter you know why coins run away rolling on their edge, and how the speed of a steam engine was controlled by a mechanical device. Systems where the dynamics is not integrable will subsequently be addressed in Chapter 7.

add more pics

6.1 *Motivation and Outline: How to deal with constraint motion?*

Almost all interesting problems in mechanics involve constraints due to rails or tracks, and due to mechanical joints of particles. The most elementary example is a swing (Figure 6.1), where a rope forces a mass M to move on a path with positions constrained to a circle with radius given by the length L of the rope. Gravity $M\,\boldsymbol{g}$ and the pulling force $\boldsymbol{F}_r$ of the rope acting act on the mass (Figure 6.2). However, how large is the latter force? At the topmost point of its trajectory the mass is at rest, and no force is needed along the rope to keep it on its track. At the lowermost point, where the swing goes with its maximum speed, there is a substantial force. Newton's formalism requires a discussion of these forces. Lagrange established an alternative approach that provides equations of motion with substantially less effort. The key idea of this formalism is to select generalized coordinates adapted to the problem.

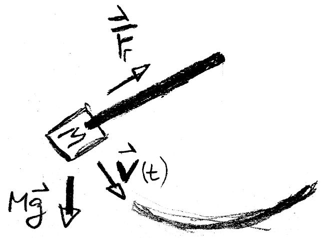


Figure 6.2: Forces acting for the motion of a swing, or its equivalent idealization of of a mathematical pendulum.

Definition 6.1: Generalized Coordinates

We consider N particles moving in D dimensions. There are forces that keep the particles on a subset of space with a dimension smaller than D, and their relative positions may be constrained by bars and joints. Due to the constrains the system only has $M < D\,N$ degrees of freedom. In this chapter we denote the positions of the particles as $\boldsymbol{x} \in \mathbb{R}^{D\,N}$, and we specify position compatible with the constraints as $\boldsymbol{x}(\boldsymbol{q}(t))$, where $\boldsymbol{q} \in \mathbb{R}^M$ are the *generalized coordinates* adapted to the constrained motion.

Example 6.1: Generalized coordinates for a pendulum

We describe the position of the mass in a mathematical pendulum by the angle $\theta(t)$, as introduced in Example 1.10. The position of the mass in the 2D pendulum plane is thus described by the vector

$$\boldsymbol{x}(t) = L \begin{pmatrix} \sin\theta(t) \\ -\cos\theta(t) \end{pmatrix} = L\,\hat{\boldsymbol{R}}(\theta(t))\,.$$

In view of the chain rule its velocity amounts to

$$\dot{\boldsymbol{x}} = L\,\dot\theta\,\partial_\theta \hat{\boldsymbol{R}}(\theta(t)) = L\,\dot\theta\,\hat{\boldsymbol{\theta}}(\theta(t)) \quad \text{with} \quad \hat{\boldsymbol{\theta}}(\theta(t)) = \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}$$

Remark 6.1. Note that $\hat{\boldsymbol{R}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$ are orthonormal vectors that describe the position of the mass in terms of polar coordinates rather than fixed-in-space Cartesian coordinates.

Theorem 6.1: Basis vectors for polar coordinates

Let $\{\hat{x}, \hat{z}\}$ be a basis of $\mathbb{R}^2$, and (R, θ) be the polar coordinates[1] associated to a point with Cartesian coordinates (x, z). Then

- $R = \sqrt{x^2 + z^2}$ is the distance of the point from the origin
- $\theta = -\arctan(x/z)$ the angle with respect to $-\hat{z}$,

We denote the vector from the origin to (R, θ) as $R \quad \hat{\mathbf{R}}(\theta)$. Then the following statements hold

a) $\hat{\mathbf{R}}(\theta)$ is a normal vector at the position (R, θ) of a circle C_R with center at the origin at radius R.

b) $\hat{\boldsymbol{\theta}} = \partial_\theta \hat{\mathbf{R}}$ is a vector tangential to C_R at the position (R, θ).

c) $\partial_\theta \hat{\boldsymbol{\theta}} = -\hat{\mathbf{R}}$.

d) For every $\theta \in [0, 2\pi)$ the vectors $\{\hat{\mathbf{R}}(\theta), \hat{\boldsymbol{\theta}}(\theta)\}$ form an orthonormal basis of $\mathbb{R}^2$.

[1] The choice of the axes and the angle reflects the notations adopted in Figures 1.2 and 1.3.

Remark 6.2. The coordinate representation of $\hat{\mathbf{R}}(\theta)$ and $\hat{\boldsymbol{\theta}}(\theta)$ in Cartesian coordinates is provided in Example 6.1.

Remark 6.3. The assertions of Theorem 6.1 also apply when the unit vectors of $\mathbb{R}^2$ are denoted as $\{\hat{x}, \hat{y}\}$, and when the angle θ denotes the angle with respect of the $\hat{x}$ axis. The different reference axis *only* changes the coordinate representation of the vectors.

Example 6.2: Generalized coordinates for a ping-pong ball

A ping-pong ball consists of N atoms located in the three-dimensional space. During a match they follow an intricate trail in the vicinity of the ping-pong players. At any time during their motion the atoms are located on a thin spherical shell with fixed positions with respect to each other. Rather than specifying the position of each atom one can therefore specify the position of the ball in terms of six generalized coordinates (Figure 6.3): Three coordinates provide its center of mass. The orientation of the ball can be provided by specifying the orientation of a body fixed axis in terms of its polar and azimuthal angle, and a third angle specifies the orientation of a point on its equator when rotating the ball around the axis.

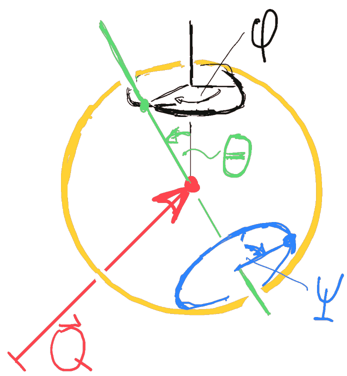


Figure 6.3: The position of a ball in space can be described in terms of a 3D vector $\boldsymbol{Q}$ that describes the center of the ball (red dot), angles θ, φ that describe the orientation in space of a fixed axes in the ball (green line), and another angle ψ that describes the position of point that is not on the axis (blue point).

Generalized coordinates describe only positions complying with the constraints of the motion, and they do not account for other positions from the very beginning. Lagrange's key observation is that constraint forces, e. g. the force on the rope of the swing, only act in a direction orthogonal to the positions described by generalized coordinates. Therefore, the constraint forces do not affect the time evolution of generalized coordinates. For the pendulum and the ping-pong ball one only has to account for gravity to find the evolution of the generalized coordinates. There is no need to deal with the force along the rope in the swing, and the atomic interaction forces that keep atoms in their positions in the ping-pong ball.

Outline

In Section 6.2 we will introduce the Lagrange formalism that will allow us without much ado to determine the EOM for generalized coordinates. Subsequently, in Section 6.3 we deal with systems with a single degree of freedom. In Section 6.4 we will see how symmetries in the dynamics allow us to eliminate a second degree of freedom based on an informed choice of generalized coordinates. Section 6.5 deals with two-particle systems and other problems where one deals with several degrees of freedom. In all cases we will eventually reduce the dynamics to one-dimensional problems.

6.1.1 Self Test

Problem 6.1. **Different reference axes for polar coordinates**
Verify the assertions of Theorem 6.1 and Remark 6.3:

a) Specify the coordinate representation of $\{\hat{\boldsymbol{R}}(\theta), \hat{\boldsymbol{\theta}}(\theta)\}$ for the case where θ denotes the angle with respect to the positive $\hat{\boldsymbol{x}}$ axis.

b) Verify the assertion of Theorem 6.1.

Problem 6.2. **Describing the orientation of dice**
We place a die on the table such that its center lies at the origin of a 3D Cartesian coordinate frame, and its axes are aligned with the coordinate axes. We characterize the configuration of the die by the number of dots on the faces pointing in the three positive coordinate directions.

a) Show that there are 24 different possibilities to place the die.

b) Determine the angles (θ, φ, ψ) (cf. Figure 6.3) that will turn a die from the configuration be $(1, 2, 3)$ to

b1) $(2, 3, 4)$ b2) $(4, 6, 2)$ b3) $(1, 3, 5)$

6.2 *Lagrange formalism*

The Lagrange formalism provides an effective approach to derive the EOM for generalized coordinates. We first provide a derivation in a Cartesian coordinate frame. Then we discuss how the EOM for generalized coordinates are determined.

6.2.1 Euler-Lagrange equations for Cartesian coordinates

In Section 5.5 we saw that a mobile will be at rest in a position characterized by the coordinate vector $\boldsymbol{x}$ when the leading order correction $\delta \boldsymbol{x} \cdot \nabla \Phi(\boldsymbol{x})$ to its potential energy $\Phi(\boldsymbol{x})$ vanishes for every perturbation $\delta \boldsymbol{x}$ of the position. In the following we denote the leading order corrections term of the Taylor expansion as variation.

Definition 6.2: Variation of a scalar function

Let $f : \mathbb{D} \times [t_I, t_E] \to \mathbb{R}$ with $\mathbb{D} \subset R^D$ be function that has continuous first derivatives for all $\boldsymbol{x} \in \mathbb{D}$. The *variation of* f for a small deviation $\delta \boldsymbol{x}$ of $\boldsymbol{x}$ such that $\boldsymbol{x} + \delta \boldsymbol{x} \in \mathbb{D}$ amounts to the linear-order term of the Taylor expansion of f,

$$\delta f(\boldsymbol{x}, t) = \delta \boldsymbol{x} \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x}, t) = \sum_{i=1}^{D} \delta x_i \, \frac{\partial f(\boldsymbol{x}, t)}{\partial x_i}$$

In Section 5.5 we showed that $\delta \Phi(\boldsymbol{x}_0) = 0$ for every critical point $\boldsymbol{x}_0$ where the system is (and remains) at rest. We now also account fort explicitly time-dependent potentials $\Phi(\boldsymbol{x}, t)$ and consider the variations $\delta \boldsymbol{x}(t)$ of time dependent trajectories $\boldsymbol{x}(t)$ with $t \in [t_I, t_F]$. Here $\delta \boldsymbol{x}(t)$ describes the deviation of the perturbed trajectory from the reference trajectory $\boldsymbol{x}(t)$ at time t, and it is understood that $\delta \boldsymbol{x}(t_I) = \delta \boldsymbol{x}(t_F) = \boldsymbol{0}$ Now we have

$$\delta \Phi(\boldsymbol{x}, t) = \delta \boldsymbol{x} \cdot \nabla_{\boldsymbol{x}} \Phi(\boldsymbol{x}, t) = -\delta \boldsymbol{x} \cdot \boldsymbol{F}(\boldsymbol{x}, t) = -\delta \boldsymbol{x} \cdot m \, \ddot{\boldsymbol{x}}$$

The velocity and acceleration for the perturbed trajectory $\boldsymbol{x} + \delta \boldsymbol{x}$ are $\dot{\boldsymbol{x}} + \delta \dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}} + \delta \ddot{\boldsymbol{x}}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(m \, \dot{\boldsymbol{x}} \cdot \delta \boldsymbol{x} \right) = m \, \ddot{\boldsymbol{x}} \cdot \delta \boldsymbol{x} + m \, \dot{\boldsymbol{x}} \cdot \delta \dot{\boldsymbol{x}} = m \, \ddot{\boldsymbol{x}} \cdot \delta \boldsymbol{x} + \delta \frac{m \, \dot{\boldsymbol{x}}^2}{2}$$

where $T = m \, \dot{\boldsymbol{x}}^2/2$ is the kinetic energy. Hence, we can express the variation of the potential as

$$\begin{aligned} \delta \Phi(\boldsymbol{x}, t) &= -\frac{\mathrm{d}}{\mathrm{d}t} \left(\delta \boldsymbol{x} \cdot m \, \dot{\boldsymbol{x}} \right) + \delta T(\dot{\boldsymbol{x}}) \\ \Rightarrow \quad \delta \big(T(\dot{\boldsymbol{x}}) - \Phi(\boldsymbol{x}, t) \big) &= -\frac{\mathrm{d}}{\mathrm{d}t} \left(\delta \boldsymbol{x} \cdot m \, \dot{\boldsymbol{x}} \right) \end{aligned}$$

The difference between the kinetic and potential energy is a total time derivative. Integrating the expression over time from t_I to t_F therefore provides

$$\int_{t_I}^{t_F} \mathrm{d}t \, \delta \big(T(\dot{\boldsymbol{x}}) - \Phi(\boldsymbol{x}, t) \big) = - \int_{t_I}^{t_F} \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} \left(\delta \boldsymbol{x} \cdot m \, \dot{\boldsymbol{x}} \right) \tag{6.2.1}$$

$$= \delta \boldsymbol{x}(t_I) \cdot m \, \dot{\boldsymbol{x}}(t_I) - \delta \boldsymbol{x}(t_F) \cdot m \, \dot{\boldsymbol{x}}(t_F) = 0 \tag{6.2.2}$$

The integral vanishes because $\boldsymbol{x}$ is fixed a the start and the end point.

Up to mathematical identities that are always true we only used Newton's law $\boldsymbol{F}(\boldsymbol{x}, t) = m\ddot{\boldsymbol{x}}$ to arrive at this conclusion. This observation is denoted as the principle of least action. Rather than on Newton axioms we may therefore base mechanics on the principle of least action.

Definition 6.3: Lagrangian

We consider a dynamics with kinetic energy $T(\dot{\boldsymbol{x}}(t))$ and potential energy $\Phi(\boldsymbol{x}(t), t)$ for trajectories $\boldsymbol{x}(t)$. The difference

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = T(\dot{\boldsymbol{x}}) - \Phi(\boldsymbol{x}, t)$$

will be called *Lagrangian* or *Lagrange function* of the dynamics.

Definition 6.4: Action of a trajectory

For a dynamics with Lagrangian $\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$ the *action* $S[\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]$ of a trajectory $\boldsymbol{x}(t)$, $t_I \leq t \leq t_F$ with velocity $\dot{\boldsymbol{x}}(t)$ is defined as

$$S[\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)] = \int_{t_I}^{t_F} \mathrm{d}t\, \mathcal{L}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t) \tag{6.2.3}$$

The *variation of the action* will be defined as

$$\delta S[\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)] = \int_{t_I}^{t_F} \mathrm{d}t\, \delta\mathcal{L}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t)$$

Axiom 6.1: Principle of least action

Let $\boldsymbol{x}(t)$ with $t_I \leq t \leq t_F$ be a trajectory from $\boldsymbol{x}(t_I)$ to $\boldsymbol{x}(t_F)$ that satisfies Newton's law $\boldsymbol{F}(\boldsymbol{x}, t) = m\ddot{\boldsymbol{x}}$ with a force that is derived from a potential $\Phi(\boldsymbol{x}, t)$. Then the variation of the action associated the trajectory will vanish

$$0 = \delta S[\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]$$

Remark 6.4. The principle is called the principle of *least* action. However, it only requires that the action has a critical point. There are many examples in physics where the action takes a saddle point, rather than a minimum.

The principle provides an alternative way to determine the EOM that proceeds as follows.

$$\begin{aligned}
0 = \delta S[\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)] &= \int_{t_I}^{t_F} \mathrm{d}t\, \delta\mathcal{L}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t) \\
&= \int_{t_I}^{t_F} \mathrm{d}t\, \left[\delta\dot{\boldsymbol{x}}\, \nabla_{\dot{\boldsymbol{x}}}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) + \delta\boldsymbol{x}\, \nabla_{\boldsymbol{x}}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right] \\
&= \int_{t_I}^{t_F} \mathrm{d}t\, \delta\boldsymbol{x}\, \left[\left(-\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\dot{\boldsymbol{x}}}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right) + \nabla_{\boldsymbol{x}}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right]
\end{aligned}$$

In the last step we performed a partial integration.[2] The integral must vanish for every choice of the variation $\delta\boldsymbol{x}$. In particular we may choose a function $\delta\boldsymbol{x}$ that takes the same sign as the square bracket whenever it does not vanish. However, in that case the integral is strictly positive unless the square bracket vanishes. This

[2] The boundary term of the partial integration vanishes,

$$\begin{aligned}
\left[\delta\boldsymbol{x}\, \nabla_{\dot{\boldsymbol{x}}}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right]_{t_I}^{t_F} &= \delta\boldsymbol{x}(t_F)\ \left[\nabla_{\dot{\boldsymbol{x}}}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right]_{t=t_F} \\
&\quad - \delta\boldsymbol{x}(t_I)\ \left[\nabla_{\dot{\boldsymbol{x}}}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right]_{t=t_F} \\
&= 0
\end{aligned}$$

provides the EOM of the dynamics in terms of the Euler-Lagrange equation.

Theorem 6.2: Euler-Lagrange equations

Let $x_i(t)$ be a coordinates of a trajectory $\boldsymbol{x}(t)$ of a dynamics with Lagrangian $\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$. Then $x_i(t)$ is a solution of the *Euler-Lagrange equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}_i}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = \frac{\partial}{\partial x_i}\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \tag{6.2.4}$$

6.2.2 *Mathematical background: variational calculus*

The principle of least action is an application of variational calculus to the action integral, Equation (6.2.3). In order to provide a better intuition of the mathematical concept of the variation, we demonstrate now how one can derive a differential equation of the shortest path on a plane.

Shortest path in a 2d plane. We describe a curve from the origin, $(0, 0)$ to the position (x_e, y_e) in the plane by a function $f(x)$ with $f(0) = 0$ and $f(x_e) = y_e$. Hence, the curve follows the coordinates $\boldsymbol{q} = (x, f(x))$, and according to Remark 3.7 the length of a curve is determined by the line integral

$$L[f(x)] = \int_0^{x_e} \mathrm{d}x\, \mathcal{D}_{\text{plane}}(f(x), f'(x))$$

$$\text{with} \quad \mathcal{D}_{\text{plane}}(f(x), f'(x)) = \left|\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x}\right| = \sqrt{1 + (f'(x))^2}$$

where $f'(x) = \mathrm{d}f(x)/\mathrm{d}x$. Paths of minimal length must therefore be solutions of the Euler-Lagrange equation

$$0 = \frac{\partial \mathcal{D}}{\partial f} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{D}}{\partial f'} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{f'(x)}{\sqrt{1 + (f'(x))^2}}$$

which implies that there is a constant K with

$$K = \frac{f'(x)}{\sqrt{1 + (f'(x))^2}} \quad \Rightarrow \quad K^2\left(1 + (f'(x))^2\right) = (f'(x))^2$$

$$\Rightarrow \quad f'(x) = \frac{K}{\sqrt{1 - K^2}} = \text{const}$$

Consequently, the shortest connection between two points in the plane is a straight line, where the slope is constant. We urge the reader to go through the steps of the derivation of the Euler-Lagrange equation for this problem, and to take not of the important requirement that the variation of the path $\delta\boldsymbol{q}$ must vanish at *both* endpoints of the trajectory. An example of a variational problem where this requirement is relaxed is provided in Problem 6.4.

Shortest path on a cylinder surface. We describe a curve on a cylinder with radius R by adopting cylinder coordinates and specifying $z(\theta)$ for the range $\mathcal{I} = [\theta_I, \theta_E]$. The values at the boundary of $\mathcal{I}$ will be denoted as $z(\theta_I) = z_I$ and $z(\theta_E) = z_E$. Hence, the curve follows the coordinates $\boldsymbol{q} = z(\theta)\,\hat{\boldsymbol{z}} + R\,\hat{\boldsymbol{r}}(\theta)$ along a path with direction

$$\boldsymbol{q}'(\theta) = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\theta} = z'(\theta)\,\hat{\boldsymbol{z}} + R\,\hat{\boldsymbol{\theta}}(\theta)$$

The length of this curve is determined by the line integral

$$L[z(\theta)] = \int_{\theta_I}^{\theta_E} \mathrm{d}\theta\,\mathcal{D}_{\text{cyl}}\big(z(\theta), z'(\theta)\big)$$

$$\text{with}\quad \mathcal{D}_{\text{cyl}}\big(z(\theta), z'(\theta)\big) = \left|\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\theta}\right| = R\,\sqrt{1 + \big(z'(\theta)/R\big)^2}$$

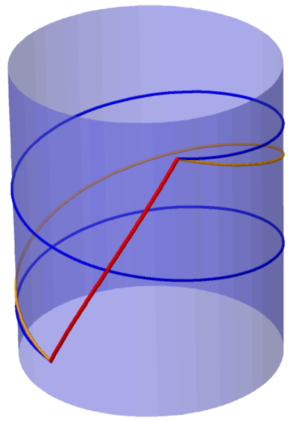


Figure 6.4: Paths of extremal length on a cylinder. In this pictures we have $\theta_E - \theta_I = \pi/3$, $h_E - h_I = 3\,R$, and we show paths with winding numbers $n \in \{0, -1, -2\}$.

The distance function $\mathcal{D}_{\text{cyl}}$ of this problem is identical to the one for the plane, up to replacing $f(x)$ by $z(\theta)$ and x by $R\,\theta$. Hence, the solutions will be paths of the form

$$z(\theta) = z_I + \frac{z_E - z_I}{\theta_E - \theta_I + 2\pi\,n}\,(\theta - \theta_I) \quad\text{with}\quad n \in \mathbb{Z}$$

When one makes sure that $\theta_E \in (\theta_I - \pi, \theta_I + \pi)$ then the solution for $n = 0$ represents the shortest path from z_I to z_E. For $\theta_E - \theta_I = \pi$ the path for $n = 0$ and $n = -1$ have the same length. All other paths represent local minima of L. Small perturbations of the path will increase the length. However, trajectories that reach the final point with a smaller number of loops around the cylinder will in general be shorter. An example is shown in Figure 6.4.

Shortest path on a catenoid. A catenoid is the surface of revolution of the hyperbolic cosine function. We describe a curve on a catenoid with radius $\cosh z$ at height z by adopting cylinder coordinates and specifying $\theta(z)$ for the range $\mathcal{I} = [z_I, z_E]$.[3] The values at the boundary of $\mathcal{I}$ are now denoted as $z(\theta_I) = z_I$ and $z(\theta_E) = z_E$. Hence, the curve follows the coordinates $\boldsymbol{q} = z\,\hat{\boldsymbol{z}} + \cosh z\,\hat{\boldsymbol{r}}(\theta(z))$ along a path with direction

[3] The surface is no longer translation invariant along the axis, but still rotation symmetric. As a consequence, $\theta(z)$ will turn out to be a cyclic variable, while a parameterization in terms of $z(\theta)$ will involve nontrivial derivatives with respect to z. You may check this in Problem 6.3.

$$\boldsymbol{q}'(z) = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}z} = \hat{\boldsymbol{z}} + \sinh z(\theta)\,\hat{\boldsymbol{r}}(\theta(z)) + \cosh z(\theta)\,z'(\theta)\,\hat{\boldsymbol{\theta}}(\theta(z))$$

The length of this curve is determined by the line integral

$$L[z(\theta)] = \int_{z_I}^{z_E} \mathrm{d}z\,\mathcal{D}_{\text{cat}}\big(\theta(z), \theta'(z)\big)$$

$$\text{with}\quad \mathcal{D}_{\text{cat}}\big(\theta(z), \theta'(z)\big) = \left|\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\theta}\right| = \sqrt{\big(1 + \theta'(z)^2\big)\,\cosh^2 z + \sinh^2 z}$$

Consequently, the Euler-Lagrange function takes the form

$$0 = \frac{\partial \mathcal{D}_{\text{cat}}}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial \mathcal{D}_{\text{cat}}}{\partial \theta'} \quad = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\theta'(z)\,\cosh^2 z}{\sqrt{\big(1 + \theta'(z)^2\big)\,\cosh^2 z + \sinh^2 z}}$$

The variable θ is cyclic and we denote the entailed conservation law as K. Rearranging terms provides

$$\theta'(z) = K \sqrt{\frac{1+\tanh^2 z}{\cosh^2 z - K^2}}$$

such that

$$\theta(z) = \theta_I + K \int_{z_I}^{z} \mathrm{d}z \sqrt{\frac{1+\tanh^2 z}{\cosh^2 z - K^2}}$$

The shortest paths on the catenoid must be determined by numerical evaluation of this integral.

make fig with paths on catenoid

Problem 6.5 extends the present discussion to situations where one minimizes the surface *area* of a soap film, rather than a feature of a one-dimensional object. Problem 6.23 addresses extremal paths on a sphere. Unless two points lie exactly on opposite sides of the sphere (like North and South pole) there are exactly two trajectories of extremal length. One of them is the shortest trajectory. The other one is a saddle point.

6.2.3 *Euler-Lagrange equations for generalized coordinates*

The Euler-Lagrange equations derive from a variational principle stating that the gradient of the Lagrange function with respect to the phase-space coordinate $\boldsymbol{\Gamma} = (\boldsymbol{x}, \dot{\boldsymbol{x}})$ must vanish for physically admissible trajectories. This holds for *all* directions in phase space. However, generalized coordinates do not qualify as a vector such that some care is needed to derive their EOM.

Example 6.3: Rollercoaster trail

The position $\boldsymbol{x}(t)$ on the trail of a rollercoaster can uniquely be described by the (dimensionless) distance ℓ along the trail that it has gone. Hence, generalized coordinate $\ell(t)$ uniquely describes the configuration $\boldsymbol{x}(\ell(t))$ of the rollercoaster at time t.

Example 6.4: Driven pendulum

A driven pendulum is a mathematical pendulum where the position of the fulcrum $\boldsymbol{X}_f$ and the length of the pendulum arm $L(t)$ are subjected to a prescribed temporal evolution. The position of the pendulum weight, $\boldsymbol{x}$, may then be described by the angle $\theta \in [0, 2\pi] = \mathbb{D}$,

$$\boldsymbol{x}(\theta, t) = \boldsymbol{X}_f(t) + R(t) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Here, the time dependence of $\boldsymbol{X}_f(t)$ and $R(t)$ reflect the temporal evolution of the time-dependent setup of the pendulum. The temporal evolution of the pendulum will be described in terms of the generalized coordinate $\theta(t)$.

Let $\boldsymbol{q}$ be the generalized coordinates of a system and $\boldsymbol{x}(\boldsymbol{q})$ the associated configuration vector of the system. It will be provided in Cartesian coordinates from the point of view of an observer who is at rest. Hence, $\boldsymbol{x}$ is a vector with all properties discussed in Chapter 2. In contrast, $\boldsymbol{q}$ will in general only be a tuple of functions that provide a convenient parameterization of valid configurations. We address the situation where the forces in the system are conservative, arising from a potential energy $\Phi\big(\boldsymbol{x}(\boldsymbol{q}), t\big)$. Moreover, we assume that the potential energy can be represented as a sum of $\Phi_c\big(\boldsymbol{x}(\boldsymbol{q}), t\big)$ and $U\big(\boldsymbol{x}(\boldsymbol{q}), t\big)$. The contribution $\Phi_c\big(\boldsymbol{x}(\boldsymbol{q}), t\big)$ accounts for forces that constraint the coordinates of the system such that they comply with positions $\boldsymbol{x}(\boldsymbol{q})$. The part $U\big(\boldsymbol{x}(\boldsymbol{q}), t\big)$ accounts for all other forces.

We will now explore the implications of the principle of least action for variations of the path that refer only to accessible coordinates. For the kth coordinate of the variation we write

$$\delta x_k = x_k(\boldsymbol{q} + \delta\boldsymbol{q}, t) - x_k(\boldsymbol{q}, t) = \sum_{\nu=1}^{d} \frac{\partial x_k}{\partial q_\nu}\, \delta q_\nu$$

and for the associated time derivative we have

$$\delta \dot{x}_k = \frac{\mathrm{d}}{\mathrm{d}t} \delta x_k = \sum_{\nu=1}^{d} \frac{\partial \dot{x}_k}{\partial q_\nu}\, \delta q_\nu + \sum_{\nu=1}^{d} \frac{\partial x_k}{\partial q_\nu}\, \delta \dot{q}_\nu$$

As a consequence the variation of the Lagrangian takes the form

$$\delta\mathcal{L} = \delta\boldsymbol{x} \cdot \nabla_{\boldsymbol{x}}\mathcal{L} + \delta\dot{\boldsymbol{x}} \cdot \nabla_{\dot{\boldsymbol{x}}}\mathcal{L} = \delta\boldsymbol{x} \cdot (\boldsymbol{F}_c + \boldsymbol{F}_e) + \delta\dot{\boldsymbol{x}} \cdot m\dot{\boldsymbol{x}}$$

where $\boldsymbol{F}_c$ represent the constraint forces. We consider variations $\delta\boldsymbol{x}$ that relate trajectories complying with the constraints such that $\delta\boldsymbol{x} \cdot \boldsymbol{F}_c = 0$. Therefore, in the setting of generalized coordinates one need not account for constraint forces.[4] We will now express the variation of the Lagrangian in terms of the variations of the generalized coordinates,

[4] This constraint is commonly denoted as *d'Alembert's principle*.

$$\begin{aligned}
\delta\mathcal{L} &= \sum_{k=1}^{D} \left[\delta x_k \, \frac{\partial\mathcal{L}}{\partial x_k} + \delta\dot{x}_k \, \frac{\partial\mathcal{L}}{\partial\dot{x}_k} \right] \\
&= \sum_{k=1}^{D} \left[\left(\sum_{\nu=1}^{d} \frac{\partial x_k}{\partial q_\nu}\, \delta q_\nu \right) \frac{\partial\mathcal{L}}{\partial x_k} + \sum_{\nu=1}^{d} \left(\frac{\partial \dot{x}_k}{\partial q_\nu}\, \delta q_\nu + \frac{\partial x_k}{\partial q_\nu}\, \delta\dot{q}_\nu \right) \frac{\partial\mathcal{L}}{\partial\dot{x}_k} \right] \\
&= \sum_{\nu=1}^{d} \delta q_\nu \sum_{k=1}^{D} \left(\frac{\partial x_k}{\partial q_\nu}\, \frac{\partial\mathcal{L}}{\partial x_k} + \frac{\partial \dot{x}_k}{\partial q_\nu}\, \frac{\partial\mathcal{L}}{\partial\dot{x}_k} \right) + \sum_{\nu=1}^{d} \delta\dot{q}_\nu \, \sum_{k=1}^{D} \frac{\partial x_k}{\partial q_\nu}\, \frac{\partial\mathcal{L}}{\partial\dot{x}_k}
\end{aligned}$$

On the other hand

$$\frac{\partial \mathcal{L}(\boldsymbol{x}(\boldsymbol{q},t),\dot{\boldsymbol{x}}(\boldsymbol{q},\dot{\boldsymbol{q}},t),t)}{\partial q_\nu} = \sum_{k=1}^{D} \left(\frac{\partial x_k}{\partial q_\nu} \frac{\partial \mathcal{L}}{\partial x_k} + \frac{\partial \dot{x}_k}{\partial q_\nu} \frac{\partial \mathcal{L}}{\partial \dot{x}_k} \right)$$

$$\begin{aligned}
\frac{\partial \mathcal{L}(\boldsymbol{x}(\boldsymbol{q},t),\dot{\boldsymbol{x}}(\boldsymbol{q},\dot{\boldsymbol{q}},t),t)}{\partial \dot{q}_\nu} &= \sum_{k=1}^{D} \frac{\partial \mathcal{L}(\boldsymbol{x}(\boldsymbol{q},t),\dot{\boldsymbol{x}}(\boldsymbol{q},\dot{\boldsymbol{q}},t),t)}{\partial \dot{x}_k} \frac{\partial \dot{x}_k}{\partial \dot{q}_\nu} \\
&= \sum_{k=1}^{D} \frac{\partial \mathcal{L}}{\partial \dot{x}_k} \frac{\partial}{\partial \dot{q}_\nu} \left(\frac{\partial x_k}{\partial t} + \sum_{\mu=1}^{d} \frac{\partial x_k}{\partial q_\mu} \dot{q}_\mu \right) \\
&= \sum_{k=1}^{D} \frac{\partial \mathcal{L}}{\partial \dot{x}_k} \frac{\partial x_k}{\partial q_\nu}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta S = \int \mathrm{d}t \delta \mathcal{L} &= \int \mathrm{d}t \sum_{\nu=1}^{d} \left(\delta q_\nu \frac{\partial \mathcal{L}}{\partial q_\nu} + \delta \dot{q}_\nu \frac{\partial \mathcal{L}}{\partial \dot{q}_\nu} \right) \\
&= \int \mathrm{d}t \sum_{\nu=1}^{d} \delta q_\nu \left(\frac{\partial \mathcal{L}}{\partial q_\nu} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_\nu} \right)
\end{aligned}$$

The equations of motion are derived from the Lagrangian, Definition 6.5, by Algorithm 6.1.

Definition 6.5: Lagrangian in generalized coordinates

The Lagrange function $\mathcal{L}$ amounts to the difference of the kinetic energy T and the potential energy U of the system,

$$\mathcal{L} = T - U = \sum_\alpha \frac{m_\alpha}{2} \dot{x}_\alpha^2(\boldsymbol{q}) - U(\boldsymbol{x}(\boldsymbol{q})) \tag{6.2.5}$$

Constraint forces are not considered.

Algorithm 6.1: Euler Lagrange EOMs

a) Identify generalized coordinates $\boldsymbol{q}$ that describe the admissible configurations of the system.

b) Determine $\boldsymbol{x}(\boldsymbol{q})$, and the resulting expression of the potential energy in terms of $\boldsymbol{q}$,

$$U(\boldsymbol{q}) = U(\boldsymbol{x}(\boldsymbol{q}))$$

c) Evaluate the kinetic energy based on the chain rule

$$T(\boldsymbol{q},\dot{\boldsymbol{q}}) = \sum_\alpha \frac{m_\alpha}{2} \dot{x}_\alpha^2(\boldsymbol{q}) = \sum_\alpha \frac{m_\alpha}{2} \left(\sum_i \frac{\partial x_\alpha}{\partial q_i} \dot{q}_i \right)^2$$

where x_α is the α-component of the configuration vector $\boldsymbol{x}$ and m_α the mass of the associated particle.

Hence, we establish the Lagrange function

$$\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) = T(\boldsymbol{q},\dot{\boldsymbol{q}}) - U(\boldsymbol{q})$$

expressed in terms of the generalized coordinates $\boldsymbol{q}$ and their time derivatives $\dot{\boldsymbol{q}}$.

d) Determine the EOM for the component q_i of $\boldsymbol{q}$ by evaluating the *Euler-Lagrange equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \tag{6.2.6}$$

In the next section we will apply the formalism to models with a single degree of freedom.

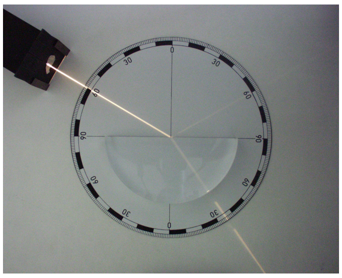
6.2.4 *Self Test*

Problem 6.3. **Shortest path on a catenoid**

In footnote 3 we pointed out that it is a good idea to parameterize the paths on a body of revolution in terms of $z(\theta)$ rather than $\theta(z)$.

Adopt the latter parameterization and work out for yourself that this leads to a second order ODE while the former one provides a conserved quantity, and subsequently immediately a first order ODE.

Problem 6.4. **Fermat's principle**



Fermat's principle states that a light beam propagates along a path minimizing the flight time. When passing from air into glass it changes direction according to Sellius' refraction law. Here, we consider a setting where the beam starts in air at the position, $(x, y) = (0, 0)$, to the top left in the figure, with coordinates where $\hat{x}$ points downwards and $\hat{y}$ to the right. The path of the light is described by a function $y(x)$. We require that beam passes from air into the glass at the position (a, u) such that it will eventually proceed through the prescribed position (b, w) in the glass. The speed of light in air and in glass will be denoted as c_A and c_G, respectively.

a) Show that the time of flight T for a (hypothetical) trajectory $y(x)$ with derivative $y'(x)$ can be determined as follows

$$T = c_A^{-1} \int_0^a \mathrm{d}x \sqrt{1 + (y'(x))^2} + c_G^{-1} \int_a^b \mathrm{d}x \sqrt{1 + (y'(x))^2} \,.$$

b) In the following we consider a glass body with a planar surface, and align the coordinates such that glass surface is aligned parallel to the y-axis. Hence, we know that the light passes from air to glass at the fixed position a, but we still have to determine u. Determine δT for a variation $y(x) + \delta y(x)$ of the trajectory. What does this imply for $\delta y(x)|_{x=0}$, $\delta y(x)|_{x=a}$ and $\delta y(x)|_{x=b}$? What does it imply for the boundary terms that arise from the integration by parts, when determining δT?

c) Show that the beam must go in a straight line in air and in glass. Show that this implies that

$$T(u) = \frac{1}{c_A} \sqrt{u^2 + a^2} + \frac{1}{c_G} \sqrt{(w - u)^2 + (b - a)^2} \,.$$

Derive Snellius' law from the condition that $0 = \mathrm{d}T(u)/\mathrm{d}u$.

d) Snellius' Law can also be directly obtained from Fermat's principle. How?

Problem 6.5. **Stability of soap films**
When a soap film is suspended between two rings, it takes a cylinder-symmetric shape of minimal surface area. We discuss here the form of the film for rings of radius R_0 and R_1 positioned at the height x_0 and x_1, respectively. At the Mathematikum in Gießen there is a nice demonstration experiment: x_0 is the surface height of soap solution in a vessel around the platform where the children are standing, and x_1 is the height of the ring pulled upwards by the children.

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http://mathematikum.df-kunde.de/Wanderausstellung/index.php?m=2&la=de&id=314

a) Let $w(x)$ be the radius of the cylinder-symmetric soap films at the vertical position x. Sketch the setup and mark the relevant notations for the problem.

b) Show that the surface area A of the soap film takes the form

$$A = \int_{x_0}^{x_1} \mathrm{d}x \, w(x) \, f(w'(x)) \, ,$$

Here, the factor $f(w'(x))$ takes into account that the area is larger when the derivative $w'(x) = \mathrm{d}x/\mathrm{d}x$ increases. Determine the function $f(w'(x))$ in this expression.

c) Show that A is extremal for shapes $w(x)$ that obey the differential equation

$$w''(x) = \frac{1 + (w'(x))^2}{w(x)} \, .$$

d) Determine the solutions of the differential equation.
Hint: Rewrite the equation into the form

$$\frac{w'(x) \, w''(x)}{1 + (w'(x))^2} = \frac{w'(x)}{w(x)} \, .$$

e) Consider now solutions with $-x_0 = x_1 = a$ and $R_0 = R_1 = R$, and denote the radius at the thinnest point of the soap film as w_0. Show that w_0 is the solution of

$$\frac{R}{a} = \frac{w_0}{a} \cosh \frac{a}{w_0} \, .$$

f) Sketch R/a as function of a/w_0. For given R and a you can then find w_0. For small separation of the rings you should find two solutions. What happens when one slowly rises the ring? Will an adult ever manage to pull up the ring to head height before the film ruptures?

6.3 *Dynamics with one degree of freedom*

We will now illustrate the application of the Lagrange formalism for three examples with a single degree of freedom of the motion:

1. The mathematical pendulum, Example 6.1, will give a first idea of how to find EOMs with the Lagrange formalism. This EOM can also easily be found by other approaches. It serves here to illustrate problems where one adopt 2D polar coordinates.

2. The motion of a pearl on a rotating ring constitutes a system with an explicit time dependence. In that case the Lagrange formalism dramatically simplifies the the derivation of the EOM. We will also discuss for this model how to account for dissipative forces and we will see how the solutions of a problem can change qualitatively upon varying a parameter. In this example we will adopt spherical coordinates.

3. The motion of a weight on a carousel we will discussed as an example of a system that needs a dedicated treatment for the description of admissible positions. The discussion will be based on cylindrical coordinates.

6.3.1 *The EOM for the mathematical pendulum*

The parameterization introduced in Example 6.1 provides the kinetic energy

$$T = \frac{M}{2}\,\dot{\boldsymbol{x}}^2 = \frac{M}{2}\,L^2\,\dot\theta^2\,\hat{\boldsymbol{\theta}}(\theta(t))^2 = \frac{M}{2}\,L^2\,\dot\theta^2$$

and the potential energy in the gravitational field

$$U = -M\boldsymbol{g}\cdot\boldsymbol{x} = -M\,L\,\hat{\boldsymbol{R}}\big(\theta(t)\big)\cdot\boldsymbol{g} = -M\,L\,g\cos\theta(t)$$

since $\boldsymbol{g} = g\,\hat{\boldsymbol{R}}(0)$.

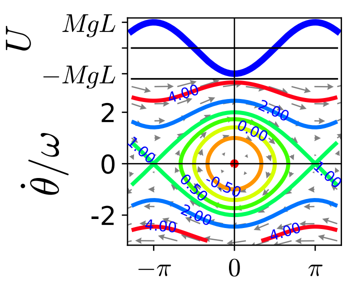
Consequently,

$$\begin{aligned}\mathcal{L} &= \frac{M}{2}\,L^2\,\dot\theta^2 + M\,g\,L\,\cos\theta(t)\\ \Rightarrow\quad M\,L^2\,\ddot\theta(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta} = -M\,g\,L\,\sin\theta(t)\\ \Rightarrow\quad \ddot\theta(t) &= -\frac{g}{L}\,\sin\theta(t)\end{aligned} \tag{6.3.1}$$

The EOM (6.3.1) can be integrated once by multiplication with $2\dot\theta(t)$

$$\begin{aligned}\dot\theta^2(t) - \dot\theta^2(t_0) &= \int_{t_0}^{t}\mathrm{d}t\,2\dot\theta\ddot\theta = \int_{t_0}^{t}\mathrm{d}t\,2\dot\theta\left(-\frac{g}{L}\,\sin\theta(t)\right)\\ &= 2\int_{\theta(t_0)}^{\theta(t)}\mathrm{d}\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{g}{L}\,\cos\theta\right) = 2\,\frac{g}{L}\,\big(\cos\theta(t) - \cos\theta(t_0)\big)\end{aligned}$$

This is a Mattheiu differential equation. For most initial conditions it can not be solved by simple means. However, the first integral provides the phase-space trajectories $\dot\theta(\theta)$ for every given set of initial conditions $\big(\theta(t_0), \dot\theta(t_0)\big)$,



$$\dot\theta = \pm\sqrt{\dot\theta^2(t_0) + \frac{2g}{L}\,\bigl(\cos\theta(t) - \cos\theta(t_0)\bigr)}$$

The phase-space portrait is shown in Figure 6.5. There are trivial solutions where the pendulum is resting without motion at its stable and unstable rest positions $\theta = 0$ and $\theta = \pi$, These positions are denoted as *fixed points* of the dynamics. There are closed circular trajectories close to the minimum, $\theta = 0$, of the potential where it is harmonic to a good approximation. These are solutions with energies $0 < 1 + E/MgL \lesssim 1$.

For larger amplitudes the amplitude of the swinging grows, and the circular trajectories get deformed. When E approaches MgL the phase-space trajectories arrive close to the tipping points $\theta = \pm\pi$ where they form very sharp edges. For θ close to $\theta = \pm\pi$ the trajectories look like the hyperbolic scattering trajectories for the potential $-a\,x^2/2$ that was discussed in Problem 4.25. When the non-dimensional energy is exactly one, the pendulum starts on top, goes through the minimum and returns to the top again. Apart from the fixed points, this is the only case where the evolution can be obtained in terms of elementary functions. For the initial condition $\dot\theta(t_i) = 0$ and $\cos\theta_i = -1$ we find

$$\omega^{-1}\,\dot\theta_H(t) = \pm\sqrt{2 + 2\,\cos\theta_H(t)} = \pm 2\,\cos\frac{\theta_H(t)}{2}$$

The same equation is also obtained for the initial condition $\theta_0 = 0$ and $\dot\theta(t_0) = \sqrt{2g/L}$ half-way on the way from the top back to the top. For this initial condition the ODE for $\dot\theta_H$ can be integrated, and we find

$$\pm 2\,\omega\,(t - t_0) = \int_0^{\theta(t)} \frac{\mathrm{d}\theta}{\cos\frac{\theta}{2}} = \ln\tan\frac{\theta+\pi}{4}$$

$$\Rightarrow \qquad \theta_H(t) = -\pi + 4\,\arctan \mathrm{e}^{\pm 2\omega\,(t-t_0)} \tag{6.3.2}$$

The $\pm$ signs account for the possibility that the pendulum can move clockwise and counterclockwise. The counterclockwise trajectory is shown in Figure 6.6. In the limit $t \to -\infty$ it starts in the unstable fixed point $\theta = -\pi$. It falls down till it reaches the minimum $\theta = 0$ at time t_0, and then it rises again, reaching the maximum $\theta = \pi$ for time $t \to \infty$. Such a trajectory is called a homocline.

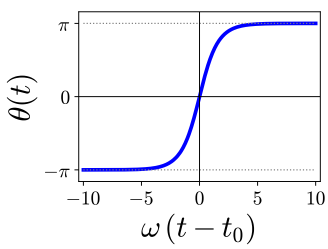


Figure 6.6: Anticlockwise moving heterocline for the mathematical pendulum.

Definition 6.6: Homoclines and Heteroclines

Homoclines and *heteroclines* are trajectories that approach a fixed points of a dynamics in their infinite past and future. A homocline returns to the same fixed point from where it started. A heterocline connects two different fixed points.

The take-home message of this example is that the minima and maxima of a potential organize the phase space flow. Close to each minimum a conservative system will have closed trajectories that represent oscillations in a potential well. The well is confined by maxima to the left and right of the minimum of the potential. When these maxima have different height there is a homoclinic orbit coming down from and returning to the shallower maximum. When they have the same height, they are connected by heteroclinic orbits. Thus, the homoclines and heteroclines divide the phase space into different domains. Initial conditions within the same domain show qualitatively the same dynamics. Initial conditions in different domains feature qualitatively different dynamics. For the mathematical pendulum the heteroclines divide phase space into three domains, up to the 2π translation symmetry of θ:

a) There are trajectories oscillating around $\theta = 0$, with energies smaller than MgL. The region of these oscillations is bounded by the heteroclines provided in Equation (6.3.2).

b) Trajectories with initial conditions lying above the anticlockwise moving heterocline will persistently rotate anticlockwise and never reverse their motion.

c) Trajectories with initial conditions lying below the clockwise moving heterocline will persistently rotate clockwise and never reverse their motion.

The general strategy for sketching phase-space plots is summarized in the following algorithm and illustrated in Figure 6.7.

Algorithm 6.2: Phase space plots

a) Identify the minima and maxima of the potential. Mark the minima as (marginally) stable fixed points with velocity zero. Mark the maxima as unstable fixed points with velocity zero.

b) Identify the fate of trajectories departing from the unstable fixed points. Identify to this end the closest positions on the potential that have the same height as the maximum. When it is another extremum the orbit will form an heterocline. Otherwise, it will be reflected and return to the initial maximum, forming a homocline. If there is no further point of the same height, the trajectory will escape to infinity.

c) Add characteristic trajectories close to the minima, in between and outside of homo- and heteroclines.

In these steps it is advisable to

- Observe the symmetries of the system. To the very least the plot is symmetric with respect to reflection at the horizontal axis, i. e. swapping the sign of the velocity.
- Observe energy conservation (if it applies): The modulus of the velocity takes a local minimum for a maximum of

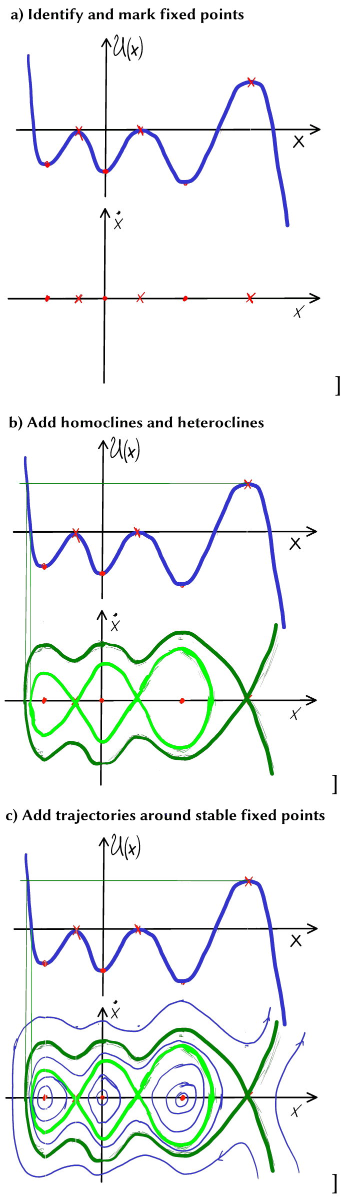


Figure 6.7: Step by step sketch of a phase-space plot.

the potential, and a local maximum for a minimum of the potential.

6.3.2 *The EOM for a pearl on a rotating ring*

We consider a pearl of mass M that can freely move on a ring. The ring is mounted vertically in the gravitational field and it spins with angular velocity Ω around its vertical symmetry axis. The setup constrains the position of the pearl to lie on a spherical shell. The position of the pearl on the ring is fully described by the angle $\theta(t)$ of the deflection of the pearl from the direction of gravity (see Figure 6.8. In addition we must specify the orientation of the ring. This will be done by the angle $\varphi(t) = \Omega t$ that enters as a parameter in this time-dependent problem. Hence, the position of the pearl is most conveniently specified in terms of polar coordinates (R, θ, φ) where R takes the constant value ℓ, and $\varphi(t) = \Omega t$ enters as a time-dependent parameter.

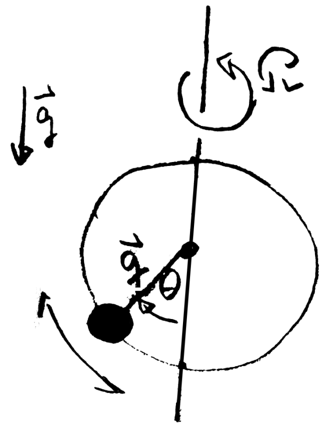


Figure 6.8: Motion of a pearl moving on a ring rotating with a fixed frequency Ω.

Theorem 6.3: Basis vectors for spherical coordinates

Let $\{\hat{x}, \hat{y}, \hat{z}\}$ be a fixed Cartesian basis of $\mathbb{R}^3$, and (R, θ, φ) be the spherical coordinates associated to a point $(x, y, z) \in \mathbb{R}^3$. Then

- $R = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin
- $\theta = \arctan(\sqrt{x^2 + y^2}/z)$ is the angle with respect to $\hat{z}$,
- $\varphi = \arctan(y/x)$ the angle with respect to $\hat{x}$ of the projection of the position into the (x, y) plane.

We denote the vector from the origin to (R, θ, φ) as $R\,\hat{\mathbf{R}}(\theta, \varphi)$. Then the following statements apply

a) $\hat{\boldsymbol{\theta}} = \partial_\theta \hat{\mathbf{R}}$ is a vector pointing along the grant circle of the unit sphere selected by (θ, φ) and $\hat{z}$

b) $\hat{\boldsymbol{\varphi}} = \hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}}$ is a vector tangential to the unit sphere at (θ, φ) and vertical to $\hat{z}$

c) For every $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ the vectors $\{\hat{\mathbf{R}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\}$ form a right-handed orthonormal basis of $\mathbb{R}^3$.

add sketch of unit vectors for spherical coordinates

Remark 6.5. The directions of the basis vectors $\{\hat{\mathbf{R}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\}$ depend on θ and φ.

Theorem 6.4: Derivatives of the basis vectors for spherical coordinates

The partial θ and φ derivatives of the basis vectors $\{\hat{\boldsymbol{R}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\}$ obey the following relations

$$\begin{aligned} \partial_\theta \hat{\boldsymbol{R}} &= \hat{\boldsymbol{\theta}} & \partial_\varphi \hat{\boldsymbol{R}} &= \sin\theta\, \hat{\boldsymbol{\varphi}} \\ \partial_\theta \hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{R}} & \partial_\varphi \hat{\boldsymbol{\theta}} &= \cos\theta\, \hat{\boldsymbol{\varphi}} \\ \partial_\theta \hat{\boldsymbol{\varphi}} &= 0 & \partial_\varphi \hat{\boldsymbol{\varphi}} &= -\sin\theta\, \hat{\boldsymbol{R}} - \cos\theta\, \hat{\boldsymbol{\theta}} \end{aligned} \tag{6.3.3}$$

Proof. The proofs of Theorems 6.3 and 6.4 are given as Problem 6.6. □

In polar coordinates the position of the pearl is

$$\boldsymbol{x}(t) = \ell\, \hat{\boldsymbol{R}}\big(\theta(t), \varphi(t)\big)$$

We adopt the Lagrange formalism to determine the equation of motion for $\theta(t)$, which is the only coordinate in this setting. (The motion of the pearl has a single degree of freedom.)

The potential energy takes the same form as for the pendulum,

$$U = -M\, \boldsymbol{g} \cdot \boldsymbol{x} = -M\, g\, \ell\, \cos\theta(t)\,.$$

The kinetic energy is obtained based on its velocity

$$\begin{aligned} \dot{\boldsymbol{x}} &= \ell\, \frac{\mathrm{d}}{\mathrm{d}t} \hat{\boldsymbol{R}}(\theta(t), \Omega t) = \ell\, \dot\theta\, \partial_\theta \hat{\boldsymbol{R}}(\theta(t), \Omega t) + \ell\, \Omega\, \partial_\varphi \hat{\boldsymbol{R}}(\theta(t), \Omega t) \\ &= \ell\, \dot\theta\, \hat{\boldsymbol{\theta}}\big(\theta(t), \Omega t\big) + \ell\, \Omega\, \sin\theta(t)\, \hat{\boldsymbol{\varphi}}\big(\theta(t), \Omega t\big) \end{aligned}$$

which provides the Lagrange function

$$\mathcal{L}(\theta, \dot\theta) = \frac{M}{2}\, \ell^2\, \dot\theta^2 + \frac{M}{2}\, \ell^2\, \Omega^2\, \sin^2\theta(t) + M\, g\, \ell\, \cos\theta(t)$$

It only differs from the expression for the spherical pendulum by the fact that $\varphi(t)$ is not a coordinate whose evolution must be determined from an EOM. Rather it is a parameter $\varphi(t) = \Omega\, t$ provided by the setting of the problem.

The motion only has a single DOF, $\theta(t)$, with EOM

$$\ddot\theta(t) = -\frac{g}{\ell}\, \sin\theta(t)\, \left(1 - \frac{\ell\, \Omega^2}{g}\, \cos\theta(t)\right) \tag{6.3.4}$$

This EOM can once be integrated by the same strategy adopted for the swing and the spherical pendulum. Thus, one finds the effective potential[5]

[5] A comprehensive discussion of the notion of an effective potential will be provided in Definition 6.11.

$$U_{\text{eff}}(\theta) = -\omega^2\, \cos\theta\, \left[1 - \left(\frac{\Omega}{\omega}\right)^2\, \cos\theta\right] \tag{6.3.5}$$

Figure 6.9 shows the effective potential and phase space portraits for different values of angular momentum, i.e. of the dimensionless control parameter $\kappa = \Omega/\omega$. For $\kappa < 1$ the phase space has the

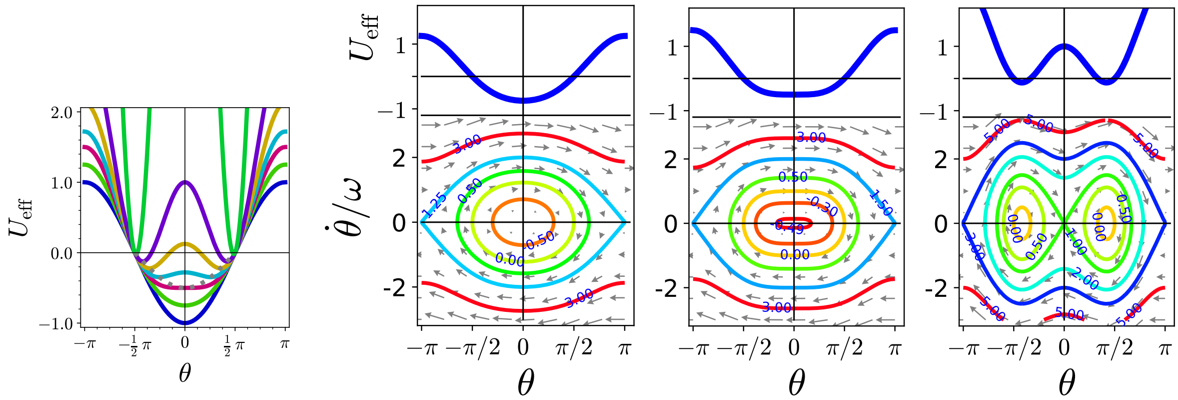


Figure 6.9: The left panel shows the effective potential for the pearl on a ring for parameter values $(\Omega/\omega) \in \{0, 2^{-1/2}, 1, 1.2, 1.5, 2, 5\}$ from bottom to top. The subsequent panels show phase-space portraits of the motion for $\Omega/\omega = 2^{-1/2}$, 1, and 2, respectively.

same structure as that of a mathematical pendulum, with a stable fixed point at $\theta = 0$. When κ passes through one, this minimum of U_{eff} turns into a maximum, and two new minima emerge at the positions

$$\theta_c = \pm \arccos \kappa^{-2} = \pm \arccos \left(\frac{\omega}{\Omega}\right)^2 \qquad (6.3.6)$$

that are indicated by a dotted gray line in the left panel of Figure 6.9. The new maximum at zero is always shallower than the maxima at $\pm\pi$. Hence, it gives rise to two homoclinic orbit that wind around the new stable fixed points. The maxima at $\pm\pi$ will further we connected by heteroclinic orbits. Hence, phase space is divided into five distinct regions. For energies smaller than $U_{\mathrm{eff}}(\theta = 0)$ the trajectories wiggle around one of the stable fixed points. They stay on one side of the ring and oscillate around the angle θ_c. There are two regions of this type because the pearl can stay on both sides of the ring. For $U_{\mathrm{eff}}(\theta = 0) < E < U_{\mathrm{eff}}(\theta = \pi)$ the trajectories show oscillations back and forth between the two sides of the ring, For $E > U_{\mathrm{eff}}(\theta = \pi)$ they rotate around the ring in clockwise or counter-clockwise direction for $\dot\theta < 0$ or $\dot\theta > 0$, respectively.

There are two take-home message from this example:

1. There are no conservation laws in the dynamics when there are explicitly time-dependent constraints. Hence, the strategies of Chapter 4 to establish and discuss the EOM can no longer be applied. However, the Lagrange formalism still provides the EOM in a straightforward manner.

2. In general, the structure of the phase-space flow changes upon varying the dimensionless control parameters of the dynamics. These changes are called bifurcations, and they are a very active field of contemporary research in theoretical mechanics. The pearl on the ring features a pitchfork bifurcation since the positions of the fixed points resemble the shape of a pitch fork (see Figure 6.10).

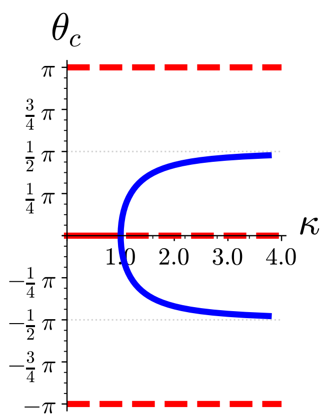


Figure 6.10: Parameter dependence of the positions of the fixed points of the rotation governor. Solid lines mark stable fixed points, and unstable fixed

Definition 6.7: Pitchfork bifurcation

Upon variation of a parameter p a stable fixed point x_* of a one-dimensional ODE may loose stability at some parameter p_c, turning into an unstable fixed point. In that case one will encounter one of the following scenarios:

supercritical pitchfork bifurcation At p_c two new stable fixed points emerge to its left and right of x_*.

subcritical pitchfork bifurcation There were unstable fixed points to the left and right of x_* that merge with x_* at the parameter p_c. Subsequently there is only a single unstable fixed point.

Remark 6.6. The supercritical pitchfork bifurcation is commonly simply denoted as pitchfork bifurcation.

Remark 6.7. In classical mechanics a pitchfork bifurcation, Figure 6.10, emerges whenever a minimum of a potential is dented to turn it into a maximum that is then surrounded minima on both sides, as shown in Figure 6.9.

The supercritical pitchfork bifurcation refers to a situation where we initially have a minimum surrounded by two maxima, and then the minimum is pushed until it disappears. When this happens the maxima approach each other, and eliminate the dent in between. An example is provided in Problem 6.8.

add fig for supercritical pitchfork

6.3.3 *Centrifugal Governor*

The sharp increase of θ_c in Equation (6.3.6), when the rotation frequency ω rises beyond Ω is used in a feedback mechanism of the governor to control the rotation speed of steam engines (Figure 6.11).

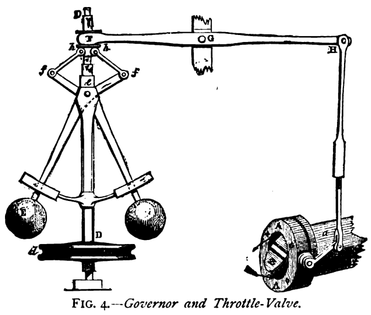


Figure 6.11: Rotational governor and throttle valve. When the rotation speed exceeds a critical value the weights move outward and the arm opens a valve that reduces pressure in the steam engine. (Image from "Discoveries & Inventions of the Nineteenth Century" by R. Routledge, 13th edition, published 1900, Public domain, via Wikimedia Commons)

Oscillations around the stable fixed points is an undesirable feature of the governor such that some dissipation is welcome. We revisit Equation (6.2.1) to extend the Lagrange formalism for forces that do not derive from a potential

$$
\begin{aligned}
0 &= -\int_{t_I}^{t_F} \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} \left(\delta \boldsymbol{x} \cdot m \, \dot{\boldsymbol{x}} \right) = -\int_{t_I}^{t_F} \mathrm{d}t \, \left(\delta \dot{\boldsymbol{x}} \cdot m \, \dot{\boldsymbol{x}} + \delta \boldsymbol{x} \cdot m \, \ddot{\boldsymbol{x}} \right) \\
&= -\int_{t_I}^{t_F} \mathrm{d}t \, \left(\delta \dot{\boldsymbol{x}} \cdot \nabla_{\dot{\boldsymbol{x}}} \mathcal{L} + \delta \boldsymbol{x} \cdot m \left(\boldsymbol{F}_d - \nabla_{\boldsymbol{x}} \Phi \right) \right) \\
&= \int_{t_I}^{t_F} \mathrm{d}t \, \delta \boldsymbol{x} \, \left(-\boldsymbol{F}_d + \nabla_{\boldsymbol{x}} \mathcal{L} - \frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\dot{\boldsymbol{x}}} \mathcal{L} \right)
\end{aligned}
$$

Thus, an additional dissipative force $\boldsymbol{F}_d = -\gamma \, \dot{\theta} \, \hat{\boldsymbol{\theta}}$ will give rise to an additional additive term in Equation (6.3.4) such that the rotational governor has an EOM

$$
\ddot{\theta}(t) = -\frac{g}{\ell} \, \sin\theta(t) \, \left(1 - \frac{\ell \, \Omega^2}{g} \, \cos\theta(t) \right) - \frac{\gamma}{M} \, \dot{\theta}(t)
$$

and the energy evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\theta}^2}{2} + U_{\mathrm{eff}}(\theta)\right) = -\frac{\gamma}{M}\,\dot{\theta}^2(t)$$

Small friction deflects the trajectories towards smaller values of the effective potential, until the system comes to rest in a stable fixed point. For $\Omega/\omega = 2$ the impact of dissipation $\gamma/M = 0.2$ and 2.0 is shown in Figure 6.12.

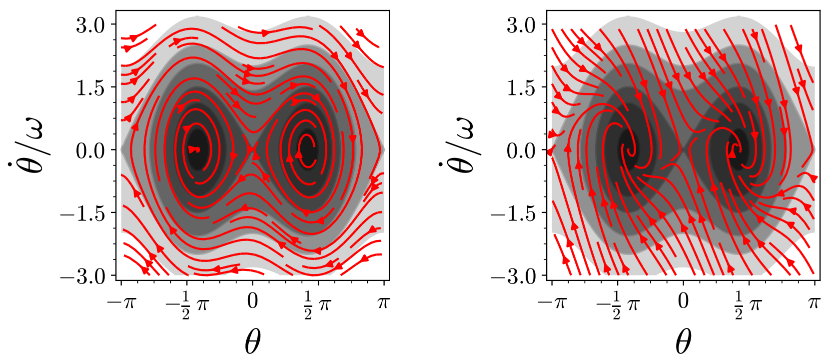


Figure 6.12: Phase-space plot for a rotational governor with rotation frequency $\Omega = 2\omega$. The colors of gray in the background show contour levels of the energy. The streamlines indicated the evolution of the dynamics for (left) weak dissipation, $\gamma = 0.2$, and (right) strong dissipation, $\gamma = 2$, dissipation. Due to dissipation the trajectories acquire a component downwards in energy.

6.3.4 Carousel

The positions in the systems that we treated so far were described in terms of polar coordinates (mathematical pendulum, Section 6.3.1) and spherical coordinates (pearl on a rotating ring, Section 6.3.2). We will now address a system where we adopt *cylindrical coordinates* to describe particle positions: the motion of the beats of a toy carousel that is shown in Figure 6.13. The carousel is composed of four cantilever beams of length R that extend outwards from a vertical axis that is rotating with angular velocity Ω. At the far end of each beam there is a pendulum attached that freely swings in outward direction. The inclination of the pendulum arm towards gravity will be denoted as θ. (Oscillations parallel to the motion of the beams are not be considered.) The pendulum arm has a length L and it carries a weight m. Due to a magnetic contact the pendulum experiences minimal friction in its motion. Henceforth we the focus on the motion of one of the beats.

We pick the origin of the coordinate system on the ration axes right on the height of the cantilever. The rotation is around the vertical axes characterized the the unit vector $\hat{z}$. Looking from the top (right panel of Figure 6.13) the pendulum arm sticks out in direction $\varphi = \Omega t$. We adopt polar coordinates, as introduced in Remark 6.3, in the horizontal plane vertical to $\hat{z}$. We denote the position of the fulcrum of the pendulum as $\boldsymbol{R} = R\,\hat{\boldsymbol{r}}(\varphi)$, and express the vector from the fulcrum to the weight as $L\,\sin\theta\,\hat{\boldsymbol{r}}(\varphi) - L\,\cos\theta\,\hat{\boldsymbol{z}}$.

add sketch of unit vectors for cylindrical coordinates

This amounts to a representation of the position of the mass in terms of *cylindrical coordinates* (see **??**).

Theorem 6.5: Basis vectors for cylindrical coordinates

Let $\{\hat{x}, \hat{y}, \hat{z}\}$ be a basis of $\mathbb{R}^3$, and $(R = \sqrt{x^2+y^2}, \varphi = \tan(y/x), z)$ be the cylindrical coordinates associated to a point with Cartesian coordinates (x, y, z). Then

- $R = \sqrt{x^2+y^2}$ is the distance of the point from the $\hat{z}$ axis
- $\varphi = \arctan(y/x)$ the angle with respect to $\hat{x}$ of the projection of the position into the (x, y) plane.

We denote the vector from the origin to (R, φ, z) as $\boldsymbol{x} = z\,\hat{\boldsymbol{z}} + R\,\hat{\boldsymbol{R}}(\varphi)$. Then, the following statements apply

a) $\hat{\boldsymbol{\varphi}} = \partial_\varphi \hat{\boldsymbol{R}}$ and $\hat{z}$ are the horizontal and vertical unit vectors tangential to the surface of a cylinder with axis pointing along $\hat{z}$

b) For every $\varphi \in [0, 2\pi)$ the vectors $\{\hat{\boldsymbol{R}}(\varphi), \hat{\boldsymbol{\varphi}}(\varphi), \hat{z}\}$ form a right-handed orthonormal basis of $\mathbb{R}^3$.

The φ-derivatives of $\hat{\boldsymbol{R}}$ and $\hat{\boldsymbol{\varphi}}$ follow the same rules as for polar coordinates, Theorem 6.1.

Proof. The proof is left to the reader. □

The position $\boldsymbol{x}$ and the velocity $\dot{\boldsymbol{x}}$ of the weight attached to the carousel are

$$\boldsymbol{x} = (R + L\,\sin\theta)\,\hat{\boldsymbol{r}}(\Omega t) - L\,\cos\theta\hat{\boldsymbol{z}}$$
$$\dot{\boldsymbol{x}} = (R + L\,\sin\theta)\,\Omega\,\hat{\boldsymbol{\varphi}}(\Omega t) + L\,\cos\theta\,\dot{\theta}\,\hat{\boldsymbol{r}}(\Omega t) + L\,\sin\theta\,\dot{\theta}\hat{\boldsymbol{z}}$$

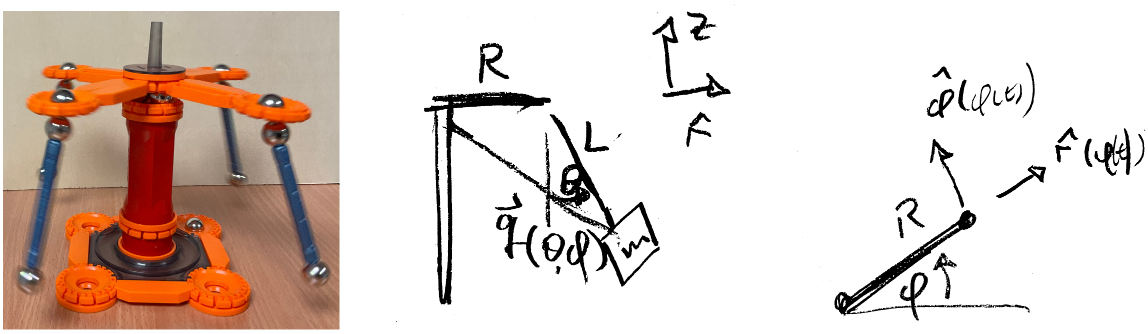


Figure 6.13: Experimental setup and description of configurations for a toy carousel.

The kinetic energy and potential energy are

$$T = \frac{m}{2}\,\dot{x}^2 = \frac{m}{2}\left[L^2\,\dot\theta^2 + (R + L\,\sin\theta)^2\,\Omega^2\right]$$

$$V = -mgL\,\cos\theta$$

and the Euler-Lagrange equation for $\theta(t)$ take the form

$$m\,L^2\,\ddot\theta(t) = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta} = m\,\Omega^2\,(R + L\,\sin\theta)\,\cos\theta - mgL\,\sin\theta$$

We introduce

- the eigenfrequency of the hanging arm, $\omega = \sqrt{g/L}$
- the ratio of frequencies, $\tau = \Omega/\omega$
- the ratio of the length of the arms, $\lambda = R/L$

and absorb ω into the dimensionless time scale. Thus, we find

$$\ddot\theta = \tau^2\,(\lambda + \sin\theta)\,\cos\theta - \sin\theta$$

which admits a conserved energy-like quantity

$$\mathcal{E} = \frac{\dot\theta^2}{2} + U_{\text{eff}}(\theta)$$

with an effective potential

$$U_{\text{eff}}(\theta) = -\frac{\tau^2}{2}\,(\lambda + \sin\theta)^2 - \cos\theta \tag{6.3.7}$$

The left panel of Figure 6.14 shows the effective potential for a fixed ratio $L/R = 4$ and different values of Ω/ω. For small frequencies, $\Omega/\omega = 0.2$, the masses are pushed outwards such that the equilibrium position is no longer at $\theta = 0$. Otherwise, the phase space plot looks like the one of a mathematical pendulum. For increasing $\tau = \Omega/\omega$ a shoulder emerges in the potential, and for $\tau > \tau_c \simeq 1.5$ this leads to the emergence of a new minimum with $-\pi/2 < \theta_- < 0$. It is separated from the previous minimum by a maximum at θ_+ with $-\theta_- \gg -\theta_+ > 0$. For $\tau > \tau_c$ there are two stable fixed points that lie in regions surrounded by homoclinic trajectories that start and end at θ_+. Further outside there are oscillating trajectories that move around both fixed points, and beyond the heteroclinic trajectories that connect the maxima of the potential one finds trajectories that keep rotating in the same direction.

When increasing the rotation frequency beyond τ_c a second stable solution emerges in the system. To find the parameters where it emerges we observe that fixed points emerge at positions θ where the effective force vanishes

$$\sin\theta = \tau^2\,(\lambda + \sin\theta)\,\cos\theta$$

$$\Rightarrow\quad \lambda_\tau(\theta) = \frac{\tan\theta}{\tau^2} - \sin\theta$$

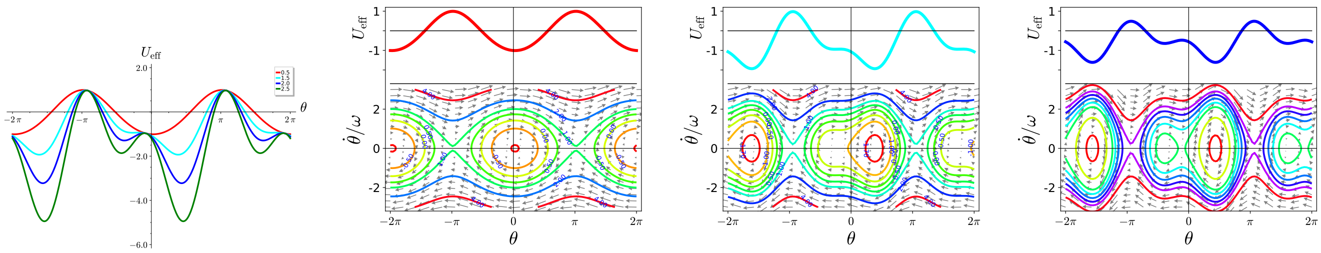


Figure 6.14: The effective potential (left) and phase space plots for the parameter values $\Omega/\omega = 0.5$, 1.5 and 2.0 (from left to right).

This equation is solved for a unique angle θ_c when $\lambda_\tau(\theta)$ is monotonic, and potential extrema of $\lambda_\tau(\theta)$ must fulfill

$$0 = \frac{d\lambda}{d\theta} = \frac{1}{\tau^2}\,\frac{1}{\cos^2\theta_c} - \cos\theta_c \qquad \Rightarrow \qquad \cos\theta_c = \tau^{-2/3}$$

Consequently,

for $\tau < 1$ the function $\lambda_\tau(\theta)$ is monotonous such that there is a unique fixed point. It is a minimum.

for $\tau > 1$ the function $\lambda_\tau(\theta)$ has a maximum and a minimum such that there can be up to three fixed point: two minima and a maximum.

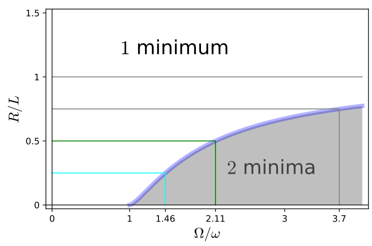


Figure 6.15: Phase diagram with positions of the bifurcations as function of τ and λ.

The range values $\lambda = R/L$ where there are two minima is bounded by the extrema $\lambda_\tau(\theta_c)$,

$$\lambda_c(\theta_c) = \sin\theta_c\left(\frac{1}{\tau^2\cos\theta_c} - 1\right) = \mp\sqrt{1-\cos^2\theta_c}\left(\frac{1}{\tau^2\cos\theta_c} - 1\right)$$
$$= \pm\left(1-\tau^{-4/3}\right)^{3/2}$$

Hence, there is a single maximum for $\tau = \Omega/\omega < 1$ and when $R/L > \left(1-\tau^{-4/3}\right)^{3/2}$ for $\tau > 1$. In Figure 6.15 this conclusion is visualized in a phase diagram where this aspect of the behavior is marked in the parameter plane of the problem, i. e. as function of $\tau = \Omega/\omega$ and $\lambda = R/L$.

Definition 6.8: Phase Diagram

A phase diagram for some property of a dynamics is a plot that shows for which parameters this property can be observed.

It is illuminating to observe that the motion of the pearl on the rotating ring is recovered as the special case $R = 0$ of the carousel dynamics. For $\lambda = 0$ the effective potential, Equation (6.3.7), has three critical points for $\tau > 1$, and two of them disappear in a pitchfork bifurcation when they all meet at $\theta = 0$ for $\tau = 1$. In terms of the condition on the forces this happens when at $\theta = 0$ the slope of $\tau^2 \sin\theta$ becomes larger than the one of $\tan\theta$. For $\tau > 0$

there will be three intersections of the two functions rather than a single one.

The scenario for $\lambda > 0$ is different. This is most easily seen by considering a value $\tau > 1$ and increasing λ. For $\lambda = 0$ there are intersections of $\tau^2 \sin\theta$ and $\tan\theta$ at $\theta \in \{0, \pm\theta_c\}$. Increasing λ amounts to a vertical displacement of the sine function. The critical points that are initially at $\pm\theta_c$ will then move right and the one in the middle moves to the left because $\lambda + \sin\theta > 0$ for $\lambda > 0$ and $\theta = 0$. Therefore, the minimum to the right will persist, and the minimum to the left will at some point merge with the maximum and they will annihilate. Such a scenario is called a saddle-node bifurcation.

Definition 6.9: Saddle-node bifurcation

We consider a one-dimensional ODE for a real variable x that depends on a parameter p. In a certain range of parameters $p \in \mathbb{U}$ the systems has a stable fixed point at x_s and an unstable fixed point at x_u. A *saddle-node bifurcations* emerges at $p_* \in \partial\mathbb{U}$ when $x_u(p)$ and $x_s(p)$ meet and annihilate at $x_* = x_u(p_*) = x_x(p_*)$.

Remark 6.8. In classical mechanics a saddle-node bifurcation emerges when the height difference between a minimum and a neighboring maximum of a potential diminishes, becomes zero in a saddle point, and subsequently, the potential will no longer have no critical point in the considered region.

6.3.5 *Self Test*

Problem 6.6. **Basis vectors for spherical coordinates**

a) Verify by explicit calculation that $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$ obey the relations

$$\hat{\boldsymbol{\theta}} = \frac{\partial \hat{\boldsymbol{R}}}{\partial \theta} \qquad \text{and} \qquad \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{R}} \times \hat{\boldsymbol{\theta}}\,,$$

and that they form an orthonormal basis.

b) How is $\hat{\boldsymbol{\varphi}}$ related to $\partial\hat{\boldsymbol{R}}/\partial\varphi$?

c) Verify also the other expressions, Equation (6.3.3), for the partial derivatives of the basis vectors.

Problem 6.7. **Phase-space analysis for a pearl on a rotating ring**

a) Evaluate $\dot{\boldsymbol{x}}(t) = \ell\, \dot{\hat{\boldsymbol{R}}}(\theta(t), \Omega t)$ based on the relations established in Problem 6.6.

b) Determine the kinetic energy T and the potential energy V of the pearl.

c) Fill in the steps in the derivation of the EOM for θ, as provided in Equation (6.3.4).

d) Verify that neither the energy nor any of the coordinates of momentum and angular momentum are conserved for this motion.

Problem 6.8. **An EOM with a subcritical pitchfork bifurcation**
Consider the equation of motion

$$\ddot{x} = x\left(x^2 - 1\right)\left(x^2 - a\right)$$

where the right-hand side is considered as a dimensionless force on a particle that resides at the dimensionless position $x \in \mathbb{R}$. The dynamics depends on the parameter $a \in \mathbb{R}$.

a) Determine the fixed points of the dynamics, and verify that $x = 0$ is a stable fixed point for $a < 0$ and an unstable fixed point for $a > 0$.

b) Verify that the dynamics has a subcritical pitchfork bifurcation at $a = 0$.

c) Sketch the bifurcation diagram, i.e. a plot analogous to Figure 6.10 where the positions of the fixed points are indicated as function of a. Use solid lines for stable fixed points and dotted lines for unstable fixed points.

d) Sketch the form of the potential for dynamics with $a < 0$, $0 < a < 1$, and $a > 1$.

Problem 6.9. **An EOM with a saddle-node bifurcation**
Consider the equation of motion

$$\ddot{x} = (1 - x)\left(x^2 + a\right)$$

where the right-hand side is considered as a dimensionless force on a particle that resides at the dimensionless position $x \in \mathbb{R}$. The dynamics depends on the parameter $a \in \mathbb{R}$.

a) Determine the fixed points of the dynamics, and verify that the EOM has three fixed points for $a < 0$, two fixed points for $a = 1$, and a single fixed point for $a < 0$.

b) Sketch the bifurcation diagram, i.e. a plot analogous to Figure 6.10 where the positions of the fixed points are indicated as function of a. Use solid lines for stable fixed points and dotted lines for unstable fixed points.

c) Sketch the form of the potential for dynamics with $a < -1$, $-1 < a < 0$, and $a > 0$.

6.4 *Several degrees of freedom and conservation laws*

In Section 6.3 we discussed the EOM of systems with one degree of freedom. In the present section this analysis is extended to systems with two and more degrees of freedom. Again the discussion will be based on examples:

1. We first discuss uniform motion where there are no fores acting on the particle. Here, we learn about the concept of cyclic observables, and how momentum conservation is related to translation invariance.

2. Subsequently, we address the case of uniform acceleration. In this case there only is translation invariance in the directions normal to the acceleration. The according components of the momentum are still preserved.

3. For systems with harmonic, Hookian forces we explore the relation between rotation invariance and the conservation of angular momentum. Moreover, we introduce the concepts of an *effective potential* and of a *rotation barrier*.

4. Finally, we discuss the spherical pendulum. It constitutes an example with a generic, non-trivial coupling of the evolution of its different DOF. We will learn how to exploit conservation laws to decompose the dynamics into distinct dynamics with a single degree of freedom.

6.4.1 *Force-free motion*

When there are no forces acting on the particle we have

$$T = \frac{m}{2}\,\dot{\boldsymbol{x}}^2 = \frac{m}{2}\,\sum_i \dot{x}_i^2$$
$$V = 0$$

Note that the Lagrange function $\mathcal{L} = T - V$ only depends on the velocity $\dot{\boldsymbol{q}}$ and not on the position $\boldsymbol{q}$. As a consequence, the Euler-Lagrange equation for the coordinate x_i requires that the acceleration of the particle vanishes,

$$m\,\ddot{x}_i = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

Alternatively, one may also express this result in the form of momentum conservation (cf. Section 3.4.3),

$$\dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\,\dot{\boldsymbol{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\dot{\boldsymbol{x}}}\mathcal{L} = \nabla_{\boldsymbol{x}}\mathcal{L} = \boldsymbol{0}$$

Momentum conservation therefore is an immediate consequence of the Euler-Lagrange equation for systems where the Lagrange function does not depend on the coordinates.

The general case is summarized in the following definition:

Definition 6.10: Cyclic coordinates

A coordinate q_i is called *cyclic* when the Lagrange function depends only on its time derivative $\dot{q}_i$, and not on q_i. In that case the associated Euler-Lagrange equation establishes a conservation law,

$$C = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} .$$

After all

$$\frac{\mathrm{d}\, C}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} = 0 .$$

Remark 6.9. The constant value of C is determined by the initial conditions on $\dot{q}_i$ and on the other coordinates.

Based on the concepts introduced in Problem 4.35, this finding generalizes as follows to a translation-invariant dynamics of N particles

Theorem 6.6: Momentum conservation and translation invariance

Let, $\boldsymbol{q} = (\boldsymbol{q}_i, i = 1 \dots N)$, be the Cartesian coordinate vector of N particles with masses m_i that reside at the positions $\boldsymbol{q}_i(t)$. We consider a dynamics with conservative forces that derive from an translation-invariant potential $V(\boldsymbol{q}, t) = \sum_{i<j} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|, t)$.
Then the total momentum $\boldsymbol{P} = \sum_i m_i\, \dot{\boldsymbol{q}}_i$ is conserved.

Proof. The kinetic energy of the dynamics is

$$T = \sum_{i=1}^{N} \frac{m_i}{2}\, \dot{\boldsymbol{q}}_i^2$$

and by assumption $V(\boldsymbol{q}, t)$ does not depend on $\dot{\boldsymbol{q}}$. Hence,

$$\sum_k \nabla_{\dot{\boldsymbol{q}}_k} \mathcal{L} = \sum_k \nabla_{\dot{\boldsymbol{q}}_k} T = \sum_i \frac{m_i}{2} \left(\sum_k \nabla_{\dot{\boldsymbol{q}}_k} \dot{\boldsymbol{q}}_i^2 \right) = \sum_{i=1}^{N} m_i\, \dot{\boldsymbol{q}}_i = \boldsymbol{P}$$

Consequently, the time derivative of $\boldsymbol{P}$ amounts to

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_k \nabla_{\dot{\boldsymbol{q}}_k} \mathcal{L} = \sum_k \nabla_{\boldsymbol{q}_k} \mathcal{L} = - \sum_{i<j} \left(\sum_k \nabla_{\boldsymbol{q}_k} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|, t) \right)$$

since T does not involve the coordinates $\boldsymbol{q}_k$. Now, we observe that

$$\nabla_{\boldsymbol{q}_k} = |\boldsymbol{q}_i - \boldsymbol{q}_j| = \begin{cases} \dfrac{\boldsymbol{q}_i - \boldsymbol{q}_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|} & \text{for} \quad k = i \\ \dfrac{\boldsymbol{q}_j - \boldsymbol{q}_i}{|\boldsymbol{q}_i - \boldsymbol{q}_j|} & \text{for} \quad k = j \\ 0 & \text{else} \end{cases}$$

We denote the derivative of $\Phi(d, t)$ with respect to its d as $\Phi'(d, t)$ and observe that the sum over k will have two non-vanishing contributions,

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = - \sum_{i<j} \left[\frac{\boldsymbol{q}_i - \boldsymbol{q}_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|}\, \Phi'_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|, t) + \frac{\boldsymbol{q}_j - \boldsymbol{q}_i}{|\boldsymbol{q}_i - \boldsymbol{q}_j|}\, \Phi'_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|, t) \right]$$

$$= \boldsymbol{0}$$

□

6.4.2 *Free flight*

A particle experiencing free flight is subject to a constant force, i. e. a force that is uniform in space and that does not depend on time. A common example is a particle of mass m close to the Earth surface. The particle is subject to a constant acceleration $\boldsymbol{g}$ (Section 4.2) that gives rise to a potential

$$V(\boldsymbol{q}) = -m\, \boldsymbol{g} \cdot \boldsymbol{q}$$

For Cartesian coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ where the first coordinate axis is aligned antiparallel to $\boldsymbol{g}$ the Lagrange function takes the from

$$\begin{aligned}
T &= \frac{m}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 \right) \\
V &= m\, g\, q_1 \\
\mathcal{L} &= T - V = \frac{m}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 \right) - m\, g\, q_1 \,.
\end{aligned}$$

In this setting q_2 and q_3 are cyclic variables such that the corresponding components of the momentum, $\boldsymbol{P} = m\, \boldsymbol{q}$, are conserved,

$$\dot{P}_2 = \dot{P}_3 = 0\,.$$

Moreover, for the first component we find

$$m\, \ddot{q}_1 = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{\partial \mathcal{L}}{\partial q_1} = m\, g$$

such that we recover the EOM (4.2.1) with the solution, Equation (4.2.2).

The impact of constant external forces on a set of N particles is discussed in Problem 6.10.

6.4.3 *Harmonic Oscillator*

A force $\boldsymbol{F}$ that is proportional to the distance $\boldsymbol{q}$ from some reference position is denoted as a harmonic, or Hookian spring force, $\boldsymbol{F} = -k\, \boldsymbol{q}$ (cf. Remark 3.4). One readily checks that it derives from a potential

$$V = \frac{k}{2}\, \boldsymbol{q}^2$$

such that the Lagrange function for a particle of mass m takes the form

$$\mathcal{L} = \frac{m}{2}\, \dot{\boldsymbol{q}}^2 - \frac{k}{2}\, \boldsymbol{q}^2$$

Accordingly, the Euler-Lagrange equations provide the EOM

$$\ddot{\boldsymbol{q}} = -\frac{k}{m}\, \boldsymbol{q}\,. \tag{6.4.1}$$

The equation for each component can be solved independently, providing

$$q_i(t) = A_i\, \sin(\alpha_i + \omega\, t) \qquad \text{with} \quad \omega = \sqrt{\frac{k}{m}}\,.$$

Here the amplitudes A_i and the phase factors α_i are integration constants that must be chosen in accordance with the initial condition $\big(\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0)\big)$.

Even though we found an explicit solution is non-trivial to sort out how the trajectories will look like. One can gain additional insight into their shape by describing the motion in terms of spherical coordinates, Theorem 6.3,

$$\begin{aligned}\boldsymbol{q} &= R(t)\,\hat{\boldsymbol{R}}\big(\theta(t), \varphi(t)\big)\\ \dot{\boldsymbol{q}} &= \dot{R}(t)\,\hat{\boldsymbol{R}}\big(\theta(t), \varphi(t)\big) + R(t)\,\dot\theta(t)\,\hat{\boldsymbol{\theta}}\big(\theta(t), \varphi(t)\big) + R(t)\,\sin\theta(t)\,\dot\varphi(t)\,\hat{\boldsymbol{\varphi}}\big(\theta(t), \varphi(t)\big)\end{aligned}$$

such that

$$\begin{aligned}T &= \frac{m}{2}\,\Big(\dot{R}^2(t) + R^2(t)\,\dot\theta^2(t) + R^2(t)\,\dot\varphi^2(t)\,\sin^2\theta(t)\Big)\\ V &= \frac{k}{2}\,R^2(t)\end{aligned}$$

The potential is rotationally symmetric. It depends only on R, and neither on θ nor on φ. Moreover, the kinetic energy also does not involve φ. Hence, φ is cyclic. It provides the conservation law

$$C_\varphi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = m\,R^2\,\sin^2\theta(t)\,\dot\varphi \tag{6.4.2}$$

Moreover, the Euler-Lagrange equation for θ provides

$$\begin{aligned}m\,R^2(t)\,\ddot\theta + 2\,m\,R\,\dot{R}\,\dot\theta &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\,R^2\,\dot\theta\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\theta}\\ &= \frac{\partial\mathcal{L}}{\partial\theta} = m\,R^2(t)\,\dot\varphi^2(t)\,\sin\theta\,\cos\theta \end{aligned}\tag{6.4.3}$$

This equation has a fixed point for $\theta = \pi/2$. We therefore *choose* the polar coordinates such that the initial position $\boldsymbol{q}$ and the initial velocity $\dot{\boldsymbol{q}}$ lie in the equatorial plane of the coordinate system, i. e. we require that $\theta(t_0) = \pi/2$ and that

$$\begin{aligned}0 &= \hat{\boldsymbol{\theta}}\left(\frac{\pi}{2}, \varphi_0\right)\cdot\dot{\boldsymbol{q}}(t_0)\\ &= \hat{\boldsymbol{\theta}}\left(\frac{\pi}{2}, \varphi_0\right)\cdot\left[\dot{R}(t_0)\,\hat{\boldsymbol{R}}\left(\frac{\pi}{2}, \varphi_0\right) + R_0\,\dot\theta(t_0)\,\hat{\boldsymbol{\theta}}\left(\frac{\pi}{2}, \varphi_0\right) + R_0\,\dot\phi(t_0)\,\hat{\boldsymbol{\varphi}}\left(\frac{\pi}{2}, \varphi_0\right)\right]\\ &= R_0\,\dot\theta(t_0)\end{aligned}$$

$$\Rightarrow\quad \dot\theta(t_0) = 0 \qquad \text{unless}\quad R_0 = 0\,,$$

and for for $R_0 = 0$ one can still choose the coordinates such that $\dot\theta(t_0) = 0$. For this choice the angular momentum will take the constant value

$$\boldsymbol{L} = R\,\hat{\boldsymbol{R}}\times m\,\left(\dot{R}\,\hat{\boldsymbol{R}} + R\,\dot\varphi\,\hat{\boldsymbol{\varphi}}\right) = m\,R^2\,\dot\varphi\,\hat{\boldsymbol{R}}\times\hat{\boldsymbol{\varphi}} = C_\varphi\,\hat{\boldsymbol{\theta}}$$

Hence, Equation (6.4.3) expresses the conservation of the modulus of angular momentum, and the motion is confined to a plane due to the conservation of the direction of the angular momentum. This is in line with the analogous observation in the Kepler problem, Section 4.7.1.

For $\theta = \pi/2$ and $\dot\theta = 0$ the EOM for $R(t)$ takes the form

$$m\,\ddot R = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot R} = \frac{\partial \mathcal{L}}{\partial R} = m\,R\,\dot\varphi^2 - k\,R$$

We employ angular-momentum conservation, $C_\varphi = m\,R^2\,\dot\varphi$, to eliminate $\dot\varphi$ in this equation. Subsequently, we introduce the dimensionless time, $\tau = \omega\,(t - t_0)$, and distance, $r = R/R_c$, based on the frequency $\omega = \sqrt{k/m}$ and the length scale $R_c = \sqrt{C_\varphi/m\,\omega}$:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{1}{r^3} - r \tag{6.4.4}$$

Multiplication by $2\,\dot r$ and integration provides

$$\dot r^2(\tau) - \dot r^2(0) = \int_0^\tau \mathrm{d}\tau\,\frac{\mathrm{d}r^2}{\mathrm{d}\tau} = -\int_0^\tau \mathrm{d}\tau\,\frac{\mathrm{d}r}{\mathrm{d}\tau}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{2} + \frac{1}{2\,r^2}\right)$$
$$= -\int_{r_0}^{r(\tau)} \mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{2} + \frac{1}{2\,r^2}\right) = -\left(\frac{r^2}{2} + \frac{1}{2\,r^2}\right)_{r_0}^{r(\tau)}$$

This entails conservation of the dimensionless energy,

$$e = \frac{\dot r^2}{2} + \frac{r^2}{2} + \frac{1}{2\,r^2} \tag{6.4.5}$$

This amounts to the EOM, (6.4.4), of a particle at position $r(\tau)$ that follows a dynamics with energy conservation, Equation (6.4.5), where it has a dimensionless kinetic energy $\dot r^2/2$ and a potential energy $r^2/2 + r^{-2}/2$. Hence, we refer to it as effective potential for the r dynamics.

Definition 6.11: Effective potential

Let $\mathcal{L}(\dot q_1, \dot q_2, \cdots, q_1)$ be a Lagrange function of a dynamics that depends only on the velocities $\dot q_1, \dot q_2, \cdots$ and on the first coordinate q_1. All other coordinates are either cyclic or fixed to specific values by other means. Then the conservation laws resulting for the dynamics of q_i with $i > 1$ can be used to eliminate the velocities $\dot q_i$ for $i > 1$. This results in an EOM for $q_1(t)$ that only involves $\ddot q_1$, $\dot q_1$, q_1, and conserved quantities with values fixed by initial conditions. Often this equation can be integrated once, and cast into the dimensionless form

$$e = \frac{\dot\Theta^2(\tau)}{2} + \Phi_{\text{eff}}(\Theta(\tau))$$

where e is a constant of motion, τ is the dimensionless time, Θ is some function of $\dot q_1$ and q_1, and $\dot\Theta$ is the τ derivative of Θ.
This maps the q_1 dynamics to the motion of a particle with a single DOF Θ, kinetic energy $\dot\Theta^2/2$, and an *effective potential* $\Phi_{\text{eff}}(\Theta)$.

Remark 6.10. If the system conserves energy, then e will often amount to the non-dimensionalized energy.

Remark 6.11. In Sections 6.3.2 and 6.3.4 we encountered settings where the angle φ was controlled by an external motor such that the energy is not conserved (cf. Problem 6.7). In this case the constraint on φ allows us to math the non-inertial dynamics of the rotating system into an autonomous dynamics with the effective potentials, Equations (6.3.5) and (6.3.7), respectively. ⌘

In the coordinate plane the trajectories of the harmonic oscillator that take the form of closed ellipses. This is most transparently confirmed by choosing the Cartesian coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ in Equation (6.4.1) such that q_3 vanishes and adopting a complex variable $z = q_1 + \mathrm{i}\, q_2$. The EOM will then take the form $\ddot{z} = -\omega^2 z$, and it has the general solution

$$z(t) = z_1\, \mathrm{e}^{\mathrm{i}\,\omega(t-t_0)} + z_2\, \mathrm{e}^{-\mathrm{i}\,\omega(t-t_0)}$$

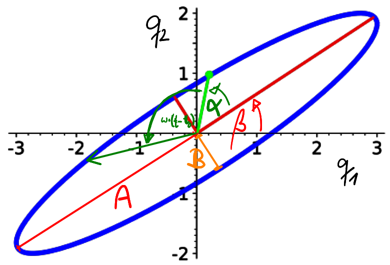


Figure 6.16: Sketch of the configuration-space trajectory of a harmonic oscillator, Equation (6.4.6). The blue line indicates the positions visited by the trajectory. The light green dot indicates the initial position at time t_0. The dark green arrow indicates the position at some time t. The particle will reside at that time at a position at a distance $|z|$ from the origin with an angle $\beta + \alpha + \omega\,(t - t_0)$ with respect to the positive q_1 axis.

We express the complex integration constants z_1 and z_2 in terms of four real numbers A, B, α, and β,

$$z_1 = \frac{A+B}{2}\, \mathrm{e}^{\mathrm{i}\,(\beta+\alpha)} \qquad\qquad z_2 = \frac{A-B}{2}\, \mathrm{e}^{\mathrm{i}\,(\beta-\alpha)}$$

and find

$$z(t) = \mathrm{e}^{\mathrm{i}\,\beta}\, \left[A\, \cos\big(\alpha + \omega\,(t-t_0)\big) + \mathrm{i}\, B\, \sin\big(\alpha + \omega\,(t-t_0)\big)\right] \tag{6.4.6}$$

The amplitudes A and B describe the length of the semi-axes of the ellipse. The angle β rotates the ellipse, i. e. it describes the orientation of the axes. The angle α is a phase factor that establishes the position on the ellipse where the particle started at time t_0. A sketch of the resulting trajectory is provided in Figure 6.16.

We observe that the trajectory keeps a minimum distance B to the origin. For the harmonic potential this distance can be calculated explicitly by observing that $\dot{R}$ vanishes at the minimum distance R_{min} such that energy conservation entails

$$E = \frac{1}{2\,m}\, \frac{C_\varphi^2}{R_{\text{min}}^2} + \frac{k}{2}\, R_{\text{min}}^2$$

$$\Rightarrow \qquad R_{\text{min}}^2 = \frac{E}{k} \left(1 - \sqrt{1 - \left(\frac{C_\varphi}{k\,E}\right)^2}\right) \tag{6.4.7}$$

Hence, the trajectory can go through the origin iff $C_\varphi = 0$.

This observation holds for every system with energy conservation and a potential that obeys cylinder symmetry.[6]

[6] Here, cylinder symmetry is a weaker condition than spherical symmetry. It is sufficient when the potential is symmetric under rotation around *some* axis. Spherical symmetry requires that it is symmetric under rotation around *every* axis.

Theorem 6.7: Rotation barrier

Let a particle of mass m follow a dynamics where the energy E and a component $C_\varphi = \hat{\boldsymbol{n}} \cdot \boldsymbol{L}$ of the angular momentum $\boldsymbol{L}$ are conserved. Then the particle will keep a minimum distance

$$R_{\min} = \frac{|C_\varphi|}{\sqrt{2m\,(E - \Phi_{\min})}} \tag{6.4.8}$$

from the axis $\hat{\boldsymbol{n}}$. Here, $\Phi_{\min}$ is a lower bound to the potential energy. The non-vanishing angular momentum C_φ sets up a *rotation barrier* preventing the system to enter a region close to the rotation axis.

Proof. We adopt cylinder coordinates (R, φ, z) with a symmetry axis $\hat{\boldsymbol{z}}$ aligned along $\hat{\boldsymbol{n}}$. Then the kinetic energy and L_n amount to

$$T = \frac{m}{2}\left(\dot{z}^2 + R^2\,\dot{\varphi}^2\right)$$
$$C_\varphi = m\,R^2\,\dot{\varphi}$$

As a consequence, we have

$$E - \varphi_{\min} \geq E - U(R,z) = T \geq \frac{m}{2}\,R^2\,\dot{\varphi}^2 = \frac{C_\varphi^2}{2m\,R^2}$$

The bound is obtained by solving this inequality for R. □

Remark 6.12. Equation (6.4.8) implies that also in general the rotation axis can only be approached for $C_\varphi = 0$.

Remark 6.13. It is not necessary that the potential $U(R,z)$ has a minimum for $R = 0$. The argument works in exactly the same way when it has a maximum.[7] An example is provided in Problem 6.12.

[7] Due to the cylinder symmetry the potential will either have a minimum or a maximum.

Remark 6.14. Energy conservation always provides an implicit equation for $R_{\min}$. In general this equation can not be solved for $R_{\min}$ such that one can only provide the inequality, (6.4.8). However, besides for the harmonic potential, Equation (6.4.7), one can also explicitly determine the solution for the Kepler potential. This derivation is given as Problem 6.11.

6.4.4 *Spherical pendulum*

The force-free motion, free flight and harmonic oscillator all feature EOM where the motion of the different degrees of freedom were not coupled. This very exceptional. As a first step towards a more generic setting we discuss the dynamics of the spherical pendulum.

The spherical pendulum describes the motion of a mass M that is mounted on a bar of fixed length ℓ whose other end is fixed to a pivot. Thus, the position of the mass is constraint to a spherical shell. We adopt spherical coordinates to describe the position as

$$\boldsymbol{x}(t) = \ell \begin{pmatrix} \sin\theta(t)\ \cos\varphi(t) \\ \sin\theta(t)\ \sin\varphi(t) \\ -\cos\theta(t) \end{pmatrix} = \ell\,\hat{\boldsymbol{R}}(\theta(t), \varphi(t))$$

Note that the angle θ is measured here with respect the negative z axis, in contrast to the definition adopted in Theorem 6.3 (see Figure 6.17). As a consequence, we have now

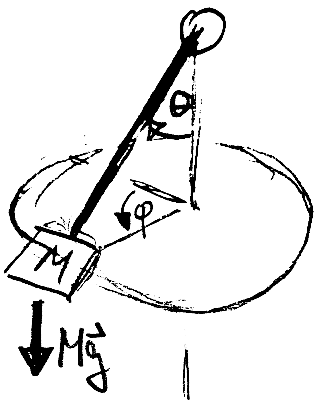


Figure 6.17: Spherical coordinates adopted to describe the motion of a spherical pendulum.

$$\hat{\boldsymbol{\theta}}(\theta,\varphi) = \begin{pmatrix} \cos\theta\ \cos\varphi \\ \cos\theta\ \sin\varphi \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{\varphi}}(\theta,\varphi) = \begin{pmatrix} -\sin\theta\ \sin\varphi \\ \sin\theta\ \cos\varphi \\ 0 \end{pmatrix}$$

with $0 < \theta < \pi$ and $0 \leq \varphi < 2\pi$. The same rules for the derivatives apply as provided in Theorem 6.4, but now $\{\hat{\boldsymbol{R}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{\theta}})$ provides a right handed coordinate system, i.e. $\hat{\boldsymbol{R}} \times \hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{\varphi}}$.

The angle θ denotes the angle between the position the mass and the gravitational field. Consequently, the potential energy in the gravitational field is obtained as

$$U = -M\,\boldsymbol{g}\cdot\boldsymbol{x} = -M\,g\,\ell\,\cos\theta(t)\,.$$

The angle φ describes in which direction the mass is deflected from the vertical line, in a plane orthogonal to the action of gravity.

For the velocity we find based on the chain rule and the derivatives of the unit vectors, Equation (6.3.3),

$$\begin{aligned} \dot{\boldsymbol{x}} &= \ell\,\dot\theta\,\partial_\theta \hat{\boldsymbol{R}}\big(\theta(t),\varphi(t)\big) + \ell\,\dot\varphi\,\partial_\varphi \hat{\boldsymbol{R}}\big(\theta(t),\varphi(t)\big) \\ &= \ell\,\dot\theta\,\hat{\boldsymbol{\theta}}\big(\theta(t),\varphi(t)\big) + \ell\,\dot\varphi\,\sin\theta(t)\,\hat{\boldsymbol{\varphi}}\big(\theta(t),\varphi(t)\big) \end{aligned}$$

The expression for $\dot{\boldsymbol{x}}$ and the orthogonality condition $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\varphi}} = 0$ for the base vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ provide the kinetic energy

$$T = \frac{M}{2}\,\dot{\boldsymbol{x}}^2 = \frac{M}{2}\,\ell^2\,\dot\theta^2(t) + \frac{M}{2}\,\ell^2\,\sin^2\theta(t)\,\dot\varphi^2(t)$$

Consequently, the Lagrange function for the spherical pendulum takes the form

$$\mathcal{L}(\theta,\varphi,\dot\theta,\dot\varphi) = \frac{M}{2}\,\ell^2\,\dot\theta^2 + \frac{M}{2}\,\ell^2\,\sin^2\theta(t)\,\dot\varphi^2(t) + M\,g\,\ell\,\cos\theta(t)$$

We observe that $\mathcal{L}$ does not depend on φ. In that case it is advisable to first discuss the EOM for φ. It takes the form

$$M\,\ell^2\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot\varphi\,\sin^2\theta(t)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$

The derivative of the Lagrange function with respect to φ vanishes because $\mathcal{L}$ does not depend on φ. Such a coordinate is called cyclic, and it always implies a conservation law. For the spherical pendulum it entails the conservation law

$$C_\varphi = \dot\varphi\,\sin^2\theta(t) = \text{const} \tag{6.4.9}$$

Comparison to Equation (6.4.2) reveals C_φ is proportional to the z-component of the angular momentum. The conservation law provides an expression of $\dot\varphi$ in terms of θ

$$\dot\varphi(t) = \frac{C_\varphi}{\sin^2\theta(t)} \tag{6.4.10}$$

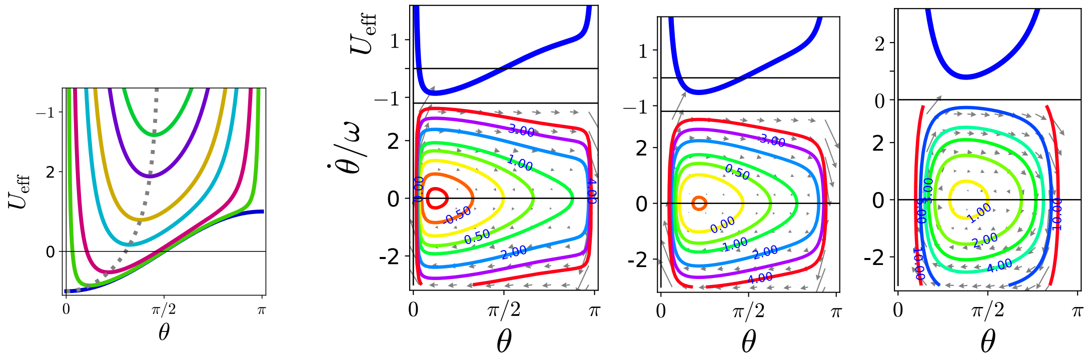


Figure 6.18: The left panel shows the effective potential for the spherical pendulum at parameter values $C_\varphi 2 \in \{0, 0.01, 0.1, 0.5, 1, 2, 3\}$ from bottom to top. The subsequent panels show phase-space portraits of the motion for $C_\varphi^2 = 0.01$, 0.1, and 1, respectively.

Let us now consider to the other coordinate of the spherical pendulum. The EOM for $\theta(t)$ takes the form

$$\begin{aligned} M\,\ell^2\,\ddot\theta(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\theta} \\ &= \frac{\partial\mathcal{L}}{\partial\theta} = M\,\ell^2\,\dot\varphi^2(t)\ \sin\theta(t)\ \cos\theta(t) - M\,g\,\ell\,\sin\theta(t)\,. \end{aligned}$$

In this equation the unknown function $\dot\varphi(t)$ can be eliminated by means of the conservation law, Equation (6.4.9), yielding

$$\ddot\theta(t) = \frac{C_\varphi^2\,\cos\theta(t)}{\sin^3\theta(t)} - \frac{g}{\ell}\ \sin\theta(t) \qquad (6.4.11)$$

For $C_\varphi = 0$ the particle swings in a fixed plane selected by $\varphi =$ const. Its motion amounts to that of the mathematical pendulum, Equation (6.3.1) .

The resulting EOM can be integrated once by multiplication with $2\dot\theta(t)$

$$\begin{aligned} \dot\theta^2(t) - \dot\theta^2(t_0) &= \int_{t_0}^{t} \mathrm{d}t\, 2\dot\theta \left(\frac{C_\varphi^2\,\cos\theta(t)}{\sin^3\theta(t)} - \frac{g}{\ell}\ \sin\theta(t) \right) \\ &= -2 \int_{\theta(t_0)}^{\theta(t)} \mathrm{d}\theta\, \frac{\mathrm{d}}{\mathrm{d}\theta} \left(-\frac{C_\varphi^2}{\sin^2\theta} + \frac{g}{\ell}\ \cos\theta \right) \end{aligned}$$

The result can be written in the form

$$\begin{aligned} E &= \frac{\dot\theta^2(t)}{2} + \Phi_{\text{eff}}(\theta(t)) = \text{const} \\ \text{where}\quad \Phi_{\text{eff}}(\theta) &= \frac{C_\varphi^2}{\sin^2\theta} - \frac{g}{\ell}\ \cos\theta\,. \end{aligned}$$

Again a closed solution for $\theta(t)$ is out of reach. However, $\Phi_{\text{eff}}(\theta)$ can serve as an effective potential for the 1DOF motion of θ with kinetic energy $\dot\theta^2/2$. This interpretation of the dynamics provides ready access to a qualitative discussion of of the solutions of the EOM based on a phase-space plot.

Figure 6.18 shows the effective potential and phase space portraits for different positive values of C_φ. Conservation of angular momentum implies that for $C_\varphi \neq 0$ the particle can no longer access the region close to its rest position at the lowermost point of the sphere. Rather it always has to go in circles around the bottom of the well, and the sign of C_φ specifies whether it moves clockwise or anti-clockwise. The divergence of the effective potential at $\theta = \pm\pi$ is called *rotation barrier*. It emerges due to a combination of the conservation of energy and angular momentum.

The effective potential has a single minimum for $0 < \theta_c(C_\varphi) < \pi/2$, and not further extrema. The minimum describes motion where the particle moves at constant height with a constant speed in a circle. When this orbit is perturbed oscillations are superimposed on the circular motion. In a projection to the plane vertical to the action of gravity, this will lead to trajectories similar to those drawn by a Spirograph, Problem 2.42.

add traces in the (x, y) plane

The take-home message of this example is that cyclic variables entail conservation laws of the dynamics. In the very same manner as for the Kepler problem they can be used to eliminate a variable from the EOM of the other coordinates. The additional contributions in the EOMs for the other coordinate(s) are interpreted as part of an effective potential.

6.4.5 *Self Test*

Problem 6.10. **Momentum conservation and free flight**

We consider a system of$N$ particles with masses m_i. Their interactions are translation invariant, deriving from a potential $V(\boldsymbol{q}, t) = \sum_{i<j} \Phi_{ij}(|\boldsymbol{q}_i - \boldsymbol{q}_j|, t)$. However, in addition to the setting considered in Theorem 6.6 we allow for external forces. For each particle the force, $\boldsymbol{F}_i$, takes a constant value, and all forces are acting in the same direction. As an example you may consider gravity $\boldsymbol{F}_i = m_i\, \boldsymbol{g}$ or a constant electric fields $\boldsymbol{E}$ for particles with charges e_i. Adapt the arguments of the proof of Theorem 6.6 to show that

a) The component of the total momentum $\boldsymbol{P}$ parallel to the external forces is no longer conserved.

b) All other components of the total momentum are still conserved.

c) How does this finding fit with the changes of T and V upon translations, Problem 4.35.

Problem 6.11. **Minimum distance of celestial bodies**

The gravitation potential between between two celestial bodies of masses m and M follows Kepler's law

$$U(R) = \frac{m\, M\, G}{R}$$

It depends only on their distance $R = |\boldsymbol{R}|$ between the bodies. The energy of the relative motion, Equation (4.9.1), is conserved.

a) Determine the distance between the bodies upon their closest approach.

b) What is the relative velocity $|\dot{\boldsymbol{R}}|$ of the bodies at that moment?

Problem 6.12. **Evolution of a particle in a Mexican-hat potential**

We explore the motion of a particle of mass m in a rotation-symmetric potential

$$\Phi(r) = \frac{m\,A}{4}\,r^2\,(r^2 - 2\,r_0^2)$$

The particle evolves in a plane where its position is specified by the polar coordinates (r, θ).

a) Sketch the potential. Where are its maxima and minima?

b) Determine the Lagrange function for this problem, and determine the equations of motion for $\theta(t)$ and $r(t)$.

   Bonus. The angular momentum and the energy of the particle are conserved. How do you see this without calculation based on the Lagrange function?

c) Determine a frequency ω, a length scale ℓ and a constant K, such that

$$\frac{\mathrm{d}^2\hat{r}}{\mathrm{d}(\omega t)^2} = \hat{r} - \hat{r}^3 + \frac{K}{\hat{r}^3}$$

   where $\hat{r}$ denotes the dimensionless (scalar) distance

$$\text{with } \hat{r}(t) = \frac{r(t)}{\ell}\,.$$

   In the following we discuss the dimensionless equations, where we absorb ω into the time scale and drop the hat to avoid clutter in the equations.

d) Multiply the equation of motion by $\dot{r}$, and rewrite it in the form

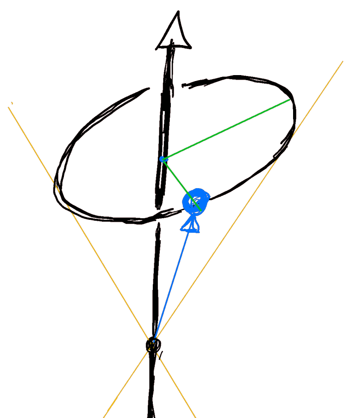
$$E = \frac{\dot{r}^2}{2} + V_{\text{eff}}(r) \qquad \text{with} \qquad V_{\text{eff}}(r) = \frac{r^4}{4} - \frac{r^2}{2} + \frac{K}{2\,r^2}\,.$$

e) Sketch the effective potential $V_{\text{eff}}(r)$ and the phase portrait of the motion for $K > 0$.

   Bonus. Why is it necessary to give a separate discussion of $K = 0$?

Problem 6.13. **Particle on a cone**

We consider a particle that can freely move on the surface of a cone with opening angle α (see sketch in the margin). We describe the particle position $\boldsymbol{q}$ in terms of cylinder coordinates (r, θ, z), where the symmetry axis of the cone is aligned with the coordinate axis $\hat{\boldsymbol{z}}$ and the origin of the coordinate system is at the cone vertex.



a) We write the particle position $\boldsymbol{q}$ as

$$\boldsymbol{q} = h(t)\,\hat{\boldsymbol{z}} + \gamma\, h(t)\,\hat{\boldsymbol{r}}(\theta)$$

Indicate the three vectors $\boldsymbol{q}$, $h(t)\,\hat{\boldsymbol{z}}$, and $\gamma\, h(t)\,\hat{\boldsymbol{r}}(\theta)$ for the blue "particle" in the sketch. How is γ related to α?

b) Determine the velocity of the particle, its kinetic energy T, the potential energy V and the Lagrangian.

c) Show that the equation of motion for θ provides a conservation law that entails

$$\dot{\theta}(t) = \frac{\lambda}{h^2(t)} \tag{6.4.12}$$

Determine λ.

⋆ d) Why is the energy $E = T + V$ is conserved for the motion of the particle?

e) Adopt Eq. (6.4.12) to eliminate $\dot{\theta}$ in the expression for the energy, and adopt dimensionless units $x = h/H$, $\tau = t/T$, and $\hat{E} = E/mgH$ such that

$$\hat{E} = \frac{1}{2}\left(\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}\right)^2 + U(x) \qquad \text{with} \qquad U(x) = \frac{1}{2\,x^2} + x$$

Determine H and T.

f) Sketch the effective potential $U(x)$.

Bonus: Which interpretation does this provide for H?

Sketch the phase-space plot for the evolution of the height $h(t)$ of the particle.

⋆ g) Assume that the lower part of the cone is chopped off at a height h_0. For which initial conditions will the particle fall out of the cone?

6.5 *Dynamics of 2-particle systems*

In Sections 6.3 and 6.4 we discussed the motion of single particles whose motion is constraint by tracks, arms and joints. Now we revisit the treatment of the Kepler problem, Section 4.6, in order to explore settings with two interacting particles. The central idea in this endeavor is a representation of the particle position as a sum of the position of the center of mass $\boldsymbol{Q}$ and the relative coordinate $\boldsymbol{R}$.

6.5.1 *The Kepler problem revisited*

We consider two celestial bodies with masses m_1 and m_2 that reside at the positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in 3D space. The bodies are modeled as point particles that interact by gravity, (cf. Example 4.10),

$$T = \frac{m_1}{2}\,\dot{\boldsymbol{x}}_1^2 + \frac{m_2}{2}\,\dot{\boldsymbol{x}}_2^2$$

$$V = -\frac{m_1 m_2 G}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|}$$

The positions $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ of the two bodies are coordinates in a 6D vector space.

Prior to calculating EOMs it is advisable to look for characteristic properties of the dynamics that will simplify the treatment of the dynamics of the 6DOF system. Here, the potential energy only depends on $\boldsymbol{R} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. This suggests to adopt a different basis that allows us to express the dynamics in terms of $\boldsymbol{R}$ and another position $\boldsymbol{Q} = a_1\, \boldsymbol{x}_1 + a_2\, \boldsymbol{x}_2$ that is linearly independent of $\boldsymbol{R}$. In that case $\boldsymbol{Q}$ will be cyclic, and it can be dealt with by a conservation law. We observe that

$$\boldsymbol{x}_1 = (\boldsymbol{Q} + a_2\, \boldsymbol{R}) / (a_1 + a_2)$$
$$\boldsymbol{x}_2 = (\boldsymbol{Q} - a_1\, \boldsymbol{R}) / (a_1 + a_2)$$

such that

$$T = \frac{1}{2}\, \dot{\boldsymbol{Q}}^2 \left(\frac{m_1 + m_2}{(a_1 + a_2)^2} \right) + \dot{\boldsymbol{Q}} \cdot \dot{\boldsymbol{R}} \left(\frac{m_1\, a_2 - m_2\, a_1}{(a_1 + a_2)^2} \right) + \frac{1}{2}\, \dot{\boldsymbol{R}}^2 \left(\frac{m_1\, a_2^2 + m_2\, a_1^2}{(a_1 + a_2)^2} \right)$$

For $a_i \propto m_i$, $i \in \{1, 2\}$ the mixed term vanishes, and the $\boldsymbol{R}$ dynamics is not coupled to the $\boldsymbol{Q}$ dynamics. To avoid clutter in the equations we pick $a_i = m_i / (m_1 + m_2)$ such that $a_1 + a_2 = 1$ such that

$$\boldsymbol{Q} = \frac{\sum_i m_i\, \boldsymbol{x}_i}{\sum_i m_i}$$

Hence, $\boldsymbol{Q}$ is the center of mass of the two-body system.

Further we introduce the total mass M and the reduced mass μ,

$$M = m_1 + m_2$$
$$\mu = m_1\, a_2^2 + m_2\, a_1^2 = \frac{m_1\, m_2^2 + m_2\, m_1^2}{(m_1 + m_2)^2} = \frac{m_1\, m_2}{m_1 + m_2}$$

This provides

$$T = \frac{M}{2}\, \dot{\boldsymbol{Q}}^2 + \frac{\mu}{2}\, \dot{\boldsymbol{R}}^2$$
$$V = -\frac{\mu M G}{|\boldsymbol{R}|}$$
$$\mathcal{L} = T - V = \frac{M}{2}\, \dot{\boldsymbol{Q}}^2 + \frac{\mu}{2}\, \dot{\boldsymbol{R}}^2 + \frac{\mu M G}{|\boldsymbol{R}|} \tag{6.5.1}$$

The generalization to more than two particles is addressed in Problem 6.14.

Equation (6.5.1) asserts that $\boldsymbol{Q}$ is a cyclic variable. I provides the conservation law

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P} \quad \text{with} \quad \boldsymbol{P} = M\, \dot{\boldsymbol{Q}} = \sum_i m_i\, \dot{\boldsymbol{x}}_i$$

and the center-of-mass motion

$$\boldsymbol{Q}(t) = \boldsymbol{Q}(t_0) + \dot{\boldsymbol{Q}}(t_0)\, (t - t_0)\,.$$

In the discussion of the harmonic oscillator, Section 6.4.3, we observed that the angular momentum is conserved when the potential is rotationally invariant. Hence, we adopt cylindrical coordinates (R, θ, z) where $\boldsymbol{R}(t_0)$ and $\dot{\boldsymbol{R}}(t_0)$ have vanishing z-coordinates,

$$\mathcal{L} - \frac{M}{2}\,\dot{\boldsymbol{Q}}^2 = \frac{\mu}{2}\left(\dot{z}^2 + \dot{R}^2 + R^2\,\dot{\theta}^2\right) + \frac{\mu M G}{R}$$

Here, z is a cyclic variable and for the specified initial conditions it vanishes for all times,

$$\ddot{z} = 0 \quad \Rightarrow \quad z(t) = z(t_0) + \dot{z}(t_0)\,(z - z_0) = 0\,.$$

Moreover, also θ is a cyclic variable. It provides the conservation law

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\,R^2\,\dot{\theta}\right)$$

$$\Rightarrow C_\theta = \mu\,R^2\,\dot{\theta} = \text{const.}$$

Finally, there is no explicit time dependence in the kinetic and potential energy. Therefore, energy, $T + V$, must be preserved. In view of $M\,\dot{\boldsymbol{Q}}^2/2\text{const}$ and $\dot{z} = 0$ this implies

$$E = \frac{\mu}{2}\left(\dot{R}^2 + R^2\,\dot{\theta}^2\right) - \frac{\mu M G}{R} = \frac{\mu}{2}\,\dot{R}^2 + \frac{C_\theta^2}{2\,\mu\,R^2} - \frac{\mu M G}{R}\,.$$

The solution of this equation was discussed in Section 4.9.

6.5.2 *Spring on rails*

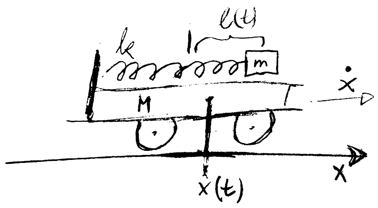


Figure 6.19: Setup of the spring on rails considered in Section 6.5.2.

We consider a cart that moves without friction on a horizontal track (see Figure 6.19). It has mass M, and at time t it is located at position $x(t)$. On the cart we attach a weight of mass m to a spring with spring constant k. It oscillates without friction in the track direction, and its displacement from the rest position is denoted as $\ell(t)$.

The kinetic energy and potential energy of the cart are

$$T_c = \frac{M}{2}\,\dot{x}^2 \qquad \text{and} \qquad V_c = 0$$

Its potential energy due to gravity vanishes because the cart moves horizontally.

We denote the rest position of the weight m as $x(t) + e$. Hence, at time t it is located at $q(t) = x(t) + e + \ell(t)$. Its kinetic energy and potential energy are

$$T_w = \frac{m}{2}\,(\dot{x} + \dot{\ell})^2 \qquad \text{and} \qquad V_w = 0$$

Also in this case the potential energy due to gravity vanishes because the weight moves horizontally. However, there is a contribution to the potential due to the extension of the spring,

$$V_s = \frac{k}{2}\ell^2$$

Altogether, this provides the Lagrangian:

$$\mathcal{L} = \frac{M}{2}\,\dot{x}^2 + \frac{m}{2}\,(\dot{x}+\dot{\ell})^2 - \frac{k}{2}\ell^2 \,.$$

Here, x is a cyclic variable,

$$0 = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\big(M\,\dot{x} + m\,(\dot{x}+\dot{\ell})\big)$$

that provides the conserved quantity

$$P = (M+m)\,\dot{x} + m\,\dot{\ell} = \frac{\mathrm{d}}{\mathrm{d}t}\,\big(M\,x + m\,(x+\ell)\big) = (m+M)\,Q$$

where

$$Q = \frac{M}{m+M}\,x + \frac{m}{m+M}\,(x+e+\ell) = x + \frac{m}{m+M}\,(e+\ell) \qquad (6.5.2)$$

is the x-component of the center of mass of the system, i. e. of the cart and the weight attached to the cart. Hence, $P = (m+M)\,\dot{Q}$ is the momentum of the horizontal center-of-mass motion. Momentum conservation implies

$$Q(t) = Q(t_0) + \dot{Q}(t_0)\,(t-t_0) \qquad (6.5.3)$$

In the following we work in the center of mass frame. Equation (6.5.2) provides

$$\begin{aligned} x &= Q - \frac{m}{m+M}\,(e+\ell) \\ \dot{x} &= \dot{Q} - \frac{m}{m+M}\,\dot{\ell} \end{aligned}$$

such that the Lagrangian takes the form

$$\begin{aligned} \mathcal{L} &= \frac{M}{2}\,\left(\dot{Q} - \frac{m}{m+M}\,\dot{\ell}\right)^2 + \frac{m}{2}\,\left(\dot{Q} - \frac{m}{m+M}\,\dot{\ell} + \dot{\ell}\right)^2 - \frac{k}{2}\,\ell^2 \\ &= \frac{m+M}{2}\,\dot{Q}^2 + \frac{2}{2}\,\left(-M\,\frac{m}{m+M} + m\,\frac{M}{m+M}\right)\dot{Q}\,\dot{\ell} \\ &+ \frac{1}{2}\,\frac{m\,M}{(m+M)^2}\,(m+M)\,\dot{\ell}^2 - \frac{k}{2}\,\ell^2 \\ &= \frac{m+M}{2}\,\dot{Q}^2 + \frac{1}{2}\,\frac{m\,M}{m+M}\,\dot{\ell}^2 - \frac{k}{2}\,\ell^2 \end{aligned}$$

As expected by analogy to the Kepler problem the total mass $m + M$ and the reduced mass $\mu = mM/(m+M)$ appear in front of the kinetic-energy contributions of the center of mass Q and the distance between the masses $\dot{e} + \dot{\ell} = \dot{\ell}$, respectively.

There is no explicit time dependence in this problem, and $\dot{Q}$ =const due to the momentum conservation, P. Hence, the energy E of the relative motion is conserved,

$$E = \frac{\mu}{2}\,\dot{\ell}^2 + \frac{k}{2}\,\ell^2$$

This amounts to the energy of a one-dimensional harmonic oscillator with frequency $\omega = \sqrt{k/\mu}$, and time evolution

$$\ell(t) = A\,\sin\big(\phi + \omega\,(t-t_0)\big)$$

where ϕ and ℓ_0 are provided by the initial conditions

$$\left.\begin{aligned}\ell(t_0) &= A\,\sin\phi\\ \dot\ell(t_0) &= A\,\omega\,\cos\phi\end{aligned}\right\}\quad\Rightarrow\quad\begin{cases}\phi = \arctan\dfrac{\omega\,\ell(t_0)}{\dot\ell(t_0)}\\ A = \ell(t_0)/\sin\phi\end{cases}$$

Hence, the position of the cart $x(t)$ is obtained as

$$\begin{aligned}x(t) &= Q(t) - \frac{m}{m+M}\,\big(e + \ell(t)\big)\\ &= Q_0 - \frac{M\,e}{m+M} + \frac{P\,(t-t_0)}{m+M} - \frac{m}{m+M}\,A\,\sin\big(\phi + \omega\,(t-t_0)\big)\end{aligned}$$

It is illuminating to take a closer look at the motion of the system for a heavy cart, $M \ll m$, and a heavy weight, $m \ll M$. To this end we select an initial condition where the cart is at rest at the rest position of the spring initially, $Q_0 = k\,M/(m+M)$. We hold the spring in a stretched position and release it at time t_0. Hence, also $P = 0$ and $\phi = \pi/2$, and

$$x(t) = -\frac{m}{m+M}\,A\,\cos\big(\omega\,(t-t_0)\big)$$

For $M \gg m$ the cart will barely move, while the weight will oscillate with amplitude A and frequency

$$\omega = \sqrt{k\,\frac{m+M}{mM}} = \sqrt{\frac{k}{m}\,(1+\varepsilon)} = \sqrt{\frac{k}{m}}\;\left(1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots\right)$$

This is the eigenfrequency of the cart-weight system when the support does not move, and, indeed, it does not move because the cart is much heavier than the weight.

In contrast, for $m \gg M$ the cart moves roughly with amplitude A, while the weight stays approximately at rest:

$$\begin{aligned}x + e + \ell &= e + \left(1 - \frac{m}{m+M}\right)\,A\,\cos\big(\omega\,(t-t_0)\big)\\ &= e + \frac{M}{m+M}\,A\,\cos\big(\omega\,(t-t_0)\big)\,.\end{aligned}$$

Hence, the weight is approximately at rest while the cart oscillates with a frequency

$$\omega = \sqrt{k\,\frac{m+M}{mM}} = \sqrt{\frac{k}{M}\,(1+\varepsilon)} = \sqrt{\frac{k}{M}}\;\left(1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots\right)\,.$$

This is the eigenfrequency of the system of a cart that is fixed to a support by the spring. The weight is much heavier than the cart, and due to its inertia it serves as a fixed support.

6.5.3 *Self Test*

Problem 6.14. **Center-of-mass coordinates for many-body systems**

Consider a system of N particles with masses m_i and position $\boldsymbol{x}_i(x)$, $i \in \{1, \cdots, N\}$, and let the potential energy be a function of the particle distances $\boldsymbol{R}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$. Show that:

a) The Lagrangian of the system can be expressed in terms of the the center of mass of the system,

$$Q = \frac{\sum_{i=1}^{N} m_i \, \boldsymbol{x}_i}{\sum_{i=1}^{N} m_i}$$

and the relative coordinates $\boldsymbol{R}_{1j}$, $j \in \{2, \dots, N\}$.

b) The center of mass is a cyclic variable, and the associated conserved quantity is the total momentum

$$\boldsymbol{P} = \sum_{i=1}^{N} m_i \, \dot{\boldsymbol{x}}_i$$

c) The other DOF can be described in terms of the particle distances $\{\boldsymbol{R}_{i1}\,,\ 1 < i \leq N\}$.
Hint: 1. Introduce $\boldsymbol{R}_i = \boldsymbol{x}_i - \boldsymbol{Q}$, insert the expression for $\boldsymbol{Q}$, and show that

$$\boldsymbol{R}_i = \frac{1}{M} \sum_{k=1}^{N} m_k \left(\boldsymbol{x}_i - \boldsymbol{x}_k \right) .$$

2. Evaluate $\boldsymbol{R}_{ij} = \boldsymbol{R}_i - \boldsymbol{R}_j$ and express the result in terms of $\{\boldsymbol{R}_{i1}\,,\ 1 < i \leq N\}$.

d) What changes when gravity is acting on the particles.

e) What happens when there is a constant external force (i. e. the same force for all particles).

Problem 6.15. **Pendulum on rails**

We set a mathematical pendulum on a cart that can move without friction horizontally in the direction of motion of the pendulum. The cart has an overall mass M. This mass will also include the pendulum frame which is considered to be a part of the cart. The pendulum comprises of a mass m that moves on an arm of length ℓ. We choose the coordinates such that the fulcrum of the pendulum is located at $(x(t), 0)$, and the mass of the pendulum at $(x + \ell \sin\theta, -\ell\cos\theta)$.

a) Sketch the setup, indicating the parameters and coordinates of the problem.

b) Determine the kinetic energy of the cart, the kinetic energy of the pendulum weight, and the potential energy of the pendulum weight.
Provide the resulting Lagrange function for the pendulum on the cart.

c) Identify the equation of motion for x, and show that it leads to a conserved quantity of the form

$$p_x = \alpha \, \dot{x} + \beta \, \dot{\theta} \, \cos\theta$$

How do α and β depend on the parameter ℓ, m, M, and g?

d) Determine the x-component of the center of mass of the pendulum weight and the cart. Which interpretation does this provide for the result of part (c)?

e) In the following we work in the center of mass frame, where $p_x = 0$. Show that the equation of motion for θ will then take the form

$$\ddot{\theta} = -\sin\theta \, \frac{1 + \mu \, \dot{\theta}^2 \, \cos\theta}{1 - \mu \, \cos^2\theta}$$

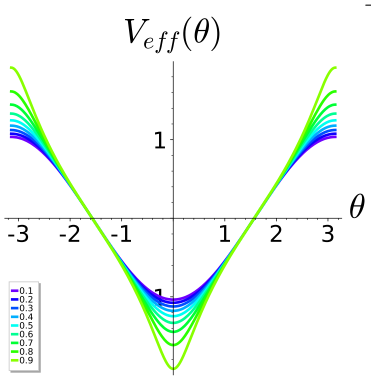
This form of the equation involves a particular form of the dimensionless time scale! Which time scale has been chosen? How does μ depend on the masses m and M?

* f) Why is it clear a priori that μ can only depend on m and M and not on the other parameters of the problem, g and ℓ?

* g) In the limit $\mu \to 0$ one recovers the equation of motion of the mathematical pendulum. Why would one expect this result?

h) We consider the case of small oscillations where terms of quadratic and higher order in θ and $\dot{\theta}$ may be dropped. Verify that the equation of motion for θ amounts in that case to the motion in an effective potential

$$V_{\text{eff}} = \frac{1}{2\sqrt{\mu}} \, \ln \frac{1 - \sqrt{\mu}\, \cos\theta}{1 + \sqrt{\mu}\, \cos\theta}$$



* i) The plot in the margin shows the effective potential for different values of μ. For small μ it amounts to $V_{\text{eff}} \simeq \cos\theta$. For $\mu \to 1$ one obtains a very narrow, very deep potential. What does this tell about the motion of the pendulum in the two limiting cases?

Underpin your conclusion for $\mu \to 1$ by showing that in this limit the dimensionless height y of the pendulum weight obeys the EOM

$$\ddot{y} = -1$$

What will be its position x?

2 coupled springs

6.6 *Worked problems: Foucault pendulum*

A Foucault pendulum is a mathematical pendulum set up on the Earth surface where we consider only small amplitude oscillations $\theta \ll 1$ with vanishing angular momentum, $C = 0$. According to Equation (6.4.11) one expects that it follows the EOM

$$\ddot{\theta}(t) = -\frac{g}{\ell} \, \sin\theta(t) \simeq -\frac{g}{\ell} \, \theta(t) \,. \tag{6.6.1}$$

This is a harmonic oscillator with frequency $\omega = \sqrt{g/\ell}$ and solution $\theta(t) = \theta_{\max} \, \sin(\alpha + \omega \, t)$. The amplitude $\theta_{\max} \ll 1$ and the offset phase α must be chosen to match boundary conditions. One expects that this motion will be confined to a fixed plane $\varphi =$const.

In contrast the plane of motion is turning slowly, and the angular speed of the turning depends on the location where one sets up a Foucault pendulum. This is a striking consequence of the fact that a laboratory on the Earth surface is *not* an inertial system since it is turning around the Earth axis once every 24 hours.

In order to account for the Earth rotation we adopt spherical coordinates $\hat{\boldsymbol{r}}(\lambda, \phi)$ with origin at the Earth center and directions indicated by the longitude λ and latitude ϕ (see sketch in the margin). Earth is rotation around the axis $\hat{z}$ with a fixed angular frequency $\Omega = 2\pi/24\,\text{h}$. Hence, ϕ =const and $\lambda = \Omega\, t$ for a pendulum at a fixed position on the Earth surface.

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Figure 6.20: Foucault pendulum in the Panthéon, Paris, France.

We denote the distance of the pendulum plane from the Earth center as R, and the deflection of the pendulum towards the East and North as $x(t)$ and $y(t)$, respectively. Consequently, the position $\boldsymbol{q}(t)$ of the pendulum weight amounts to

$$\boldsymbol{q}(t) = R\, \hat{\boldsymbol{r}}\big(\lambda(t), \phi(t)\big) + x(t)\, \hat{\boldsymbol{\lambda}}\big(\lambda(t)\big) + y(t)\, \hat{\boldsymbol{\phi}}\big(\lambda(t), \phi(t)\big)$$

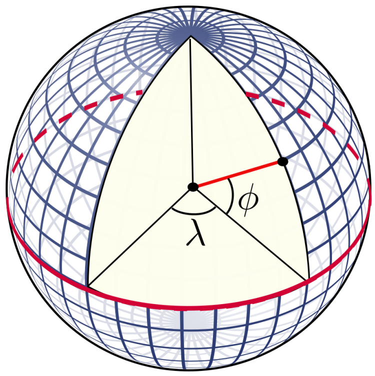


Figure 6.21: Choice of the angles indicating the longitude $\lambda \in \{-\pi/2, \pi/2\}$ and latitude $\phi \in \{-\pi/2, \pi/2\}$ of positions on Earth.

Therefore, the temporal change of each unit coordinate vector must be orthogonal to $\hat{z}$, and it must also be orthogonal to the vector in order to preserve normalization. Therefore,[7]

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{r}}(t) = \Omega\, \hat{z} \times \hat{\boldsymbol{r}}(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{\lambda}}(t) = \Omega\, \hat{z} \times \hat{\boldsymbol{\lambda}}(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{\phi}}(t) = \Omega\, \hat{z} \times \hat{\boldsymbol{\phi}}(t)$$

and

$$\dot{\boldsymbol{q}} = \Omega\, \hat{z} \times \boldsymbol{q} + \dot{x}\, \hat{\boldsymbol{\lambda}}(t) + \dot{y}\, \hat{\boldsymbol{\phi}}(t)$$

This provides the Lagrange function as follows the potential energy and the kinetic energy

$$V = M\, \omega^2\, \left(x^2(t) + y^2(t)\right)$$

$$T = \frac{M}{2}\, \dot{\boldsymbol{q}}^2 = \frac{M\, \Omega^2}{2}\, \left(\hat{z} \times \boldsymbol{q}\right)^2 + M\, \Omega\, \left(\hat{z} \times \boldsymbol{q}\right) \cdot \left(\dot{x}\, \hat{\boldsymbol{\lambda}}(t) + \dot{y}\, \hat{\boldsymbol{\phi}}(t)\right) + \frac{M}{2}\, \left(\dot{x}^2 + \dot{y}^2\right)$$
$$= \frac{M\, \Omega^2\, \cos^2\phi}{2}\, \boldsymbol{q}^2(t) + M\, \Omega\, \Big(\hat{z} \times \boldsymbol{q}\Big) \cdot \left(\dot{x}\, \hat{\boldsymbol{\lambda}}(t) + \dot{y}\, \hat{\boldsymbol{\phi}}(t)\right) + \frac{M}{2}\, \left(\dot{x}^2(t) + \dot{y}^2(t)\right)$$

where we used that $|\hat{z} \times \boldsymbol{q}| = |\boldsymbol{q}|\, \sin\angle(\hat{z}, \boldsymbol{q}) = |\boldsymbol{q}|\, \sin\phi$. Hence, the Lagrange function takes the form

$$\mathcal{L} = \frac{M}{2}\, \left(\dot{x}^2(t) + \dot{y}^2(t)\right) + M\, \Omega\, \left(\hat{z} \times \boldsymbol{q}\right) \cdot \left(\dot{x}\, \hat{\boldsymbol{\lambda}}(t) + \dot{y}\, \hat{\boldsymbol{\phi}}(t)\right)$$
$$+ \frac{M\, \Omega^2\, \cos^2\phi}{2}\, \left(R^2 + x^2(t) + y^2(t)\right) - \frac{M\, \omega^2}{2}\, \left(x^2(t) + y^2(t)\right)$$

For $x(t)$ we obtain the following contributions to the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = M\, \ddot{x} + M\, \Omega\, \Big[\left(\hat{z} \times \dot{\boldsymbol{q}}\right) \cdot \hat{\boldsymbol{\lambda}} + \left(\hat{z} \times \boldsymbol{q}\right) \cdot \left(\hat{z} \times \hat{\boldsymbol{\lambda}}\right)\Big]$$
$$= M\, \ddot{x} + M\, \Omega\, \left(\hat{\boldsymbol{\lambda}} \times \hat{z}\right) \cdot \left(\dot{\boldsymbol{q}} - \hat{z} \times \boldsymbol{q}\right)$$
$$= M\, \ddot{x} + M\, \Omega\, \left(\hat{\boldsymbol{\lambda}} \times \hat{z}\right) \cdot \left(\dot{x}\, \hat{\boldsymbol{\lambda}}(t) + \dot{y}\, \hat{\boldsymbol{\phi}}(t)\right)$$

[7] The reader is urged to check that by an explicit calculation.

and

$$\frac{\partial \mathcal{L}}{\partial x} = M\,\Omega\,\left(\hat{z}\times\hat{\boldsymbol{\lambda}}\right)\cdot\left(\dot{x}\,\hat{\boldsymbol{\lambda}}(t)+\dot{y}\,\hat{\boldsymbol{\phi}}(t)\right) - M\left(\omega^2-\Omega^2\,\cos^2\phi\right)\,x(t)$$

This provides the EOM

$$\ddot{x} = 2\,\Omega\,\left(\hat{z}\times\hat{\boldsymbol{\lambda}}\right)\cdot\left(\dot{x}\,\hat{\boldsymbol{\lambda}}(t)+\dot{y}\,\hat{\boldsymbol{\phi}}(t)\right) - \left(\omega^2-\Omega^2\,\cos^2\phi\right)\,x(t)$$

Moreover, for $y(t)$ we obtain the same expression up substituting $y(t)$ for $x(t)$ and $\hat{\boldsymbol{\phi}}$ for $\hat{\boldsymbol{\lambda}}$,

$$\ddot{y} = 2\,\Omega\,\left(\hat{z}\times\hat{\boldsymbol{\phi}}\right)\cdot\left(\dot{x}\,\hat{\boldsymbol{\lambda}}(t)+\dot{y}\,\hat{\boldsymbol{\phi}}(t)\right) - \left(\omega^2-\Omega^2\,\cos^2\phi\right)\,y(t)$$

Consequently,

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ 0\end{pmatrix} = -2\,\Omega\,\hat{z}\times\begin{pmatrix}\dot{x}\\ \dot{y}\\ 0\end{pmatrix} - \left(\omega^2-\Omega^2\,\cos^2\phi\right)\begin{pmatrix}x\\ y\\ 0\end{pmatrix}$$

Here, we padded the component for the motion in vertical direction by zeros. Now we observe, that $\hat{z} = \sin\phi\,\hat{\boldsymbol{r}} + \cos\phi\,\hat{\boldsymbol{\phi}}$ and we observe that the contribution to of $\cos\phi\,\hat{\boldsymbol{\phi}}$ to the cross product will give rise to a vector in $\hat{\boldsymbol{r}}$ direction, which we do not consider here. Consequently,

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = -2\,\Omega\,\sin\phi\,\begin{pmatrix}-\dot{y}\\ \dot{x}\end{pmatrix} - \left(\omega^2-\Omega^2\,\cos^2\phi\right)\begin{pmatrix}x\\ y\end{pmatrix}$$

This EOM is most transparently solved by introducing the complex variable $z = x + \mathrm{i}\,y$, and observing that $-\mathrm{i}\dot{z} = -\dot{y} + \mathrm{i}\,\dot{x}$. Hence,

$$\ddot{z} = \ddot{x} + \mathrm{i}\ddot{y} = 2\,\mathrm{i}\,\Omega\,\sin\phi\,\dot{z} - \left(\omega^2-\Omega^2\,\cos^2\phi\right)z$$

This linear, homogeneous differential equations in solved by superpositions of exponential functions, $\exp(\gamma\,t)$ where γ obeys the characteristic equation

$$0 = \gamma^2 - 2\,\mathrm{i}\,\Omega\,\sin\phi\,\gamma + \left(\omega^2-\Omega^2\,\cos^2\phi\right)$$

$$\begin{aligned}\Rightarrow\quad \gamma_\pm &= \mathrm{i}\,\Omega\,\sin\phi \pm \mathrm{i}\,\sqrt{\Omega^2\,\sin\phi^2 + \omega^2 - \Omega^2\,\cos^2\phi}\\ &= \mathrm{i}\,\left(\Omega\,\sin\phi \pm \sqrt{\omega^2-\Omega^2\,\cos(2\phi)}\right)\\ &\simeq \pm\mathrm{i}\,\omega + \mathrm{i}\,\Omega\,\sin\phi\end{aligned}$$

with corrections to the frequency ω that are smaller to the leading order by a factor of $(\omega/\Omega)^2$.

The Foucault pendulum is started at a position $z(t_0)$ with velocity $\dot{z}(t_0)$. For this initial condition we obtain the solution

$$z(t) = z_0\,\exp\!\big(\mathrm{i}\,(\Omega\,\sin\phi)\,(t-t_0)\big)\,\cos\!\big(\omega\,(t-t_0)\big)$$

The phase factor of $z_0 \in \mathbb{C}$ accounts for the initial orientation of the displacement. The evolution of the phase factor $\exp\!\big(\mathrm{i}\,(\Omega\,\sin\phi)$ induces a rotation of the plain of motion with a frequency of $\Omega\,\sin\phi$.

At the poles this rotation entails that the orientation of the oscillation is fixed for an outside observer, while the Earth rotates underneath.

At the equator there is no rotation.

At mid latitudes one observed a rotation frequency of $\Omega \sin\phi$. Note that the sine factors entails that the Foucault pendulum moves clockwise on the Northern hemisphere and counter-clockwise on the Southern hemisphere.

6.7 Problems

6.7.1 Rehearsing Concepts

Problem 6.16. **Two masses hanging at a rubber band**

Two weights of the same mass m are attached on opposite ends of a rubber band that is hanging over a roll. The weights are at height h_1 and h_2. They move only vertically, either one up and one down at a fixed length of the band, or stretching the band, or releasing tension on the band. We assume that friction and the mass of the band are negligible.

a) Sketch the problem, and indicate the relevant parameters and coordinates.

b) We describe the problem by adopting the coordinates $H = h_1 + h_2$ and $D = h_1 - h_2$. Verify that the Lagrange function will then take the following form

$$\mathcal{L}(H, D, \dot{H}, \dot{D}) = \frac{\mu}{2}\left(\dot{H}^2 + \dot{D}^2\right) - mg\,H - \frac{k}{2}\,H^2$$

Here k is the elastic module of the rubber band, and μ is an effective mass. How is μ related to m?
The expression for the $\mathcal{L}$ adopts a particular choice of the origin used to specify h_1 and h_2. Which choice has been used?

c) Determine the equations of motion for H and D.

d) Solve the equations of motion and interpret the result. For which values of H and D will you trust the result?

Problem 6.17. **Kitchen pendulum**

We consider a pendulum that is built from two straws (length L_1 and L_2), two corks (masses m_1 and m_2), a paper clip, and some Scotch tape (see picture to the right). It is suspended from a shashlik skewer, and its motion is stabilized by means of the spring taken from a discharged ball-pen. Hence, the arms move vertically to the skewer. We denote the angle between the arms as α, and the angle of the right arm with respect to the horizontal as $\theta(t)$.

Figure 6.22: Setup of the kitchen pendulum.

a) Determine the kinetic energy, T, and the potential energy, V, of the pendulum. Argue that T and V can only depend on θ and $\dot{\theta}$, and determine the resulting Lagrangian $\mathcal{L}(\theta, \dot{\theta})$.

b) Determine the EOM of the pendulum.

c) Find the rest positions of the pendulum, and discuss the motion for small deviations from the rest positions. Sketch the according motion in phase space.

d) The EOM becomes considerably more transparent when one selects the center of mass of the corks as reference point. Show that the center of mass lies directly below the fulcrum when the pendulum it at rest.

e) Let ℓ be the distance of the center of mass from the fulcrum, and $\varphi(t)$ be the deflection of their connecting line from the vertical. Determine the Lagrangian $\mathcal{L}(\varphi, \dot{\varphi})$ and the resulting EOM for $\varphi(t)$.

f) Do you see how the equations for $\ddot{\theta}(t)$ and $\ddot{\varphi}(t)$ are related?

Problem 6.18. **The driven pendulum**

We consider a (mathematical) pendulum of mass m with a pendulum arm of length ℓ. The deflection of the arm with respect to its rest position is denoted as $\theta(t)$. The fulcrum of the pendulum is moving vertically, residing at the position $z(t)$ at time t.

a) Sketch the setup of for this problem, and mark the relevant physical parameters and coordinates.

b) Determine the kinetic energy T and the potential energy V of the pendulum mass.

c) We assume that the mass of the pendulum arm does not play a significant role for the motion. Use the Lagrange formalism to determine the equation of motion for θ.

Can you imagine why $\ddot{\theta}(t) = 0$ for $\ddot{z} = -g$?

d) Discuss the stability of the fixed points of the motion for $\ddot{z} > -g$ and $\ddot{z} < -g$. Sketch corresponding phase-space plots for constant $\ddot{z}$.

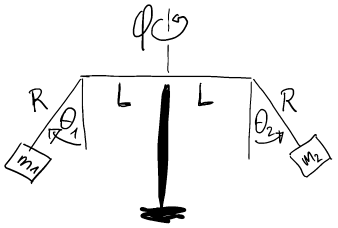
⋆ e) Consider now the T-periodic motion with

$$z(t) = \begin{cases} a\,t\,(t - T/2) & \text{for} \quad 0 < t < T/2 \\ a\,(T - t)\,(t - T/2) & \text{for} \quad T/2 < t < T \end{cases}$$

What does this imply for $\ddot{z}$? How would the trajectories close to $\theta = 0$ and $\theta = \pi$ look like for $a > g$?

Problem 6.19. **Freely rotating carousel**

We consider a carousel with two arms of length R that are attached to cantilevers of length L. The arms can swing outwards and inwards, and we assume that they move without friction and can take full turns. Their outwards deflection will be denoted as $\theta_1(t)$ and $\theta_2(t)$, respectively. At time t the cantilever turns towards the



direction $\varphi(t)$ in the horizontal plane. We consider free rotation; there is no external driving controlling the evolution of φ. We describe the system in cylinder coordinates, $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{\varphi}}$, with $\hat{\boldsymbol{z}}$ aligned antiparallel to gravity $\boldsymbol{g}$.

a) Provide a sketch to illustrate that the positions $\boldsymbol{q}_i$ of mass $i \in \{1, 2\}$ can be expressed as,

$$\boldsymbol{q}_i = (L + R\,\sin\theta_i)\,\hat{\boldsymbol{r}} - R\,\cos\theta_i\,\hat{\boldsymbol{z}}\,,$$

and take the derivative to show that

$$\dot{\boldsymbol{q}}_i = R\,\dot{\theta}_i\,\cos\theta_i\,\hat{\boldsymbol{r}} + (L + R\,\sin\theta_i)\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}} + R\,\dot{\theta}_i\,\sin\theta_i\,\hat{\boldsymbol{z}}\,.$$

b) Establish the kinetic energy, T, the potential energy V, and the Lagrange function.

c) Verify that φ is a cyclic variable and show that the associated conserved quantity takes the form

$$K = f(\theta_1, \theta_2)\,\dot{\varphi}$$

Provide the explicit form of $f(\theta_1, \theta_2)$.

d) Determine the equations of motion for $\theta_i(t)$, and select an appropriate time scale (the same for both angles!) such that

$$\ddot{\theta}_i = -\sin\theta_i + \frac{k^2}{\big(f(\theta_1, \theta_2)\big)^2}\,\cos\theta_i\,(\lambda + \sin\theta_i)$$

Which time scale has been adopted?
How do k and λ depend on the masses, lengths scales, and K?

⋆ e) We have found that the dynamics of θ_1 and θ_2 are coupled due to the appearance of $f(\theta_1, \theta_2)$. Compare this dynamics of the angles to the case where the rotation frequency $\dot{\varphi}$ is controlled by a motor — like for a carousel. How does the coupling differ? What does this imply for the time evolution?

f) The discussion of the coupled dynamics is too involved for the exam. In the following we therefore consider the case where no weight is attached to arm two (i.e. $m_2 = 0$). Show that $f(\theta_1, \theta_2)$ will then be proportional to $(\lambda + \sin\theta_1)^2$. Show that this implies that

$$\ddot{\theta}_1 = -\sin\theta_1 + \frac{\kappa^2}{(\lambda + \sin\theta_1)^\nu}\,\cos\theta_1$$

What is the relation between κ, k, and the system parameters? Which value do you find for the value of ν?

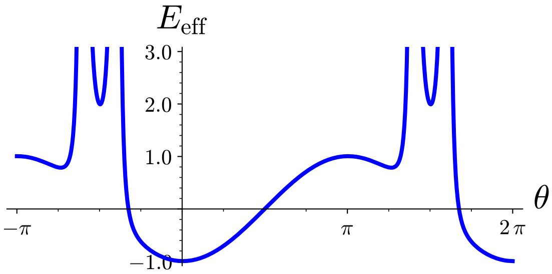
⋆ g) Show that the dynamics is governed by an effective potential

$$U_{\text{eff}}(\theta_1) = -\cos\theta_1 + \frac{c}{(\lambda + \sin\theta_1)^2}$$

Determine the constant c in this expression.

h) When λ is large the prefactor of the $\cos\theta_1$ is almost constant. Sketch the phase-space dynamics!

i) For all $\lambda < 1$ the dynamics in phase-space changes dramatically, no matter how small one chooses κ. For instance the following graph shows the effective potential for $\kappa = 0.1$ and $\lambda = 0.95$.



Sketch the phase-space dynamics for this parameter value.

Bonus: How does the arm move close to the stable fixed points?

Why does the effective potential change so dramatically for $\lambda < 1$? What is the physical origin of this change?

j) Finally we consider the case of two equal masses on a carousel with $\lambda \gg 1$. Further we consider the situation where the two masses are oscillating more or less in phase such that $\theta_1 = \theta + \delta$ and $\theta_2 = \theta - \delta$ with $\delta \ll 1$. Show that the equations of motion can then be approximated as

$$\ddot{\theta} \pm \ddot{\delta} = -\cos\theta \mp \delta\,\sin\theta + \frac{\kappa^2}{2}\,\frac{(\lambda + \sin\theta) \pm \delta\,\cos\theta}{(\lambda + \sin\theta)^4}$$

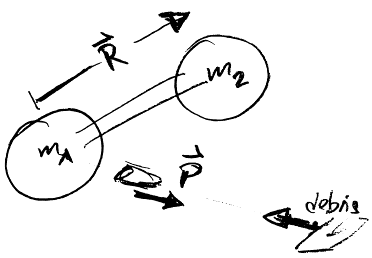
Take the sum and the difference of the two equations to show that θ evolves in the same way as for the case where only one mass was attached, while

$$\ddot{\delta} \simeq -\delta\,\left(\sin\theta - \frac{\kappa\,\cos\theta}{(\lambda + \sin\theta)^4}\right]$$

What does this imply when θ resides at the fixed point θ_c where also $\theta_c \ll 1$.

Problem 6.20. **Space-station lost**

We consider a cluster of two coupled space stations (masses m_1 and m_2) that are connected by a supply channel whose mass may be neglected as compared to m_1 and m_2. The supply channel takes the form of a stiff tube that cannot bend. However, it is attached to the stations in such a way that the distance R between the stations may change; with a restoring force $-k\,(R - \ell)$ towards the rest length ℓ.



We consider an emergency where one of the stations is approached by a heavy piece of space debris (cf. sketch in the margin). To prevent damage the debris is intercepted by a space torpedo that leaves the endangered station with momentum $\boldsymbol{P}$. We choose a coordinate frame where the station is at rest initially. How will it move after launching the torpedo?

We approach the problem by first determining the equations of motion of the coupled space stations. Subsequently, we discuss the situation at hand.

a) Determine the center of mass $\boldsymbol{Q}$ of the coupled space stations, and its equation of motion.

   Before launching the torpedo the center of mass is at rest at the origin of our coordinate system. How does it move after the launch?

b) The launch of the torpedo induces a torque that leads to a rotation of the coupled space stations. We will now discuss this rotation in their center of mass frame, i.e., we discuss the evolution of the vector $\boldsymbol{R}(t)$ that describes their orientation in space.

   Show that the angular momentum $\boldsymbol{L} = \boldsymbol{R}(t) \times \mu\dot{\boldsymbol{R}}(t)$ is preserved.

⋆ c) Why does this imply that the coupled space station will rotate in the plane defined by the initial position of m_1 and m_2, and the approaching piece of debris?

d) Determine the Lagrange function for the motion of $\boldsymbol{R}$ in the plane selected by angular-momentum conservation, and determine the equation of motion for $\boldsymbol{R}(t)$.

e) Show that the conservation of energy E and the modulus of angular momentum imply that

$$E = a\dot{R}^2 + \frac{b}{R^2} + c\,R^2 - d\,R + e$$

   and determine the positive real constants a, b, c, d, and e in terms of the masses m_1 and m_2, the rest length ℓ of the supply channel, strength k of the restoring force, and the angular momentum $|\boldsymbol{L}| = L$.

f) Determine the effective potential $\Phi_{\text{eff}}(R)$ for the evolution of the distance $R(t)$, and sketch the solutions of its equation of motion in phase space.

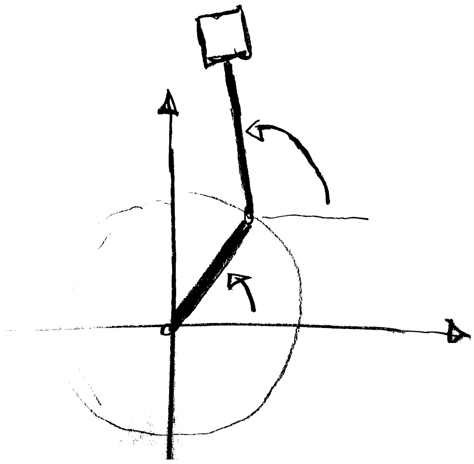
⋆ g) Mark the initial condition of $R(t)$ in the phase-space plot, i.e. its value and speed immediately after launch of the torpedo, and discuss the entailed evolution of $\boldsymbol{R}(t)$.

⋆ h) How does the minimum *energy* required to stop its rotation depend on the direction of approach of the debris?

Problem 6.21. **Horizontal driven double pendulum**

A double pendulum is a mathematical pendulum with another pendulum attached to its loose end (see sketch). We will consider here a horizontal double pendulum where the arms run in a horizontal plane perpendicular to gravity. Moreover, we will control the position of the outer pendulum by a motor such that the angle between the x-axis and the outer arms revolves with with a *fixed, constant* frequency Ω.

a) Label the quantities in the sketch:



The pendulum is moving in the (x, y) plane. How is gravity oriented?
The inner arm has length R. Its fixed end lies in the origin of the (x, y)-plane, and it is pointing in direction $\theta(t)$ with respect to the positive x axis.
The outer arm has a length L. It connects the loose end of the inner arm and a weight of mass M. The angle between this arm and the x-axis is Ωt.

b) Consider the vectors $\hat{\boldsymbol{r}}(\phi) = \hat{\boldsymbol{x}}\,\cos\phi + \hat{\boldsymbol{y}}\,\sin\phi$ and $\hat{\boldsymbol{\phi}}(\phi) = \frac{\mathrm{d}}{\mathrm{d}\phi}\hat{\boldsymbol{r}}(\phi)$. Show that for each value of ϕ they form an orthonormal pair of unit vectors in the (x, y)-plane.

c) Express the position $\boldsymbol{q}$ of the weight with respect to the origin as a linear combination of $\hat{\boldsymbol{r}}(\theta)$ and $\hat{\boldsymbol{r}}(\Omega t)$; i.e. determine a and b such that

$$\boldsymbol{q}(t) = a\,\hat{\boldsymbol{r}}(\theta) + b\,\hat{\boldsymbol{r}}(\Omega t)$$

d) Determine $\dot{\boldsymbol{q}}(t)$ and the kinetic energy of the horizontal driven double pendulum.

e) Determine (i) the potential energy, (ii) the Lagrange function, and (iii) the equation of motion for $\theta(t)$.

f) Show that – with an appropriate choice of the time scale – the dimensionless equation of motion for $\alpha(t) = \theta(t) - \Omega\,t$ takes the form

$$\ddot{\alpha}(t) = -\sin\alpha(t)$$

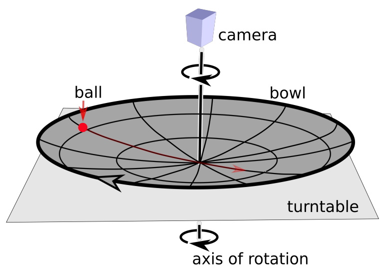
Which time scale was chosen to obtain this result?

g) Show that $E = \dfrac{\dot{\alpha}^2}{2} - \cos\alpha$ is a constant of motion.

h) Determine the stable and unstable fixed points of the equation of motion. Sketch its solutions in phase space.

⋆ i) The results of f) and g) imply that $\alpha(t)$ behaves like a mathematical pendulum. Interpret its solutions:

- Which motion of the double pendulum is described by the stable fixed points?
- Which track does the double pendulum follow for closed trajectories around a stable fixed point?
- Which motion is described the the unstable fixed points?
- How do trajectories look like where $\dot{\alpha}$ never changes its sign?
- How does the system's homoclinic trajectory look like in terms of $\theta(t)$?



Problem 6.22. **Coriolis forces on the turntable**

The effect of Coriolis forces on a particle can beautifully be described by the motion of a ball moving in bowl placed on a turntable (cf. sketch in the margin). Let m be the mass of the ball, Ω the rotation frequency of the turntable, and $z(r)$ the cylinder-symmetric height profile of the bottom of the bowl, as function of the distance r from the rotation axis. We will disregard effects due to friction and the rolling of the ball.

a) We select a bowl with a height profile that gives rise to a harmonic force $\boldsymbol{F} = -k\boldsymbol{r}$ towards the center of the turntable, and we choose Ω such that the centripetal force balances the force $\boldsymbol{F}$. Determine Ω.

b) First, we describe the motion of the ball from the perspective of an observer standing at rest next to the turntable. The position of the ball is specified by the complex number $r(t)$, that describes the distance to the rotation axis. Show that the motion of the ball takes the form of an ellipse

$$r(t) = A\,\mathrm{e}^{\mathrm{i}(\theta+\phi)}\,\mathrm{e}^{\mathrm{i}\Omega t} + B\,\mathrm{e}^{\mathrm{i}(\theta-\phi)}\,\mathrm{e}^{-\mathrm{i}\Omega t}\,, \qquad \text{with} \quad \theta, \phi, A, B \in \mathbb{R}\,.$$

How are the parameter of the ellipse related to θ, ϕ, A, and B?

c) Now, we describe the motion of the ball from the perspective of a camera that is mounted on the turntable, and follows the motion in the frame of reference moving with the turntable. In this case we describe the position of the ball by the complex number $q(t)$. Demonstrate that the equations of motion take the form

$$\frac{\delta^2 \boldsymbol{q}(t)}{\delta t^2} = -2\Omega\,\mathrm{i}\,\frac{\delta \boldsymbol{q}(t)}{\delta t}\,.$$

Determine the solution of this equation.

d) At time t_0 we release the ball with velocity $\dot{r}(t_0) = 0$ at the position $r(t_0) = R \in \mathbb{R}$. Determine the resulting solutions $r(t)$ and $q(t)$. How are the solutions related?

6.7.2 *Mathematical Foundation*

Problem 6.23. **Shortest path on a sphere**

We describe the position on the surface of a three-dimensional sphere by the angle θ with its "North pole", and the azimuthal angle φ in the horizontal plane. A trajectory on the sphere with radius R can then be specified as $\boldsymbol{q}(t) = R\,\hat{\boldsymbol{r}}\big(\theta(t), \varphi(t)\big)$, or alternatively by $\theta(\varphi)$ or $\varphi(\theta)$. We will now derive conditions for a path of extremal length on the sphere.

a) Without restriction of generality we restrict our discussion to spheres with unit radius, $R = 1$. Why is this admissible?

b) Show that the length of the path from $\big(\theta(t_i), \varphi(t_i)\big) = (\theta_i, \varphi_i)$ to $\big(\theta(t_e), \varphi(t_e)\big) = (\theta_e, \varphi_e)$ amounts to

$$L = \int_{t_i}^{t_e} \mathrm{d}t \left|\dot{\hat{\boldsymbol{r}}}\big(\theta(t), \varphi(t)\big)\right| = \int_{\varphi_i}^{\varphi_e} \mathrm{d}\varphi \sqrt{\sin^2\theta(\varphi) + \left(\frac{\mathrm{d}\theta(\varphi)}{\mathrm{d}\varphi}\right)^2}$$

$$= \int_{\theta_i}^{\theta_e} \mathrm{d}\theta \sqrt{1 + \sin^2\theta \left(\frac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}\right)^2}$$

Under which conditions do the expressions apply? Why and when do they provide the same length?

c) A necessary condition for the extremality of L is that the variation δL vanishes for the integrals that have been defined in (b). Introduce the variation $\theta(\varphi) + \varepsilon\,\delta\theta(\varphi)$ into the second representation of the length (i.e., the one involving an integral over φ), calculate δL, and determine the resulting differential equation for paths $\theta(\varphi)$ of extremal length.

d) Repeat the same steps for the variation $\varphi(\theta) + \delta\varphi(\theta)$ and the representation of the length in terms of $\varphi(\theta)$. Determine the resulting differential equation for paths $\varphi(\theta)$ of extremal length. Which derivation is simpler? How could you have seen this *before* performing the calculations?

e) The result of (d) can be integrated once. Show that this results in the following first order differential equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\theta} = \frac{\cos\alpha}{\sin\theta} \left[\sin^2\alpha - \cos^2\theta\right]^{-1/2} .$$

where $\sin\alpha$ is an integration constant.

f) Verify that the following function is a solution of the differential equation

$$\varphi(\theta) = \varphi_0 - \arcsin\frac{\cos\alpha\ \cos\theta}{\sin\alpha\ \sin\theta} .$$

Hint: Express $\sin(\varphi - \varphi_0)$ as a function of θ. Take the θ derivative of both sides of the equation. Subsequently, you can determine $\varphi'(\theta)$ by eliminating $\cos(\varphi - \varphi_0)$ by means of the known expression for $\sin(\varphi - \varphi_0)$.

g) Show that all the coordinates

$$\boldsymbol{q}(\theta) = (\sin\theta\,\cos\varphi(\theta), \sin\theta\,\sin\varphi(\theta), \cos\theta)$$

of the trajectory obtained in (f) are orthogonal to the vector $(0, \sin\alpha, \cos\alpha)$.
Then: How does the path look like on the sphere?

h) The path from the initial point to the final point on the sphere is not unique! One of the solutions is the shortest path on the sphere. What type of an extremum does the other path represent?

⋆ i) Consider the following ODE for a path on the sphere

$$\dot{\hat{\boldsymbol{r}}}(\theta(t), \varphi(t)) = \left(\hat{\boldsymbol{q}}_1 \times \hat{\boldsymbol{q}}_2\right) \times \hat{\boldsymbol{r}}(\theta(t), \varphi(t))$$

where $\hat{\boldsymbol{q}}_1$ and $\hat{\boldsymbol{q}}_2$ are two distinct points on the sphere. Show that for the initial condition $\hat{\boldsymbol{q}}_1$ the trajectory will proceed through $\hat{\boldsymbol{q}}_2$.

How long does it take to arrive at $\hat{\boldsymbol{q}}_2$?

Does the trajectory represent an extremal path on the sphere? Under which conditions would it be a path of minimal length?

add torus and cone

Problem 6.24. **1D dynamical systems with pitchfork and saddle-node bifurcations**

We consider the dynamical systems

$$\dot{x}_a(t) = a + x^2(t) \tag{6.7.1}$$

We will analyze now how the solutions $x(t)$ change upon varying the parameter $a \in \mathbb{R}$.

a) With no loss of generality we will assume that $t_0 = 0$ and $x_0 = x(t_0) = 1$. Why is this admissible?
Hint: 1. How would the solutions differ when one chooses $t_0 = 1$ rather than $t_0 = 0$?
2. Consider the evolution of $x(t)/x_0^2$ with a suitable change of the time unit and the parameter a.

b) With no loss of generality we will assume that $a \in \{-1, 0, 1\}$.
Why is this admissible?
Hint: For $a \neq 0$ you may divide the ODE by $\sqrt{a^2} = (\sqrt{|a|})^2$.

c) Solve the differential equation by separation of variables and partial fraction decomposition.

d) Plot the solutions $x_a(t)$ for $a = -1$, $a = 0$, and $a = 1$.
What does this plot tell about the solutions for general t_0, x_0, and a?

e) Determine the fixed points of Equation (6.7.1) and plot the bifurcation diagram.
Can you see how the structure of the bifurcation diagram fits with the explicit solutions obtained in d)?

f) Repeat the analysis for the system

$$\dot{x}(t) = a\,x(t) + x^3(t)$$

6.7.3 *Transfer and Bonus Problems, Riddles*

Problem 6.25. **The steel-can pendulum**

We consider a pendulum that is built from a steel can with two heavy magnets of identical mass M attached inside (see the figure in the margin). When the can is released it starts oscillating. When pushed, it wobbles along a straight path. The angle between the two magnets is denoted as α. We choose our coordinates in such a way that it takes angles in $[0, \pi]$. We describe the configuration of the can by the angle θ that follows how it is rolling, and we choose the origin of θ such that the magnets are positioned at $\pm\alpha/2$.

a) Assume that the mass of the can may be neglected as compared to that of the magnets. Determine the kinetic energy, T, and the potential energy, V, of the can. Use the Lagrange formalism to determine the equation of motion for θ.

b) Determine the equilibrium position of the can for $\alpha \notin \{0, \pi\}$, and provide a physical argument why they are stable or unstable. Sketch the according motion in phase space.

c) Discuss the case $\alpha = \pi$. How does the can move in this case? Sketch the trajectories in phase space.

d) For $\alpha = 0$ you should find the following equation of motion

$$0 = 2\ddot{\theta}\,\big(1 - \cos\theta\big) + \dot{\theta}^2 \sin\theta + 2\,\omega^2\,\sin\theta \qquad (6.7.2)$$

How does ω depend on the radius R of the can, the mass M, and the gravitational acceleration g?

⋆ e) Verify that the EOM for $\alpha = 0$ simplifies considerably upon introducing the variable $x = \cos(\theta/2)$:

$$\ddot{x} = \frac{\omega^2}{2}\,x \qquad (6.7.3)$$

Hint: 1. Evaluate $\ddot{x}$.
2. Eliminate $\cos\theta$ and $\sin\theta$ by observing that $1 - \cos\theta = 2\sin^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.
3. Rewrite the remaining expression in terms of x.

f) Determine the solution of Equation (6.7.3) for the initial condition as shown in the picture where the steel can is released at rest with some displacement θ_0.

g) Determine the solution of Equation (6.7.3) when the steel can is started at $\theta_0 = \pi/2$ with an initial angular speed $-\Omega$.

Sketch the phase-space plot for the case $\alpha = 0$.

Lagrange points

Take Home Message
Hints for Exam Preparation

The aim of the present course has been to give a first gimps into scientific modeling. It focussed on mechanics problems. Firstly, they are easily visualized. Secondly, they readily provide interesting mathematical challenges when one strives for a comprehensive description. Thus, they provide a unique set of problems to get acquainted with the use of mathematics as a language to address scientific problems. The involved mathematical concepts can further be underpinned in forthcoming mathematics classes. Further physical problems will be addressed in forthcoming experimental and theoretical physics lectures.

What are the next steps to be taken? To begin with you should re-read the script and revisit the exercise sheets in order to prepare for the exam. Take a particular look at exercises that were challenging at the first encounter. In doing so you should focus on understanding the rules of the game, and hands-on application of the mathematical formalism, rather than understanding the concepts in full depth. The concepts might be dealt with again in other classes. Most likely they will not put as much emphasis, however, on practicalities about the careful and efficient setup of the mathematical setup for concrete calculations.

Best wishes, success and fun for your further studies!

A
Physical constants, material constants, and estimates

$$1\,\text{year} \simeq \pi \times 10^7\text{s} \qquad \text{(A.0.1)}$$

A.1 Solar System

The solar system has 1.0014 solar masses, which amounts to 1.991×10^{30} kg.
The Earth-Sun distance is $1\,\text{AU} \simeq 500\,\text{light second} \simeq 1.5 \times 10^{11}$ m.

object	**Sun**	**Mecury**	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**	**Uranus**	**Neptun**
distance	0.005	0.387098	0.723332	1	1.523679	5.2044	9.5826	19.2184	30.11
radius	109	0.3829	0.9499	1	0.533	11.209	9.449	4.007	3.883
mass	333,000	0.055	0.815	1	0.107	317.8	95.159	14.536	17.147
period		0.240846	0.615198	1	2.1354	11.862	29.4571	84.0205	164.8

Table A.1: Properties of Sun and planets of our solar system, provided in multiples of the Earth values. The distance referes to the semi-major axis in AU. For the sun the distance denotes the sun surface, i. e. its radius.

object	**Moon**	**Ceres**	**Pluto**	**Eris**
distance	0.00257	2.769	39.482	67.864
radius	0.2727	0.073	0.1868	0.1825
mass	0.0123	0.00016	0.00218	0.0028
period	0.08085	4.61	247.94	559.07

Table A.2: Properties of the Moon and dwarf planets of our solar system. The properties of the Moon refer to its distrance to and period around Earth. Ceres is the largest object in the meteorite belt between Mars and Jupiter. Eris is a dwarf planet in the Kuiper belt that is larger in mass than Pluto.

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