Bachelor Thesis

Spontaneous Symmetry Breaking

Johannes Zierenberg

Faculty of Physics and Earth Sciences
University of Leipzig
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Supervisor:
Roland Kirschner

Second Referee:
Klaus Sibold

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1 Introduction

In the last century much of physics has been built on the properties of symmetry. For example, Special Relativity, which was built on space-time symmetries, or internal conserved charges discovered due to the existence of internal symmetries. Symmetries are always related to the conservation of generalized charges, which makes them attractive. In Nature however, they are not always observed. Many symmetries prefer massless particles, but we only observe a few, one of them the photon. At the same time, the massive elementary particles with different properties have different masses. The idea that the underlying symmetry, which is important for interactions, must be somehow hidden, gave rise to different possible explanations.
One explanation is that the symmetry is simply not observed on the scale of the experiment. An example is the hydrogen atom which appears in low energy physics as an entity. Today, we know that it consists of a proton and an electron. These two are only observed as particles, when we apply enough energy to separate them. Otherwise the symmetry related to charge is not observed at all. The important difference to the following considerations is that we do not experience an asymmetry.

However, in this thesis the concept of symmetry breaking is of more importance. When symmetry is broken, it undergoes a transition from a symmetric system to an asymmetric system. This can occur in two ways, through the explicit or the spontaneous symmetry breaking. *Explicit symmetry breaking* happens if a symmetry is broken by additional objects. Just as a scar in a man's face breaks the face' reflection symmetry explicitly, the small difference in mass between a neutron and a proton breaks their symmetry of exchange explicitly as well. In the case of a symmetric Lagrangian we can add terms that themselves do not obey the symmetry of the Lagrangian. This leaves the new Lagrangian asymmetric. In the case of *Spontaneous symmetry breaking* the underlying system or Lagrangian remains symmetric but the symmetry is hidden when observing the vacuum state, as will be shown later. An example from biology would be the houses of snails [4]. In nature we find that all houses have a spiral structure which is in general right turning. This observation is called *spiral tendency of vegetation* (Goethe). There are few exceptions, snails with left turning houses as a consequence of mutation. However, they are not capable of reproduction with other right turning snails due to important DNA differences. At first sight, it seems to be an asymmetric law of nature that

snails always have right turning houses. It is an example of spontaneous symmetry breaking because nature can produce the other turning snails. In principle both house types are possible (symmetric system). We could say that at the beginning there were two small populations of snails with different house types. But in evolution the snails with the one type of houses dominated and the other ones did not find appropriate reproduction partners and died out. Which type of house survived was of random choice, it could have been exactly the other way around. This example shows that the underlying system can be symmetric even though reality first appears asymmetric and is described by asymmetric laws.
Physical examples of spontaneous symmetry breaking are the ferromagnet and the electroweak interaction and will be explained later on.

This bachelor thesis will give an introduction into the field of spontaneous symmetry breaking. The 2 main chapters each start with a brief introduction to the global or local symmetries, which is based on the books "Fields, Symmetries, and Quarks" [8] and "Quantenmechanik, Symmetrien" [11]. After the introduction of some important terminology, we come to the case of global continuous symmetries leading to the Goldstone [2] theorem. Goldstone and Nambu [9] were the first to find massless particles in specific models. These particles are called the Goldstone- or sometimes Nambu-Goldstone bosons and the proof of their existance was then given by Goldstone, Salam and Weinberg [7]. This proof and the derivation of the number of appearing Goldstone bosons can be found in the books "Quantum Field Theory" [10] and "The Quantum Theory of Fields" [12] and are performed in detail in this thesis. Afterwards the theorem is demonstrated on the $U(1)$ and $U(1) \otimes SU(2)$ example.
Chapter 4 will discuss the spontaneous breakdown of local symmetries, since Goldstone bosons are absent whenever the broken symmetry is local [6],[1],[3]. The phenomenon called Higgs mechanism, will be introduced with the help of a simple $U(1)$ example followed by an application to a $U(1) \otimes SU(2)$ invariant system. The strong dependence of the Higgs mechanism on the Goldstone theorem will be shown by argumentation and by comparison of the global and local results. At the end, following a summary of my discussion, I will give some insight into the physical importance of these concepts.

2 Preliminaries

2.1 Notations

We will shortly introduce the notion used in this thesis. In general the unit convention is used, where we set the speed of light and Planck's number equal to one, meaning that we choose them as units for velocity and action

$$c = 1 = \hbar. \tag{2.1}$$

It follows that length and time have the same dimensions and the relativistic energy - momentum relations becomes

$$E^2 = p^2 + m^2. \tag{2.2}$$

This leads immediately to the use of four vectors and four operators since time and space are connected. The four vectors are denoted by a greek index, whereas three-component vectors are denoted by an arrow

$$A_\mu = (A_0, -\vec{A}). \tag{2.3}$$

Four vectors with subscripts are called *covariant vectors*, *contravariant vectors* are four vectors with superscripts, defined by

$$A^\mu = g^{\mu\nu} A_\nu \tag{2.4}$$

where $g^{\mu\nu} = g_{\mu\nu}$ is the *metric tensor*, that can be given in the Minkowski space (space and ct-time axes) in Cartesian coordinates

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.5}$$

The space-time four vector, the four-momentum operator and the four-current are then given by

$$\hat{x}^\mu = (t,\vec{x}) = (x^0,\vec{x}), \tag{2.6}$$

$$\hat{p}_\mu = i\partial^\mu = i\frac{\partial}{\partial x^\mu} = i(\frac{\partial}{\partial t},\nabla) \tag{2.7}$$

$$j^\mu = (\rho,\vec{j}). \tag{2.8}$$

The scalar product of four-vectors is defined by the the scalar product of a covariant and a contravariant four-vector. It is very important that under exchange of covariant and contravariant vectors this scalar product remains invariant

$$AB = A^\mu B_\mu = A_\mu B^\mu = g_{\mu\nu}A^\nu B^\mu. \tag{2.9}$$

Then the square of four-vectors is invariant under Lorentz transformations

$$A^2 = A_\mu A^\mu = A_0^2 - \vec{A}^2. \tag{2.10}$$

2.2 Euler-Lagrange Equation and Noether Theorem for Classical Fields

All of our further considerations are going to be connected to fields, which will be described by

$$\phi_\alpha(x) \quad \text{with} \quad x = x^\mu = (t,\vec{x}), \tag{2.11}$$

where α labels the different fields appearing in the theory. In classical mechanics we obtained our equation of motion from the Lagrangian via the Ansatz of Hamilton's principle of least action. This can be used in full analogy for fields. In this case the field amplitude at a coordinate x^μ can be considered as the coordinate of the theory. For an adequate discussion of classical field theory see [8] or [5]

We will see that for fields ϕ_α we obtain an analogous outcome as for generalized variables q_i. For the further short derivation of the Euler-Lagrange equation and the Noether Theorem we will drop the index in the field variables.

Let us consider a variation of the Lagrangian $\mathscr{L}(\phi,\partial_\mu\phi,x^\mu) \to \mathscr{L}'(\phi',\partial_\mu\phi',x'^\mu)$ under the following condition that simplify calculations but maintain applicability of the results:

$$\mathscr{L}'(\phi',\partial_\mu\phi',x'^\mu) = \mathscr{L}(\phi',\partial_\mu\phi',x'^\mu).$$

We consider the following transformations

$$\textbf{coordinate transformation:} \quad x^\mu \to x'^\mu = x^\mu + \delta x^\mu \tag{2.12}$$

$$\textbf{field variable transformation:} \quad \phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) \tag{2.13}$$

where the δ indicate infinitesimal variations in the coordinates or fields. This restricts the application of the Noether theorem, we want to derive at this point, to continuous transformations only. Asking for a vanishing variation of the action we obtain

$$\delta S = \int \delta\mathscr{L} \mathrm{d}^4x = \int \mathscr{L}'(\phi', \partial_\mu\phi', x'^\mu)\mathrm{d}^4x' - \int \mathscr{L}(\phi, \partial_\mu\phi, x^\mu)\mathrm{d}^4x = 0.$$

The transformed space volume element d^4x' can be transformed, recalling the transformation of the coordinates, by the corresponding Jacobian

$$J(x', x) = \det\left(\frac{\partial x'^\mu}{\partial x^\lambda}\right) = \det(\delta^\mu_\lambda + \partial_\lambda(\delta x^\mu)) = 1 + \partial_\mu(\delta x^\mu).$$

Inserting this in the previous expression for the variation of the action we get

$$\delta S = \int \left[\mathscr{L}(\phi', \partial\phi', x^\mu)J(x', x) - \mathscr{L}(\phi, \partial\phi, x^\mu)\right] \mathrm{d}^4x = \int \left[\delta\mathscr{L} + \mathscr{L}\partial_\mu(\delta x^\mu)\right] \mathrm{d}^4x$$

with $\delta\mathscr{L} = \mathscr{L}(\phi', \partial\phi', x^\mu) - \mathscr{L}(\phi, \partial\phi, x^\mu)$, $\mathscr{L}(\phi', \partial\phi', x^\mu) = \mathscr{L}$ and changing the notation $\phi' \to \phi$. Expressing the variation of $\mathscr{L}$ in terms of partial variations we obtain

$$\delta S = \int \left[\frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathscr{L}}{\partial x^\mu}\delta x^\mu + \mathscr{L}\partial_\mu(\delta x^\mu)\right] \mathrm{d}^4x.$$

Uniting the last two terms by the product rule and using the commuting properties of δ and ∂_μ to apply $\partial_\mu(\delta\phi) = \delta(\partial_\mu\phi)$, we arrive at

$$\delta S = \int \left[\frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \partial_\mu(\mathscr{L}\delta x^\mu)\right] \mathrm{d}^4x. \tag{2.14}$$

The second term might be rewritten by reverse use of the product rule in order to collect terms

$$\begin{aligned}\delta S &= \int \left[\frac{\partial \mathscr{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) + \partial_\mu(\mathscr{L}\delta x^\mu)\right]\mathrm{d}^4x \\ &= \int \mathrm{d}^4x \left[\left\{\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\right\}\delta\phi + \partial_\mu\left\{\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi + \mathscr{L}\delta x^\mu\right\}\right] \\ &= 0. \end{aligned} \tag{2.15}$$

In order that the integral vanishes for all variations we first observe the **Euler-Lagrange Equation** to be valid for field variables

$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right) = 0. \tag{2.16}$$

Further the second term in (2.15) has to vanish. We enhance the description of this term by introducing a variation with R infinitesimal parameters ε_r, $r = 1, 2, ..., R$. Then the changes in x^μ and ϕ are given by

$$\delta x^\mu = \varepsilon_r X_r^\mu \quad \text{and} \quad \delta\phi = \varepsilon_r \psi_r, \tag{2.17}$$

where X_r^μ is a matrix and ψ_r is a set of numbers. In this notation we obtain the following condition that needs to be satisfied for every parameter ε_r

$$\partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\psi_r + \mathscr{L}X_r^\mu\right) = \partial_\mu J_r^\mu = 0 \tag{2.18}$$

This is of the form of a continuity equation resulting in the conservation of charges. We obtain an array of continuity equations with the parameter r. This means that for each infinitesimal parameter that describes a transformation under which the Lagrangian is invariant, we obtain a conserved current J_r^μ and therefore a conserved charge defined by the integral over the spacelike hypersurface σ_μ.

$$Q_r = \int_\sigma J_r^\mu \mathrm{d}\sigma_\mu \tag{2.19}$$

This is the quintessence of the **Noether Theorem** mainly that we obtain as many conserved currents as the number of symmetries of the Lagrangian, defined by the number of infinitesimal parameters of the transformation under which the Lagrangian remains invariant.

These results allow us to approach the Lagrangians in a way known from classical mechanics, regarding the field as a generalized coordinate. It furthermore provides the use of theorems known from classical mechanics, for instance the mentioned Noether theorem. So for our purposes we can reduce the inconvenient formalism of classical field theory to a more practical mechanical formalism.

3 Global Symmetry Breaking

In this chapter the origin and the consequences of global symmetry breaking will be discussed. This will be done, starting with exploring general global symmetry transformations of the $U(N)$ and $SU(N)$ transformation groups followed by a general discussion of the Goldstone theorem. The consequences of this theorem will be shown and we finish with two examples.

3.1 Global Symmetries

In order to understand what global symmetries are, we have to understand the meaning of global transformations. The translation back to symmetries is simple: A global symmetry is achieved if a system or a state is invariant under a certain global transformation.

Global transformations are characterised by applying the same transformation everywhere in our space, independent of the coordinates. An easy example is the rotation of a coordinate system by an angle θ, because the transformation rotates every position vector in our coordinate system by the same angle, independent on its direction and length. Transformation operators can be collected in groups. We can define the Lie-groups as continuous groups, whose elements are given by operators $\hat{U}(\alpha_1, ..., \alpha_n; \vec{r})$ depending on n parameters and possibly on the coordinates. In the case of global symmetries these Lie-groups do not have a $\vec{r}$ dependence. One parameter of the group is in general defined such that $\hat{U}(0) = \mathbf{1}$, where $\mathbf{1}$ is the identity matrix. Then we can generalize the operators of a Lie-group in the form

$$\hat{U}(\vec{\alpha}) = \mathrm{e}^{-i\alpha_k \hat{Q}_k}, \tag{3.1}$$

where the α_k are parameters, with $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$ and the $\hat{Q}_k$ are the basic (unknown) operator functions of the group. In general we will use the Einstein sum convention, i.e. double appearing indices are summed up. This leads to the identification of the *generators* $\hat{Q}_k$ by partial derivatives of the operator with respect to the corresponding parameter

$$-i\hat{Q}_k = \frac{\partial}{\partial \alpha_k} \hat{U}(\vec{\alpha}) \Big|_{\vec{\alpha}=0}. \tag{3.2}$$

This will be of use later, when we discuss the number of Goldstone bosons. From the infinitesimal expansion we can obtain commutator relations for our Lie-algebra, which will not be discussed in this context. For a detailed discussion I refer to W.Greiner "Quantenmechanik, Symmetrien"[11]

We will now discuss two specific global symmetry groups, $U(N)$ and $SU(N)$. The $U(N)$ group, meaning unitary group in n-dimensions, is formed with respect to matrix multiplication by unitary quadratic $n \times n$ matrices

$$\hat{U} = \mathrm{e}^{i\hat{H}} \quad , \quad \text{with} \tag{3.3}$$

$$\hat{U}^{\dagger}\hat{U} = \mathbf{1}. \tag{3.4}$$

$\hat{H}$ is a hermitian quadratic matrix, so that

$$H_{ii}^{*} = H_{ii} \qquad \text{and} \tag{3.5}$$

$$H_{ij}^{*} = H_{ji} \tag{3.6}$$

for all i, j. This leads to real eigenvalues for the operators and to N^2 real parameters of $\hat{H}$ and $\hat{U}$. $U(N)$ is a compact Lie-group because the multiplication of any two matrices of the group results in a matrix that belongs to $U(N)$. The trace of an hermitian matrix can be shown to be always real (3.5), so that for unitary matrices it follows from (3.4) that

$$\det\hat{U} = \mathrm{e}^{i\mathrm{tr}(\hat{H})} = \mathrm{e}^{i\alpha}$$
$$|\det(\hat{U})|^2 = (\det(\hat{U}))^{*}\det(\hat{U}) = \det(\hat{U}^{\dagger})\det(\hat{U}) = (\det(\hat{U}))^{-1}\det(\hat{U}) = 1. \tag{3.7}$$

The $SU(N)$ group is called the special unitary group in n dimensions and requires in addition to the $U(N)$ conditions that

$$\det\hat{U} = +1. \tag{3.8}$$

This makes it a subgroup of $U(N)$, but it is still a continuous compact Lie-group, with $N^2 - 1$ real parameters.

We recall the $U(N)$ group has N^2 parameters and therefore also N^2 generators, where the $SU(N)$ group has $N^2 - 1$ parameters and generators. If you now consider global symmetries again, we can see that a $U(N)$ symmetry is achieved, if a system or a state is invariant under any global transformation belonging to $U(N)$. The same holds true respectively with $SU(N)$ or any other transformation group. In this context we sometimes talk about symmetry groups, referring to the invariance under the transformations of that specific group.

In general one can check for the invariance under a transformation group by inserting the transformation and observing for changes. In the case of the global $U(1)$ symmetry, meaning invariance under a constant phase shift, the Lagrangian is automatically invariant if it only consists of terms of $\phi^*\phi$. Then the phase terms $e^{i\alpha}$ cancel with their complex conjugate and since they are independent on position they do not contribute to the derivatives. If we consider a $SU(2)$ symmetry which is a rotation in the two dimensional isospace, we only have to require that the ϕ_i are interchangeable in the Lagrange function. Thus the Lagrangian is required to be symmetric with respect to commutation of the ϕ_i. This will be clear when it comes to working with symmetric Lagrange functions later on.

3.2 The Meaning of Vacuum

In a quantum mechanical notation the vacuum state $|0\rangle$ shall be defined as the state of the system for which the energy is minimal or in other words

$$\langle 0| H |0\rangle = min. \tag{3.9}$$

If the Lagrangian is invariant under a continuous symmetry, then in many cases also the vacuum possesses the same symmetry. If this holds, we say that the symmetry is realized in the "Wigner mode" and it holds

$$e^{-i\varepsilon Q} |0\rangle = |0\rangle \rightarrow Q |0\rangle = 0, \tag{3.10}$$

where Q is a generator of the specific symmetry transformation under consideration and ε is an infinitesimal parameter. This follows from the fact that the exponential function can be written in terms of cos() and sin(). These can again be Taylor expanded in the first components in the region of identity such as

$$e^{-i\varepsilon Q} = 1 - i\varepsilon Q + ... \tag{3.11}$$

In the following we will consider a Lagrangian $\mathscr{L}$ that is invariant under a certain symmetry operation but the vacuum state $|0\rangle$ is not.

One classical example for this occurrence is the ferromagnet. The atoms inside interact through a spin-spin interaction, which is scalar and invariant under rotations. Above the critical Curie temperature T_C the spins are randomly oriented and the ground state is symmetric; but below T_C all spins align parallel and the actual ground state of the ferromagnet is no longer symmetric under rotation. This means that the rotational symmetry of the Hamiltonian of the system is "spontaneously" broken down to cylindrical symmetry determined by the direction of magnetisation. This direction is totally random and all infinitely many other possibilities have, in absence of an external field, the same energy. The vacuum, or ground state is said to be degenerate and does not share the symmetry of the system.

In order to be well prepared we now introduce the notion of *physical fields*. In physics we expect fields, that vary around the vacuum state, having a non-vanishing vacuum expectation value. This fulfills the idea of the vacuum, being the ground state of the system and thus being the focus of the problem. In other words this means that the field shall describe excitations above the vacuum state. Only in this case we can talk about fields, that carry an actual physical meaning, especially mass terms. A classical example would be the harmonic oscillator in a potential that does not obtain its minimum at the origin. In this case we would either shift our potential or rewrite our variables in such a way, that they describe the known harmonic oscillator; we would introduce physical variables. To shortly show the analogous procedure for fields, we consider a field $\rho(x)$ in a one-dimensional linear system, whose vacuum state lies at a distance a from the origin of the coordination system

$$\langle 0| \rho |0\rangle = a. \tag{3.12}$$

Then the physical field $\rho'(x)$ is defined in such a way, that the vacuum state is vanishing and it describes variations around the vacuum

$$\rho(x) = \rho'(x) + a \quad \rightarrow \quad \langle 0| \rho' |0\rangle = 0. \tag{3.13}$$

We see that this is really analog to the classical example of shifting our variables in order to obtain deviations around the minimum.

3.3 Goldstone Theorem

Now we are prepared to discuss the Goldstone theorem. There are two formulations that will be given in this section, the classical and the quantum mechanical one. After the formulation the predictions of the theorem will be proven with the quantum mechanical approach. Due to the consistence principle this will immediately lead to the verification of the classical approach as well.

Classical Formulation: Consider the Lagrangian $\mathscr{L}$ to be invariant under a (continuous) global symmetry transformation. If furthermore the vacuum state does not share the same global symmetry we say that the symmetry is "hidden", or that the symmetry is realized in the "Goldstone mode". In this case there will appear massless particles, so called *Goldstone bosons*.[1]

Quantum Mechanical Formulation: Assume a field operator $\hat{\phi}(x)$ exists, whose vacuum expectation value $\langle 0|\hat{\phi}(x)|0\rangle$ is not zero and that is not a singlet under the transformation of some symmetry group of the Lagrangian, i.e. the transformation that leaves the Lagrangian invariant results in new vacuum states s.t.

$$Q|0\rangle \neq 0. \tag{3.14}$$

Then, since $[Q,H]=0$, the vacuum states are degenerate and there must exist massless particles in the spectrum of states.

The existence of massless particles is not a trivial conclusion and therefore will be proven now. The proof is based on the books [8] and [10]. We recall that we set $c=1=\hbar$, where c is the speed of light and $\hbar$ is the Planck's constant. We begin with the requirement that the Lagrangian is invariant under a continuous global transformation. Noether's theorem (see (2.19)) states that a Lagrangian with a continuous symmetry implies the existance of a conserved current

$$\partial^\mu \hat{j}_\mu = 0. \tag{3.15}$$

This leads instantaneously to the conservation of a charge, at constant time defined by

$$\hat{Q} = \int \mathrm{d}^3x \hat{j}_0(x). \tag{3.16}$$

As was shown earlier, the global transformation can then be applied by the unitary operator

$$\hat{U} = \mathrm{e}^{-i\varepsilon\hat{Q}},$$

where ε is an infinitesimal parameter and $\hat{Q}$, the conserved charge, is the generator.
We consider the operator $\hat{\phi}(y)$ that is not invariant under this transformation i.e. $[\hat{Q},\hat{\phi}(y)]\neq 0$. Since $\hat{\phi}(y)$ is not a singlet under the transformation there must exist another operator $\hat{\phi}'(y)$ such that

$$[\hat{Q},\hat{\phi}'(y)] = \hat{\phi}(y) \neq 0. \tag{3.17}$$

[1] In cases of supersymmetry breaking there may also appear particles with spin $\frac{1}{2}$ [10]

We furthermore suppose that the vacuum expectation value of $\hat{\phi}(y)$ is non-vanishing. Together with the definition of the charge (3.16) and the commutator for $\hat{\phi}(y)$ we obtain

$$\langle 0|\,\hat{\phi}(y)\,|0\rangle = \langle 0|\,[\hat{Q},\hat{\phi}'(y)]\,|0\rangle = \int \mathrm{d}^3x\,\langle 0|\,[\hat{j}_0(x),\hat{\phi}'(y)]\,|0\rangle \neq 0. \tag{3.18}$$

For convenience we will drop the operator notation via hats in the rest of the proof.

For later use we will now prove the time independence of this non-vanishing expectation value. Therefore we make use of the equation (3.15), which leads to

$$\partial^0[j_0(x),\phi'(y)] = -\partial^i[j_i(x),\phi'(y)] = -\nabla[\vec{j}(x),\phi'(y)]. \tag{3.19}$$

Now we integrate the vacuum expectation value of this commutator over the whole volume and in a second step apply Gauss law in order to convert the volume integral into a surface integral

$$\begin{aligned}\frac{\partial}{\partial x_0}\int \mathrm{d}^3x\,\langle 0|\,[j_0(x),\phi'(y)]\,|0\rangle &= -\int \mathrm{d}^3x\,\langle 0|\,[\nabla\vec{j}(x),\phi'(y)]\,|0\rangle \\ &= -\int \mathrm{d}\vec{S}\,\langle 0|\,[\vec{j}(x),\phi'(y)]\,|0\rangle \\ &= 0.\end{aligned} \tag{3.20}$$

The last step follows from the fact, that $\vec{j}$ and ϕ' are local operators and the point x is on a surface, whereas the point y is somewhere in the whole volume. We can make the distance between x and y as large as we need by choosing a corresponding large volume. In this case the commutator vanishes and the integral becomes zero. Thus the non-vanishing vacuum expectation values (3.18) are *time independent*.

In order to fully understand the following derivations we will at this point discuss the consequences of a translation transformation of $j_0(x)$. From quantum mechanics we know that the momentum operator p is responsible for a translation transformation. For clarity we will shortly use the four-vector indication with μ again

$$\begin{aligned} j_0(x) &= \mathrm{e}^{-ix_\mu p^\mu} j_0(0)\mathrm{e}^{ix_\mu p^\mu} \\ \Rightarrow j_0(0) &= \mathrm{e}^{ix_\mu p^\mu} j_0(x)\mathrm{e}^{-ix_\mu p^\mu} \\ &= (1+ix_\mu p^\mu+\ldots)\,j_0(x)(1-ix_\mu p^\mu+\ldots) = j_0(x)+ix_\mu[p^\mu,j_0(x)]+\ldots \end{aligned} \tag{3.21}$$

The last step follows from Taylor expansion of the e-function. Remember that x_μ is the space time four-vector and p^μ is the four-momentum operator. Furthermore we consider $|0\rangle$ and $|n\rangle$ to be eigenstates of p^μ with eigenvalues 0 and p_n^μ. The latter is a four-vector again with the index n in order to indicate to which eigenstate it belongs and to show that it is not an operator. By Taylor expansion of the $j_0(x)$ operator we obtain

$$j_0(0) = j_0(x) - x_\mu \partial^\mu j_0(x)\Big|_{x=0}. \tag{3.22}$$

Comparing the coefficients, we obtain the relation

$$[p^\mu, j_0(x)] = -i\partial^\mu j_0(x). \tag{3.23}$$

Taking the matrix elements of this expression we get

$$\langle 0| p^\mu j_0(x) - j_0(x) p^\mu |n\rangle = -i\partial^\mu \langle 0| j_0(x) |n\rangle, \tag{3.24}$$

and since $|0\rangle$ and $|n\rangle$ are eigenstates to the operator p^μ with eigenvalues 0 and p_n^μ this expression reduces to a set of partial differential equations solvable by the separation of variables Ansatz

$$-p_n^\mu \langle 0| j_0(x) |n\rangle = -i\partial^\mu \langle 0| j_0(x) |n\rangle \tag{3.25}$$

$$\Rightarrow \langle 0| j_0(x) |n\rangle = \langle 0| j_0(0) |n\rangle \, \mathrm{e}^{-ip_n^\mu x_\mu}. \tag{3.26}$$

Analogously this method leads to the following expression for the matrix elements on the other side of the diagonal

$$p_n^\mu \langle n| j_0(x) |0\rangle = -i\partial^\mu \langle n| j_0(x) |0\rangle \tag{3.27}$$

$$\Rightarrow \langle n| j_0(x) |0\rangle = \langle n| j_0(0) |0\rangle \, \mathrm{e}^{ip_n^\mu x_\mu}. \tag{3.28}$$

Now we are prepared to approach the relation (3.18) again. We start by inserting a total set of states $|n\rangle$ in between j_0 and ϕ' such that

$$\int \mathrm{d}^3x \langle 0| [j_0(x)\phi'(y)] |0\rangle = \sum_n \int \mathrm{d}^3x [\langle 0| j_0(x) |n\rangle \langle n| \phi'(y) |0\rangle - \langle 0| \phi'(y) |n\rangle \langle n| j_0(x) |0\rangle] \tag{3.29}$$

Applying the translation transformations (3.26) and (3.28) as well as the definition of the δ-function

$$\delta(p-p') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i(p-p')q}\mathrm{d}q = \delta(p'-p) \tag{3.30}$$

and $p_n^\mu x_\mu = p_n^0 x_0 - \vec{p}_n\vec{x}$ we obtain

$$\begin{aligned}
\int \mathrm{d}^3x \quad & \langle 0|\,[j_0(x)\phi'(y)]\,|0\rangle \\
= \quad & \sum_n \int \mathrm{d}^3x \Big[\langle 0|\, j_0(0)\,|n\rangle \langle n|\,\phi'(y)\,|0\rangle\, \mathrm{e}^{-ip_n^\mu x_\mu} - \langle 0|\,\phi'(y)\,|n\rangle \langle n|\, j_0(0)\,|0\rangle\, \mathrm{e}^{ip_n^\mu x_\mu} \Big] \\
= \quad & \sum_n \Big[\langle 0|\, j_0(0)\,|n\rangle \langle n|\,\phi'(y)\,|0\rangle\, \mathrm{e}^{-ip_n^0 x_0} \int \mathrm{e}^{i\vec{p}_n\vec{x}}\mathrm{d}^3x \\
& - \langle 0|\,\phi'(y)\,|n\rangle \langle n|\, j_0(0)\,|0\rangle\, \mathrm{e}^{ip_n^0 x_0} \int \mathrm{e}^{-i\vec{p}_n\vec{x}}\mathrm{d}^3x \Big] \\
= \quad & (2\pi)^3 \sum_n \delta^3(\vec{p}_n) \Big[\langle 0|\, j_0(0)\,|n\rangle \langle n|\,\phi'(y)\,|0\rangle\, \mathrm{e}^{-iE_n x_0} - \langle 0|\,\phi'(y)\,|n\rangle \langle n|\, j_0(0)\,|0\rangle\, \mathrm{e}^{iE_n x_0} \Big] \\
\neq \quad & 0
\end{aligned}$$

where we have rewritten p_n^0 by E_n.

We can now obtain several conclusions out of this last equation. First we see, that the δ-function limits the $|n\rangle$ states, giving them zero momentum. By the use of the relativistic energy expression

$$E_n = \sqrt{p_n^2 + m_n^2}, \tag{3.31}$$

and remembering that $c = 1$, we conclude that $E_n = m_n$. Since these states have an e-function term that is time dependent (x_0 refers to the time coordinate), but on the other hand equation (3.20) tells us that the whole expression is time independent we conclude

$$\langle 0|\, j_0(0)\,|n\rangle = 0 \quad \forall m_n \neq 0 \tag{3.32}$$

Further from our considerations (3.18) we know that the expression cannot be zero, so there must exist states that refer to *massless particles*

$$m_n = 0 \quad \langle 0|\, j_0(0)\,|n\rangle \langle n|\,\phi'(y)\,|0\rangle \neq 0. \tag{3.33}$$

Then also $\langle 0|\, j_0\,|n\rangle \neq 0$ holds which means, that the quantum numbers of $|n\rangle$ have to agree with those of $j_0\,|0\rangle$, due to orthogonality. This means, that the massless particles carry the quantum numbers of the generator. It can be also concluded, since j_μ is either an axial or polar vector and therefore j_0 carries no internal angular momentum that the massless particles are scalar bosons.

3.4 Physical Consequences - The Goldstone Bosons

Now that we have proven the existence of massless particles as a consequence of global symmetry breaking, the question of the number of appearing Goldstone bosons arises. This will be the essence of the next part, returning to a general potential and expanding it around its vacuum state. By our general assumptions we will obtain certain conditions that can be interpreted with the help of the existing symmetry groups.

We consider the potential V to be invariant under transformations from the symmetry group G, so that for $g \in G$ we obtain

$$V(\phi') = V(U(g)\phi) = V(\phi), \quad \phi' \neq \phi \quad \text{in general.} \tag{3.34}$$

We will furthermore assume that the vacuum state is invariant under a subgroup H of G, so with $g \in H$

$$\phi_0' = U(g)\phi_0 = \phi_0. \tag{3.35}$$

The idea is now to expand the potential around its vacuum state and to evaluate $V(\phi_0') = V(\phi_0)$. Since ϕ_0 is the vacuum state, meaning that the potential is minimal, the first derivative of V with respect to all vacuum state components vanishes,

$$\frac{\partial V}{\partial \phi_i}\Big|_{\phi=\phi_0} = 0. \tag{3.36}$$

Then the Taylor expansion yields

$$V(\phi) = V(\phi_0) + \frac{1}{2}\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\Big|_{\phi=\phi_0}(\phi_i - \phi_{0i})(\phi_j - \phi_{0j}) + \ldots. \tag{3.37}$$

We know that $V(\phi_0)$ is the minimum of the potential which leads to the following relation where M_{ij} is called the mass matrix, since it is standing in front of the term of second order

$$M_{ij} = \left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right)\Big|_{\phi_0} \geq 0. \tag{3.38}$$

Now we use the condition that the vacuum state can be invariant under some transformation of H. In general it is not invariant under a transformation belonging to G, whereas the potential is invariant under any transformation of ϕ belonging to G.

$$\begin{aligned} V(\phi_0) = V(U(g)\phi_0) = & V(\phi_0) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)\Big|_{\phi_0}([U(g)\phi]_i - \phi_{0i})([U(g)\phi]_j - \phi_{0j}) + \dots \\ & \Rightarrow \left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)\Big|_{\phi_0}\delta\phi_i\delta\phi_j = 0. \end{aligned} \tag{3.39}$$

In the last equation we denoted the variation of the field under some transformation from the vacuum state as $\delta\phi$. Now we have to distinguish between two cases depending on whether $g \in H$ or $g \in G/H$, where the G/H is a coset and not a subgroup since it does not contain the identity transformation.

In the first case, if $g \in H$ we know by the assumption of the invariant vacuum state

$$\phi_0' = \phi_0 \quad \Rightarrow \delta\phi_i = 0 \quad \forall i. \tag{3.40}$$

So equation (3.39) is fulfilled. On the other hand, if $g \in G/H$, then $U(g)\phi_0 \neq \phi_0$ in general. We therefore consider the variation of ϕ_j from the vacuum state by transformation given by

$$\delta\phi_j = [U'(0)\phi_0]_j = \left[\left(\frac{\partial U}{\partial\varepsilon_i}\right)_{\varepsilon_i=0}\phi_0\right]_j \delta\varepsilon_i \neq 0, \tag{3.41}$$

where ε_i are the infinitesimal transformation directions. This follows from the assumption that $U = \mathrm{e}^{-iT_i e_i}$, T_i being a set of matrices obeying the Lie algebra of the transformation group. When we apply a Taylor expansion of $U(g)$ around the identity ($e_i = 0$) we obtain for a first order approximation that the first derivative is responsible for small changes. $U(0)$ resembles the identity.

We furthermore know that there a certain components $\delta\phi_l$ that will remain zero. This is due to the fact, that ϕ_{0l} are non-zero components of the vacuum state, other than the ϕ_{0j} in general which are zero in order that the ground state is invariant under a transformation belonging to H, if the specific component is not invariant. It then follows that the variation from a zero state is much larger then from another state. We can consider $\delta\phi_l = 0$ such that

$$M_{ij}\delta\phi_j = 0. \tag{3.42}$$

The non-zero components of the field $\delta\phi$ have zero mass in order to fulfill equation (3.39). These are the Goldstone bosons. We can see that the number of massless fields is a question of group theory. In all cases we can expect as many Goldstone bosons as there are *broken* *symmetries* [10]. If we construct in the beginning as many fields as we have generators in G we can furthermore state that the number of fields, whose mass is not required to be zero is given by the number of generators of H, or the dimension of the Lie algebra of H. This does not necessarily mean that the mass of those fields is not zero, though. Then the number of massless fields can be given by the numbers of generators which are not in H, or the dimension of the Lie algebra of G/H. This result does clearly not depend on the representation of G.

At the end, we can for generality consider the case of no spontaneously broken symmetry. In this case the vacuum state is a singlet, which is unique and invariant under the transformation of G. So $H = G$ and there are no massless fields appearing. If on the other hand we break all symmetries such that the vacuum is not invariant under any subgroup of G we will observe a number of Goldstone bosons that is equal to the number of generators of G.

3.5 Examples

It is important to notice that the following examples will be discussed using the semi-classical approach to spontaneous symmetry breaking. This means that we seek the minimum of the potential energy, which resembles the minimum of the total energy. The vacuum state expressed by fields with physical meaning, i.e. that vanish at the vacuum, is reinserted. The rewritten Lagrangian can then be analysed according to second order field terms, that reveal the masses of the field quanta.

3.5.1 Breaking of a $U(1)$ Symmetry

For a $U(1)$ invariant system the ϕ^4 theory is chosen, which includes the Klein-Gordon Fields and a self-interaction term λ, in order to obtain a symmetry in the Lagrangian:

$$\begin{aligned} \mathscr{L} &= (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 && (3.43) \\ &= (\partial_\mu \phi)(\partial^\mu \phi^*) - V(\phi, \phi^*), && (3.44) \end{aligned}$$

where m, which is usually the mass, shall be in this case a parameter only and the components of the contravariant four-gradient are given by

$$\partial^\mu = (\frac{\partial}{\partial t}, -\nabla).$$

$\mathscr{L}$ is invariant under the global $U(1)$ transformation,

$$\phi \rightarrow \phi' = \mathrm{e}^{i\Lambda}\phi \quad (\Lambda = \text{const.}) \tag{3.45}$$

since it only consists of $\phi^*\phi$ terms.

$$\begin{aligned} \mathscr{L}' &= (\partial_\mu \phi')(\partial^\mu \phi^{*\prime}) - m^2 \phi^{*\prime}\phi' - \lambda(\phi^{*\prime}\phi')^2 \\ &= (\partial_\mu \mathrm{e}^{i\Lambda}\phi)(\partial^\mu \mathrm{e}^{-i\Lambda}\phi^*) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \\ &= \mathscr{L} \end{aligned}$$

The ground state can now be obtained by minimizing the potential. We get

$$\begin{aligned} \frac{\partial V}{\partial|\phi|} &= 2m^2|\phi| + 4\lambda|\phi|^3 = 0 \\ \frac{\partial^2 V}{\partial|\phi|^2} &= 2m^2 + 12\lambda|\phi|^2 = 0. \end{aligned} \tag{3.46}$$

By solving the equation for $m^2 > 0$ a minimum at $\phi = \phi^* = 0$ is obtained, which turns to a maximum for $m^2 < 0$. The minimum in the latter case is given by

$$\phi^*\phi = |\phi|^2 = -\frac{m^2}{2\lambda} = a^2, \tag{3.47}$$

In quantum theory ϕ becomes an operator such that the minimal condition refers to the vacuum expectation value $|\langle 0|\,\phi\,|0\rangle|^2 = a^2$. If we consider instead of the complex ϕ its linear combination out of real fields, such that $\phi = \phi_1 + i\phi_2$, all the minima lie on a circle with radius a. It is convenient to continue working in polar coordinates.

$$\phi(x) = \rho(x)\mathrm{e}^{i\theta(x)} \tag{3.48}$$

For the further discussion we choose our vacuum, because all cases are equivalent, to be

$$\langle 0|\,\phi\,|0\rangle = a; \quad a \in \mathbb{R} \tag{3.49}$$

$$\Rightarrow \langle 0|\,\rho\,|0\rangle = a; \quad \langle 0|\,\theta\,|0\rangle = 0. \tag{3.50}$$

We now see very clearly that the vacuum state is connected to all of the other degenerate vacua by the $U(1)$ symmety transformation and furthermore not invariant under the symmetry of the potential. The $U(1)$ symmetry is broken. The vacuum is given by a particular combination of values of the fields ρ and θ. Now considering physical fields, as introduced above we have to modify our field into

$$\phi(x) = (\rho'(x) + a)\mathrm{e}^{i\theta(x)} \tag{3.51}$$

$$\Rightarrow \langle 0|\,\rho'\,|0\rangle = \langle 0|\,\theta\,|0\rangle = 0. \tag{3.52}$$

If we now express the Lagrangian in terms of the new physical fields, we can draw conclusions about them from the structure of individual contributions. Plugging into $\mathscr{L}$ the field ϕ gives for the kinetic part from (3.43)

$$\begin{aligned}
(\partial_\mu \phi)(\partial^\mu \phi^*) &= (\partial_\mu \rho' \mathrm{e}^{i\theta} + (\rho' + a)i\mathrm{e}^{i\theta}\partial_\mu \theta)(\partial^\mu \rho' \mathrm{e}^{-i\theta} + (\rho' + a)i\mathrm{e}^{-i\theta}\partial^\mu \theta) \\
&= (\partial_\mu \rho')(\partial^\mu \rho') + (\rho' + a)(\partial_\mu \theta)(\partial^\mu \theta) + (\rho' + a)i(\partial_\mu \theta \partial^\mu \rho' - \partial_\mu \rho' \partial^\mu \theta) \\
&= (\partial_\mu \rho')(\partial^\mu \rho') + (\rho' + a)(\partial_\mu \theta)(\partial^\mu \theta)
\end{aligned}$$

since in the mixed terms we multiply two four-vectors resulting in the same scalars. For the potential term from (3.43) we obtain under the use of (3.47)

$$\begin{aligned}
V &= m^2\phi^*\phi + \lambda(\phi^*\phi)^2 = m^2(\rho' + a)^2 + \lambda(\rho' + a)^4 \\
&= -2\lambda a^2(\rho'^2 + 2\rho' a + a^2) + \lambda(\rho'^4 + 4a\rho'^3 + 6a^2\rho'^2 + 4a^3\rho' + a^4) \\
&= \lambda(\rho'^4 + 4a\rho'^3 + 4a^2\rho'^2 - a^4).
\end{aligned}$$

We notice a term in the potential quadratic in ρ', which indicates that ρ' has a mass

$$m^2_{\rho'} = 4\lambda a^2, \tag{3.53}$$

according to the definition of the potential. Furthermore, there is no such term in θ^2, so θ is a *massless* field.

After spontaneous symmetry breaking we observe instead of two massive scalar fields ϕ_1 and ϕ_2 being the real parts of ϕ, one massive field ρ' and one massless field θ. It can be reasoned that one needs energy to displace ρ' against the restoring forces of the potential, whereas θ does not experience restoring forces corresponding to displacement along the potential valley.

3.5.2 Non-Abelian Example

We are now prepared to approach a physically more relevant example. For this we will consider a Lagrangian invariant under a global $U(1) \otimes SU(2)$ symmetry, which is part of the basis for the electro-weak interaction theory. This is the case of two complex scalar boson fields ϕ_1

and ϕ_2 forming a $SU(2)$ doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} \tag{3.54}$$

The Lagrangian can be obtained from Klein-Gordon equation $(\partial_\mu \partial^\mu + m^2)\varphi = 0$. The Lagrangian for a complex field $\phi = (\varphi_1 + i\varphi_2)$, where φ_i are two real scalar fields with the same mass, is then given by

$$\begin{aligned} \mathscr{L} &= ((\partial_\mu \varphi_1)(\partial^\mu \varphi_1) - m^2\varphi_1^2) + ((\partial_\mu \varphi_2)(\partial^\mu \varphi_2) - m^2\varphi_2^2) \\ &= (\partial_\mu \phi)^*(\partial^\mu \phi) - m^2\phi^*\phi \end{aligned} \tag{3.55}$$

such that the Euler-Lagrange equation (2.16) results in the Klein-Gordon equation. Since we have two of those fields ϕ_i that actually interact we can construct the following Lagrangian, where λ^2 characterises the interaction term and is assumed to be positive in order that the potential describes a stable system around the vacuum, so

$$\mathscr{L} = (\partial_\mu \phi_1)^*(\partial^\mu \phi_1) + (\partial_\mu \phi_2)^*(\partial^\mu \phi_2) - m_1^2\phi_1^*\phi_1 - m_2^2\phi_2^*\phi_2 - \frac{1}{2}\lambda^2(|\phi_1|^2 + |\phi_2|^2)^2. \tag{3.56}$$

With $m_1 = m_2 = \frac{\mu}{\sqrt{2}}$ we can rewrite this Lagrangian in terms of ϕ and $\phi^\dagger$, where the latter one is the adjoint of the field, meaning the complex conjugate transposed.

$$\begin{aligned} \mathscr{L} &= (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^\dagger\phi - \frac{1}{2}\lambda^2(\phi^\dagger\phi)^2 \\ &= (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - V(\phi^\dagger\phi) \end{aligned} \tag{3.57}$$

We notice that the Lagrangian is invariant under a global $U(1)$ transformation, because we have only pairs of $\phi_i^*\phi_i$ appearing. Furthermore the Lagrangian is also invariant under a $SU(2)$ transformation since the ϕ_i are indistinguishable and therefore can be interchanged.

Now μ is to be considered as a parameter only, so we can differentiate the potential with respect to $|\phi|$ in order to obtain the extremal relations

$$\frac{\partial V}{\partial|\phi|} = \mu^2|\phi| + 2\lambda^2|\phi|^3 = 0 \tag{3.58}$$

$$\frac{\partial^2 V}{\partial|\phi|^2} = \mu^2 + 6\lambda^2|\phi|^2 \neq 0. \tag{3.59}$$

We immediately receive from these equations that for $\mu^2 > 0$, where the Lagrangian describes two real particles of mass $\frac{\mu}{\sqrt{2}}$, the minimum of the potential will be at $\phi = 0$. This will be the state of lowest energy, the vacuum state and in this case the vacuum possesses all the symmetries of the Lagrangian.

In the case $\mu^2 < 0$ however the state $\phi = 0$ describes a local maximum point. The ground state can then be obtained from equation (3.58) as a second real solution and is given by

$$|\phi_0|^2 = (\phi^\dagger \phi)_0 = -\frac{\mu^2}{2\lambda^2} = a^2. \tag{3.60}$$

Obviously there is now a set of degenerate vacuum states lying on a circle of radius a in the ϕ_1-ϕ_2 plane. They are connected by a $SU(2)$ transformation, which rotates the doublet space. The full symmetry of the Lagrangian was given by $U(1) \otimes SU(2)$, which has four generators. Notice that we have four real scalar fields in our Lagrangian. Now this original symmetry is *broken* down to a $U(1)$ symmetry of the vacuum state, since the vacuum state now gets transformed by the $SU(2)$ transformation into another vacuum state with the same energy. Since the $SU(2)$ group has three generators, three massless particles can be expected.

We can choose our vacuum state in any way, such that $|\phi| = a$ is fulfilled so we set

$$\langle 0| \phi |0\rangle = \phi_0 = \begin{pmatrix} 0 \\ a \end{pmatrix}. \tag{3.61}$$

Since the fields of interest are physical fields, in order to obtain relevant informations out of our equations, we vary our original field such that

$$\phi = \phi_0 + \chi, \tag{3.62}$$

where

$$\chi = \begin{pmatrix} \chi_1 + i\chi_2 \\ \chi_3 + i\chi_4 \end{pmatrix} \tag{3.63}$$

$$\Rightarrow \phi = \begin{pmatrix} \chi_1 + i\chi_2 \\ a + \chi_3 + i\chi_4 \end{pmatrix} \tag{3.64}$$

such that $\langle 0| \chi |0\rangle = 0$.
We can now insert ϕ again into our Lagrangian and we will obtain some information about the masses of our fields. The differential part will certainly not contribute to the identification

of massless fields

$$(\partial^\mu \phi^\dagger)(\partial_\mu \phi) = \sum_{i=1}^{4} (\partial^\mu \chi_i)(\partial_\mu \chi_i), \tag{3.65}$$

whereas the potential takes a new form, considering that $\lambda = -\frac{\mu^2}{2a^2}$ in the third step

$$\begin{aligned} V &= \frac{1}{2}\mu^2 \phi^\dagger \phi + \frac{1}{2}\lambda^2 (\phi^\dagger \phi)^2 \\ &= \frac{1}{2}\mu^2 (\chi_1^2 + \chi_2^2 + \chi_4^2 + (a + \chi_3)^2) + \frac{1}{2}\lambda^2 (\chi_1^2 + \chi_2^2 + \chi_4^2 + \chi_3^2 + a^2 + 2a\chi_3)^2 \\ &= \frac{1}{2}\mu^2 \left(\left(\sum_i \chi_i^2 \right) + a^2 + 2a\chi_3 \right) - \frac{\mu^2}{4a^2} \left(2a^2 \left(\sum_i \chi_i^2 \right) + 4a^3 \chi_3 + a^4 + 4a^2 \chi_3^2 + \ldots \right). \end{aligned} \tag{3.66}$$

We neglected terms of higher order than three, because they are irrelevant for the discussion of the masses. We can see that many terms up to second order will cancel and we will obtain

$$V = \frac{1}{4}\mu^2 a^2 - \mu^2 \chi_3^2 + \ldots \tag{3.67}$$

Since $\mu^2 < 0$ only the field χ_3 is left in the potential with a term of second order and therefore the only physical field that obtains a mass

$$m_{\chi_3} = \sqrt{-2\mu^2}. \tag{3.68}$$

We considered two particles with the same mass $|\mu|/\sqrt{2}$ and obtain only one field that carries the mass of all. In addition we receive, as expected, three massless fields χ_1, χ_2, χ_3, which are identical to the original φ_i with the respective indices.

This is a nice end of the discussion of global symmetry breaking so far and we will consider now local symmetries and the Higgs formalism, keeping in mind that the Goldstone theorem leads to the existence of as many massless particles as there are broken symmetries.

4 Local Gauge Symmetry Breaking

This chapter will deal with the local case of spontaneous symmetry breaking. The Higgs mechanism is going to be explained on several examples and at the end we will provide a comparison between the two cases of global and local symmetry breaking. As in the previous chapter we start off with a brief introduction into local symmetries and in this context also the appearance of local gauge fields and how they are implemented into the Lagrangian.

4.1 Local Gauge Symmetry

When we introduced global symmetries, they were defined by the invariance of a Lagrangian under a global transformation belonging to some symmetry group. In the local case this is a little bit different. Again we demand an invariance under a local symmetry transformation $U(\alpha_1, \ldots \alpha_N; \vec{r})$ depending on the position $\vec{r}$. In order to sattisfy this invariance, there will appear terms involving gauge fields W_μ^a in the Lagrangian, in form of a covariant derivative D_μ instead of ∂_μ and a field Lagrange term $\mathscr{L}_W$. The gauge fields will similtaneously transform, depending on U and are coupled to the field due to the covariant derivative.

4.1.1 General Considerations

In the general global case the transformation was expressed by an unitary operator U with the generators M^a and Λ^a being constant parameters

$$U = \mathrm{e}^{-i\Lambda^a M^a}.$$

Consider that $\mathscr{L}_0$ is invariant under this global transformation. We obtain a local unitary operator by making Λ^a space dependent.

$$U(x) = \mathrm{e}^{-i\Lambda^a(x) M^a} \tag{4.1}$$

where now a Lagrangian $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_W$ is invariant under this transformation if $\partial_\mu \to D_\mu$. The appearance of a covariant derivative can be obtained from the geometry of gauge fields, for which we refer to [10]. In general it holds

$$D_\mu \phi = \partial_\mu \phi + igM^a W_\mu^a \phi, \tag{4.2}$$

where g is a constant to adjust the dimensions and W_μ^a is an additional field or potential. This defines the covariant derivative of an arbitrary field ϕ transforming under an arbitrary group with generator M^a. It can already be seen, that for each generator of the local transformation group one gauge term field term each is needed. It is most likely that in order to be locally invariant a vector gauge field gets generated with as many components as generators of the symmetry group, i.e the dimension of its Lie algebra.
In the case of $U(1)$ we set $M = 1$ and $g \to e$, in order to obtain the known result

$$U(1): \quad D_\mu = \partial_\mu + ieA_\mu; \quad A_\mu = W_\mu \tag{4.3}$$

$$U(N): \quad D_\mu = \partial_\mu + igM^a W_\mu^a = \partial_\mu + igW_\mu. \tag{4.4}$$

In general we set $M^a W_\mu^a = W_\mu$ for convenience, keeping in mind that we treat a matrix W_μ.

We can create further conditions on the gauge fields in order to specify them. We demand that a Lagrangian shall be invariant under a given local transformation $U(x)$. In general this Lagrangian will have some terms that will not involve any gauge fields. These have to be invariant under $U(x)$ only. Furthermore there will appear contributions consisting only of gauge fields in the Lagrangian which will be constructed to be invariant under the gauge field transformation. Which leads us to the condition on this transformation, a similarity transformation of the covariant derivative $D'_\mu = UD_\mu U^{-1}$, such that the kinetic term of the Lagrangian is invariant under the local transformation, including the gauge field transformation. Recalling the form invariance of ∂_μ we obtain

$$\begin{aligned} D'_\mu \phi' &= UD_\mu \phi \\ \Rightarrow (\partial_\mu + igW'_\mu)U\phi &= U(\partial_\mu + igW_\mu)\phi = U(\partial_\mu \phi) + ig(UW_\mu)\phi \\ \Rightarrow (\partial_\mu U)\phi + U(\partial_\mu \phi) + igW'_\mu U\phi &= U(\partial_\mu \phi) + ig(UW_\mu)\phi \\ \Rightarrow (\partial_\mu U)U^{-1}U\phi + igW'_\mu U\phi &= ig(UW_\mu)U^{-1}U\phi \\ \Rightarrow W'_\mu &= UW_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}. \end{aligned} \tag{4.5}$$

With the use of $U^{-1}U = \mathbf{1}$ and therefore its derivative being zero, we can conclude from the rule of product differentiation

$$W'_\mu = UW_\mu U^{-1} - \frac{i}{g}(\partial_\mu U^{-1})U. \tag{4.6}$$

The Lagrange term of the gauge field depends on the symmetry group but we remain as general as possible considering the $U(N)$, $SU(N)$ symmetry groups of primary interest in this thesis. They have the following properties from commutation relation and normalization

$$[M^l, M^m] = if^{lmn}M^n; \qquad f^{lmn} - \text{antisymmetric structure constant} \tag{4.7}$$
$$\mathrm{tr}(M^l M^m) = \frac{1}{2}\delta^{lm}. \tag{4.8}$$

$SU(N)$ has N^2-1 generators, each of which requires a gauge field to compensate the arising terms in local invariance. From this follows that W_μ is a vector field with N^2-1 internal components. The following Ansatz for the gauge Lagrangian can be formulated, where $G^{\mu\nu}$ is yet unknown

$$\mathscr{L}_W = -\frac{1}{4}G^{l\mu\nu}G^l_{\mu\nu}. \tag{4.9}$$

which can be interpreted as a scalar product in the $SU(N)$ space. From the normalization condition we obtain

$$\begin{aligned}
\mathscr{L}_W &= -\frac{1}{4}G^{l\mu\nu}G^l_{\mu\nu} = -\frac{1}{2}\sum_{l,m=1}^{N^2-1} G^{l\mu\nu}tr(M^l M^m)G^m_{\mu\nu} \\
&= -\frac{1}{2}\sum_{l,m=1}^{N^2-1}\sum_{i,j=1}^{N} G^{l\mu\nu}(M^l)_{ij}(M^m)_{ji}G^m_{\mu\nu} = -\frac{1}{2}\sum_{i,j=1}^{N}(G^{\mu\nu})_{ij}(G_{\mu\nu})_{ji} \\
&= -\frac{1}{2}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}),
\end{aligned} \tag{4.10}$$

where we used $M^a G^a_{\mu\nu} = G_{\mu\nu}$ again. Since the trace is invariant under a unitary similarity transformation we suppose that $G_{\mu\nu}$ transforms like

$$G'_{\mu\nu} = UG_{\mu\nu}U^{-1}. \tag{4.11}$$

This is satisfied by the matrix Ansatz

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu], \tag{4.12}$$

or given in components

$$G^l_{\mu\nu} = \partial_\mu W^l_\nu - \partial_\nu W^l_\mu - i f^{lmn} g W^m_\mu W^n_\nu. \tag{4.13}$$

The proof of this Ansatz is straight forward. From the unitary condition $U^{-1}U = \mathbf{1}$ it follows

$$(\partial_\mu U^{-1})U + U^{-1}(\partial_\mu U) = 0.$$

Under extensive use of this relation, we obtain from (4.6)

$$\begin{aligned}
\partial_\mu W'_\nu &= (\partial_\mu U)W_\mu U^{-1} + U(\partial_\mu W_\nu)U^{-1} + UW_\nu(\partial_\mu U^{-1}) \\
&\quad - \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) - \frac{i}{g}U(\partial_\mu \partial_\nu U^{-1}) \\
&= UU^{-1}(\partial_\mu U)W_\mu U^{-1} + U(\partial_\mu W_\nu)U^{-1} + UW_\nu U^{-1}U(\partial_\mu U^{-1}) \\
&\quad - \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) - \frac{i}{g}U(\partial_\mu \partial_\nu U^{-1}) \\
&= -U(\partial_\mu U^{-1})UW_\mu U^{-1} + U(\partial_\mu W_\nu)U^{-1} + UW_\nu U^{-1}U(\partial_\mu U^{-1}) \\
&\quad - \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) - \frac{i}{g}U(\partial_\mu \partial_\nu U^{-1}) \\
&= U(\partial_\mu W_\nu)U^{-1} - [U(\partial_\mu U^{-1}), UW_\nu U^{-1}] - \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) - \frac{i}{g}U(\partial_\mu \partial_\nu U^{-1}).
\end{aligned}$$

Then $G_{\mu\nu}$ transforms like

$$\begin{aligned}
G'_{\mu\nu} =& \partial_\mu W'_\nu - \partial_\nu W'_\mu + ig[W'_\mu, W'_\nu] \\
=& U(\partial_\mu W_\nu)U^{-1} - U(\partial_\nu W_\mu)U^{-1} - [U(\partial_\mu U^{-1}), UW_\nu U^{-1}] + [U(\partial_\nu U^{-1}), UW_\mu U^{-1}] \\
& - \frac{i}{g}((\partial_\mu U)(\partial_\nu U^{-1}) - (\partial_\nu U)(\partial_\mu U^{-1})) + ig[UW_\mu U^{-1} \\
& - \frac{i}{g}(\partial_\mu U^{-1})U\nu, UW_\nu U^{-1} - \frac{i}{g}(\partial_\nu U^{-1})U].
\end{aligned}$$

With $-\frac{i}{g}[(\partial_\mu U^{-1})U\nu, (\partial_\nu U^{-1})U] = \frac{i}{g}((\partial_\mu U)(\partial_\nu U^{-1}) - (\partial_\nu U)(\partial_\mu U^{-1}))$ and the other mixed commutator terms vanishing as well we obtain

$$\begin{aligned}
G'_{\mu\nu} &= U(\partial_\mu W_\nu - \partial_\nu W_\mu)U^{-1} + igU[W_\mu, W_\nu]U^{-1} \\
&= UG_{\mu\nu}U^{-1}.
\end{aligned}$$

This makes the constructed gauge field Lagrangian term invariant under the gauge field transformation (4.6) and shows that in $\mathscr{L}_W$ no mass term of the form $\frac{1}{2}m^2 W^{l\mu} W^l_\mu$ appears.

We obtain a Lagrangian $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_W$, with covariant derivative (4.2) and the gauge Lagrangian term $\mathscr{L}_W = -\frac{1}{4}G^{l\mu\nu}G^l_{\mu\nu}$. This is invariant under a local gauge transformation according to (4.1) and (4.6) with respect to the field and the gauge field respectively.

4.1.2 $U(1)$ Example

In the well known case of a $U(1)$ symmetry group of electromagnetic gauge fields we set $g = e$ and $M^a = 1$ to obtain from (4.1)

$$U(x) = \mathrm{e}^{-i\Lambda(x)} \tag{4.14}$$
$$\partial_\mu(U^{-1}) = i(\partial_\mu\Lambda(x))\mathrm{e}^{i\Lambda(x)}. \tag{4.15}$$

We can immediately obtain the covariant derivative and the gauge field transformation from (4.2) and (4.6) respectively, renaming $W_\mu = A_\mu$, with A_μ being the four vector of the potentials of the electric and magnetic fields

$$D_\mu\phi = (\partial_\mu + ieA_\mu)\phi \tag{4.16}$$
$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\Lambda. \tag{4.17}$$

In the case of $U(1)$ we rename $G_{\mu\nu} = F_{\mu\nu}$ and with $f^{lmn} = 0$, we can construct the gauge field from (4.12)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.18}$$

These are well known results from electrodynamics. We will now consider a global example, turn it into a local one and check the invariance under the gauge transformation. We know from Chapter 3 that the the following Lagrangian has a $U(1)$ symmetry

$$\mathscr{L} = (\partial_\mu\phi)(\partial^\mu\phi)^* - \frac{1}{2}\mu^2(\phi^*\phi) - \frac{1}{2}\lambda(\phi^*\phi)^2. \tag{4.19}$$

As discussed above for the local case we need to replace the partial derivative with the covariant derivative and add the gauge field term such that

$$\mathscr{L} = (\partial_\mu + ieA_\mu)\phi(\partial^\mu - ieA^\mu)\phi^* - V(\phi^*\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.20}$$

To check the invariance under $\phi' = \mathrm{e}^{-i\Lambda(x)}\phi$ we will proceed step by step. First, the potential $V(\phi^*\phi)$ is obviously invariant since the exponential terms cancel in the products of the field and its complex conjugate $\phi'^*\phi' = \phi^*\phi$. The gauge field is also invariant under the gauge transformation as we see by inserting (4.17) into (4.18)

$$F'_{\mu\nu} = \partial_\mu A_\nu + \frac{1}{e}\partial_\mu\partial_\nu\Lambda - \partial_\nu A_\mu - \frac{1}{e}\partial_\nu\partial_\mu\Lambda = F_{\mu\nu}. \tag{4.21}$$

Now we only need to verify the term involving the derivative, where the gauge field and the field couple. It is important to notice that the partial derivative now also acts on the transformation, so we obtain

$$\begin{aligned} D'_\mu\phi' D'^{\mu*}\phi'^\mu &= (\partial_\mu + ieA'_\mu)\mathrm{e}^{-i\Lambda(x)}\phi(\partial^\mu - ieA'^\mu)\mathrm{e}^{i\Lambda(x)}\phi^* \\ &= \mathrm{e}^{-i\Lambda(x)}(\partial_\mu + ieA'_\mu - i\partial_\mu\Lambda(x))\phi\mathrm{e}^{i\Lambda(x)}(\partial^\mu - ieA'^\mu + i\partial^\mu\Lambda(x))\phi^* \\ &= (\partial_\mu + ieA_\mu)\phi(\partial^\mu - ieA^\mu)\phi^* = D_\mu\phi D^{\mu*}\phi^\mu. \end{aligned} \tag{4.22}$$

We have shown on a simple example that our general considerations hold. The $SU(2)$ case will be considered later, when it comes to an application of the Higgs mechanism.

4.2 The Higgs Mechanism

The Higgs mechanism is best introduced by demonstration on an example. For this we will go back to the $U(1)$ symmetry. In the previous section we just derived the corresponding Lagrangian (see (4.20)). In short notation

$$\mathscr{L} = (D^\mu\phi)^*(D_\mu\phi) - V(\phi^*\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.23}$$

with the known potential $V = \frac{1}{2}\mu(\phi^*\phi) + \frac{1}{2}\lambda(\phi^*\phi)^2$. Global symmetry breaking revealed the existence of one Goldstone boson. The local transformation now looks like (4.14). As already shown in Chapter 3 the minimum of the potential for $\mu^2 < 0$ is obtained at $\phi_0^2 = a^2 = -\frac{\mu^2}{2\lambda^2}$. Expanding around the vacuum state in order to obtain physical fields we rewrite our Lagrangian in terms of

$$\tilde{\phi}(x) = a + \phi(x) = a + \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) = \frac{1}{\sqrt{2}}(\sqrt{2}a + \varphi_1 + i\varphi_2) \tag{4.24}$$

If we omit the terms of higher order in ϕ_i and A_μ (higher than 3) and under the use of $(\partial_\mu \varphi_2)A^\mu = (\partial^\mu \varphi_2)A_\mu$ we can derive

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + [(\partial^\mu - ieA^\mu)\tilde{\phi}][(\partial_\mu + ieA_\mu)\tilde{\phi}] - V(\tilde{\phi}^*\tilde{\phi}) \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial^\mu \varphi_1)(\partial_\mu \varphi_1) + \frac{1}{2}(\partial^\mu \varphi_2)(\partial_\mu \varphi_2) + \sqrt{2}e(\partial_\mu \varphi_2)A^\mu \\ &\quad + e^2 A^\mu A_\mu a^2 - \frac{1}{2}\mu^2(a^2 + \frac{1}{2}(\varphi_1^2 + \varphi_2^2) + \sqrt{2}a\varphi_1) \\ &\quad - \frac{\lambda^2}{2}(a^4 + a^2(\varphi_1^2 + \varphi_2^2) + 2\sqrt{2}a^3\varphi_1 + 2a^2\varphi_1^2 + \ldots).\end{aligned} \tag{4.25}$$

With $\lambda^2 = -\frac{\mu^2}{2a^2}$ we obtain

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial^\mu \varphi_1)(\partial_\mu \varphi_1) + \frac{1}{2}(\partial^\mu \varphi_2)(\partial_\mu \varphi_2) + \sqrt{2}ea(\partial_\mu \varphi_2)A^\mu \\ &+ e^2 A^\mu A_\mu a^2 + \frac{1}{2}\mu^2\varphi_1^2 - \frac{1}{4}\mu^2 a^2 + \ldots\end{aligned} \tag{4.26}$$

We obtain the known result that φ_1 remains massive under the choice of our vacuum state to be real. Its mass is $m_{\varphi_1} = \sqrt{-\mu^2}$. In addition, the gauge field obtained a mass term. The second scalar field φ_2 appears again massless but we notice a coupling term between it and the gauge field. To receive a clearer physical statement of this potential we get rid of the coupling term. This is done by a proper choice of gauge. From (4.14) we know for infinitesimal transformations, considering the expansion only up to the linear term,

$$\phi' = \frac{1}{\sqrt{2}}(1 - i\Lambda(x))(\sqrt{2}a + \varphi_1 + i\varphi_2) = \frac{1}{\sqrt{2}}\Big((\sqrt{2}a + \varphi_1 + \Lambda\varphi_2) + i(-\Lambda\varphi_1 - \sqrt{2}a\Lambda + \varphi_2)\Big)$$

and by comparison of the real and imaginary parts of ϕ' we come to

$$\Rightarrow \qquad \varphi_1' = \frac{1}{\sqrt{2}}(\varphi_1 + \Lambda\varphi_2) \tag{4.27}$$

$$\varphi_2' = \frac{1}{\sqrt{2}}(\varphi_2 - \Lambda\varphi_1 - \sqrt{2}a\Lambda). \tag{4.28}$$

We choose Λ in such a way that the transformed field takes the form $\phi' = \frac{1}{\sqrt{2}}(\sqrt{2}a + \varphi_1)$. Explicitly

$$\Lambda(x) = \frac{\varphi_2}{\sqrt{2}a + \varphi_1 + i\varphi_2} \approx \frac{\varphi_2}{\sqrt{2}a}, \tag{4.29}$$

where in the last step we assume that our physical field $\phi = \varphi_1 + i\varphi_2$ does only small changes in magnitude around the vacuum state. By this choice of Λ we consequently transformed our gauge field according to (4.17) into

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \Lambda \approx A_\mu + \frac{1}{\sqrt{2}ae}\partial_\mu \varphi_2. \tag{4.30}$$

Inserting this new gauge field and the field $\phi' = \frac{1}{\sqrt{2}}(\sqrt{2}a + \varphi_1)$ into our Lagrangian (4.23), we end up with (rewriting $A' \to A$)

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial^\mu \varphi_1)(\partial_\mu \varphi_1) + e^2 A^\mu A_\mu a^2 + \frac{1}{2}\mu^2\varphi_1^2 + \mathscr{O}(T^2) \tag{4.31}$$

where T stands for all fields. From this it is observable that φ_2 completely vanished. We obtained a Lagrangian that is describing two massive fields and their interactions, namely φ_1 and A_μ with masses mentioned earlier. Again it is important to notice that we approached this problem semi-classically as in the global case. The only difference in the local case ist, that it was possible to transform our vacuum state by a proper choice of gauge to simplify the problem at hand.

Compared to the global case, where we had one Goldstone boson for the one broken symmetry we now obtain exactly one massive gauge field for the one broken symmetry. It can be correlated that these appearances are closely related. The quanta of the remaining massive scalar field φ_1 are then called the *Higgs bosons*.

The general case is in full analogy to the global case of symmetry breaking. We will assume a Lagrangian that is satisfying the Goldstone theorem. So it has a certain global symmetry but its vacuum state does not share it. If we demand invariance under a local symmetry, there will appear gauge fields as a consequence, as shown in the previous section, in terms of covariant derivatives and additional field terms.

Analogue to the global case we continue by expanding in terms of the the physical field χ around the vacuum state ϕ_0 setting the vacuum expectation value of the physical fields zero such that

$$\phi = \phi_0 + \chi. \tag{4.32}$$

Inserting this into the Lagrangian gives similar effects as in the Goldstone mode. The fields describing the Goldstone bosons in the global case are still massless but now coupled to massive gauge fields. The massive fields from the global scenario remain massive. In order to rule out any mass acquiring of the massless field through the coupling we can choose it to equal zero by an appropriate gauge. We can formulate a general rule for the local case.

Higgs mechanism: Consider a Lagrangian with global symmetries, not shared by the vacuum state. When we introduce a local symmetry invariance, then the spontaneous symmetry breaking no longer causes massless Goldstone bosons but massive gauge fields. The number of massive gauge fields in local symmetry breaking corresponds to the number of massless Goldstone bosons in the global case. One says that the massless gauge fields "eats" the massless Goldstone boson in order to become massive.

The fact that some massless gauge fields acquire mass can be understood as follows. From our symmetry section we know that the gauge fields appear in order to compensate terms arising from the local transformations. That is why there appear as many gauge fields as we have generators in the Lie algebra of the symmetry group for example in the $SU(N)$ case $N^2 - 1$. As a consequence of the spontaneous symmetry breaking this symmetry is reduced and the number of generators decreases. Consequently, the number of massless gauge fields has to decrease, but they cannot simply vanish. At the same time we know, there will appear as many Goldstone bosons as symmetries are broken. So the massless gauge fields simply take up the massless Goldstone bosons to acquire mass. That is why there are as many massive gauge fields as symmetries are broken.
When we consider the case that we constructed as many fields as generators we can again introduce a symmetry group G under which the Lagrangian is invariant and a subgroup H which provides the only symmetry transformations for the ground state. Then the number of massive gauge fields is given by the number of generators belonging to G/H. In other words, the dimension of the Lie algebra of G/H. On the other side $\dim H$ gives the number of massive scalar fields.

4.3 Non-Abelian Example

We will now turn to an example out of the Non-Abelian gauge theory, also called Yang-Mills theory. The symmetry group under consideration will be the $U(1) \otimes SU(2)$ group. This example is mainly the foundation of the Weinberg-Salam model for electroweak interaction, which will be briefly described later.

Recalling the $SU(2)$ example from 3.5.2 we start with some relevant properties of the symmetry group $SU(2)$. The generators are given by

$$M^a = -\frac{1}{2}\sigma^a, \tag{4.33}$$

where σ^a represent the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.34}$$

The symmetry group brings some properties from normalization and its algebra, as mentioned before

$$\mathrm{tr}(\sigma^l \sigma^m) = 2\delta^{lm}, \tag{4.35}$$

$$[\sigma^l, \sigma^m] = 2i\varepsilon_{lmn}\sigma^n, \tag{4.36}$$

$$\{\sigma^l, \sigma^m\} = 2\delta^{lm}\mathbf{1}, \tag{4.37}$$

where the latter bracket is the anticommutator, vanishing for $l \neq m$.
We obtain the transformation operator from (4.1)

$$U(x) = \mathrm{e}^{\frac{1}{2}\sigma^a \Lambda^a(x)}. \tag{4.38}$$

The covariant derivatives follow from (4.2) with $W_\mu = M^a W_\mu^a$ as

$$D_\mu \phi = \partial_\mu \phi + igW_\mu \phi \quad \text{and} \quad (D\phi)^\dagger = \partial_\mu \phi^\dagger - ig\phi^\dagger W_\mu. \tag{4.39}$$

Considering the Lagrangian term of the free gauge field $\frac{1}{4}G^{\mu\nu}G_{\mu\nu}$, we obtain for the $SU(2)$ part

$$\mathscr{L} = (D^\mu \phi)^\dagger (D_\mu \phi) - \frac{1}{2}\mu^2 \phi^\dagger \phi - \frac{1}{2}\lambda^2 (\phi^\dagger \phi)^2 - \frac{1}{4}G^{l\mu\nu}G^l_{\mu\nu} \tag{4.40}$$

with the gauge field tensor (see(4.12))

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu].$$

The scalar field ϕ has a non-vanishing vacuum expectation value and will later on take the role of the *Higgs field*.
The $U(1)$ symmetry group also contributes to the Lagrangian in the known way from the previous chapter, such that the covariant derivative receives an extra term involving the $U(1)$ gauge field A_μ

$$D_\mu = \partial_\mu + igW_\mu + i\frac{g'}{2}A_\mu. \tag{4.41}$$

Then also the Lagrange function of the free $U(1)$ field has to be added to the total Lagrangian involving $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The total Lagrange function invariant under the $U(1) \otimes SU(2)$

symmetry is then (V is the known potential)

$$\mathscr{L} = (D^\mu \phi)^\dagger (D_\mu \phi) - V(\phi^\dagger \phi) - \frac{1}{4} G^{l\mu\nu} G^l_{\mu\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{4.42}$$

As we know from Subsection 3.5.2 μ^2 is merely a parameter. For $\mu^2 > 0$ this Lagrangian describes the standard Yang-Mills theory, namely two massive scalar mesons of mass μ and one massless gauge boson. If however $\mu^2 < 0$, we know from the global case that the potential has its minimum, describing the vacuum state, at

$$|\phi_0|^2 = -\frac{\mu^2}{2\lambda^2} = a^2. \tag{4.43}$$

The vacuum states are degenerate and connected by a $SU(2)$ transformation. Therefore we can choose our particular vacuum state to be

$$\langle 0| \phi_0 |0\rangle = \phi_0 = \begin{pmatrix} 0 \\ a \end{pmatrix}. \tag{4.44}$$

Expanding our scalar field around this vacuum we introduce the physical Higgs fields χ_i such that

$$\phi = \begin{pmatrix} \chi_1 + i\chi_2 \\ (a+\chi_3) + i\chi_4 \end{pmatrix}. \tag{4.45}$$

From the global example we expect χ_1, χ_2, χ_4 to be massless Goldstone bosons. We can then use our gauge freedom to transform this field via a local $SU(2)$ gauge transformation (4.38) by elimination of χ_1, χ_2, χ_4 into

$$\phi(x) = \begin{pmatrix} 0 \\ a+\chi_3 \end{pmatrix}. \tag{4.46}$$

Inserting this into our Lagrangian we arrive at

$$\begin{aligned} \mathscr{L} =& \left(\partial^\mu \phi^\dagger - \phi^\dagger (igW^\mu + i\frac{g'}{2} A^\mu)\right) \left((\partial_\mu + igW_\mu + i\frac{g'}{2} A_\mu)\phi\right) - V(\phi^\dagger \phi) \\ & - \frac{1}{4} G^{l\mu\nu} G^l_{\mu\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ =& (\partial^\mu \phi)^\dagger (\partial_\mu \phi) - ig\phi^\dagger W^\mu (igW_\mu)\phi - ig\phi^\dagger W^\mu (i\frac{g'}{2} A_\mu)\phi - i\frac{g'}{2} A^\mu \phi^\dagger igW_\mu \phi \\ & + \frac{g'^2}{4} \phi^\dagger A^\mu A_\mu \phi - \frac{1}{2}\mu^2 \phi^\dagger \phi - \frac{1}{2}\lambda^2 (\phi^\dagger \phi)^2 - \frac{1}{4} G^{l\mu\nu} G^l_{\mu\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \dots \end{aligned} \tag{4.47}$$

The scalar products between the terms involving the partial derivative of the field and a gauge field cancel due to $\phi^\dagger = \phi$ from (4.46) and the definition of the scalar product (2.9). We neglected terms of higher order as usual and refer to (3.67) for the outcome of the potential with the new field. Replacing $W_\mu = \frac{\sigma^l}{2} W^l_\mu$ and recallig $\phi^\dagger\phi = (a+\chi_3)^2$ we obtain

$$\begin{aligned}\mathscr{L} = &(\partial^\mu \chi_3)(\partial_\mu \chi_3) + \mu^2\chi_3^2 - \frac{1}{4}G^{l\mu\nu}G^l_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &+ \frac{g^2}{4}W^{l\mu}W^l_\mu(\phi^\dagger\sigma^l\sigma^m\phi) + \frac{gg'}{4}W^{l\mu}A_\mu(\phi^\dagger\sigma^l\phi) + \frac{gg'}{4}A^\mu W^l_\mu(\phi^\dagger\sigma^l\phi) \\ &+ \frac{g'^2}{4}A^\mu A_\mu(a+\chi_3)^2 + \dots\end{aligned} \tag{4.48}$$

As mentioned earlier the anticommutator of σ^l and σ^m vanishes for $l \neq m$ so that the elements of the sum $(\phi^\dagger\sigma^l\sigma^m\phi)$ for $l \neq m$ vanish. Furthermore with the action of the pauli matrices (4.34) and the structure of ϕ in mind (see (4.46)) it follows that

$$\phi^\dagger\sigma^{1,2}\phi = 0 \quad \text{and} \tag{4.49}$$

$$\phi^\dagger\sigma^3\phi = -(a+\chi_3)^2. \tag{4.50}$$

The norm of pauli matrices is always one i.e. $(\sigma^i)^2 = 1$. The terms proportional to g^2, g'^2 and gg' then take the form, keeping in mind the four scalar product properties $W^{3\mu}A_\mu = W^3_\mu A^\mu$ and the commutation relation $[A_\mu, W^{3\mu}] = 0$

$$\begin{aligned}&\frac{g^2}{4}W^{l\mu}W^l_\mu(\phi^\dagger\sigma^l\sigma^m\phi) + \frac{gg'}{4}W^{l\mu}A_\mu(\phi^\dagger\sigma^l\phi) + \frac{gg'}{4}A^\mu W^l_\mu(\phi^\dagger\sigma^l\phi) + \frac{g'^2}{4}A^\mu A_\mu(a+\chi_3)^2 \\ &= \frac{g^2}{4}W^{l\mu}W^l_\mu(a+\chi_3)^2 - \frac{gg'}{2}W^{3\mu}A_\mu(a+\chi_3)^2 + \frac{g'^2}{4}A^\mu A_\mu(a+\chi_3)^2 \\ &= \frac{g^2}{4}(W^{1\mu}W^1_\mu + W^{2\mu}W^2_\mu)(a+\chi_3)^2 + \frac{g^2}{4}W^{3\mu}W^3_\mu(a+\chi_3)^2 - \frac{gg'}{2}W^{3\mu}A_\mu(a+\chi_3)^2 \\ &\quad + \frac{g'^2}{4}A^\mu A_\mu(a+\chi_3)^2.\end{aligned} \tag{4.51}$$

Neglecting terms of higher order again, starting at order three, we neglect the $\chi_3 T_\mu T^\mu$ terms, T_μ standing for all gauge fields. The Lagrangian then becomes

$$\begin{aligned}\mathscr{L} = &(\partial^\mu \chi_3)(\partial_\mu \chi_3) + \mu^2\chi_3^2 - \frac{1}{4}G^{l\mu\nu}G^l_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &\frac{g^2}{4}(W^{1\mu}W^1_\mu + W^{2\mu}W^2_\mu)a^2 + \frac{a^2}{4}(gW^{3\mu} - g'A^\mu)(gW^3_\mu - g'A_\mu) + \dots\end{aligned} \tag{4.52}$$

In our total Lagrangian the fields W^3 and A are coupled. In order to obtain a more physical Lagrangian we decouple it by introduction of the *Weinberg angle* θ_W

$$\tan\theta_W = \frac{g'}{g}. \tag{4.53}$$

Consequently we write

$$\begin{aligned}(gW^{3\mu} - g'A^\mu) &= g(W^{3\mu} - \frac{g'}{g}A^\mu) = g(W^{3\mu} - \tan(\theta_W)A^\mu) \\ &= \frac{g}{\cos\theta_W}(\cos\theta_W W^{3\mu} - \sin\theta_W A^\mu) \\ &= \frac{g}{\cos\theta_W}Z^\mu,\end{aligned} \tag{4.54}$$

with the new field Z_μ and its *orthogonal* field B_μ given by

$$Z_\mu = \cos\theta_W W^{3\mu} - \sin\theta_W A^\mu \quad \text{and} \tag{4.55}$$

$$B_\mu = \sin\theta_W W^{3\mu} + \cos\theta_W A^\mu. \tag{4.56}$$

These two fields decouple in the equation (4.52). It then follows from $\cos^2(x)+\sin^2(x)=1$ that

$$\cos^2(\theta_W) = \frac{g^2}{g^2+g'^2} \tag{4.57}$$

and we obtain for the coupling part of the Lagrangian

$$(gW^{3\mu} - g'A^\mu)(gW^3_\mu - g'A_\mu) = (g^2+g'^2)Z^\mu Z_\mu. \tag{4.58}$$

The gauge field contribution terms from $G^{\mu\nu}G_{\mu\nu}$ can be split by the trace properties of the pauli matrices. We neglect the commutator term in (4.12) because after multiplication it always results in terms of higher order than two

$$\begin{aligned}\frac{1}{2}\mathrm{tr}(G^{l\mu\nu}G^m_{\mu\nu}) &= \frac{1}{2}\mathrm{tr}\left((\partial_\mu\frac{1}{2}\sigma^l W^l_\mu - \partial_\nu\frac{1}{2}\sigma^l W^l_\mu)(\partial^\mu\frac{1}{2}\sigma^m W^{m\mu} - \partial^\nu\frac{1}{2}\sigma^m W^{m\mu})\right) \\ &= \frac{1}{8}\mathrm{tr}(\sigma^l\sigma^m)(\partial_\mu W^l_\mu - \partial_\nu W^l_\mu)(\partial^\mu W^{m\mu} - \partial^\nu W^{m\mu}) \\ &= \frac{1}{4}(\partial_\mu W^l_\mu - \partial_\nu W^l_\mu)(\partial^\mu W^{l\mu} - \partial^\nu W^{l\mu}).\end{aligned} \tag{4.59}$$

with the help of (4.35). The Einstein sum convention applies. From (4.55) we obtain

$$W^{3\mu} = \cos\theta_W Z_\mu + \sin\theta_W B_\mu \tag{4.60}$$

$$A^\mu = -\sin\theta_W Z_\mu + \cos\theta_W B_\mu \tag{4.61}$$

When we insert the equations for fields W^3 and A into $\frac{1}{4}G^{l\mu\nu}G^l_{\mu\nu}$ and $\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ we can rewrite the Lagrangian in terms of Z and B. With the help of a new notation we can *demonstrate* the change of the gauge Lagrangian terms of interest

$$\begin{aligned}
&(\partial_\mu W^3_\mu - \partial_\nu W^3_\mu)(\partial^\mu W^{3\mu} - \partial^\nu W^{3\mu}) + (\partial_\mu A_\mu - \partial_\nu A_\mu)(\partial^\mu A^\mu - \partial^\nu A^\mu)\\
&\quad := (\partial W_\mu)(\partial W^\mu) + (\partial A_\mu)(\partial A^\mu)\\
&\quad = \cos^2\theta_W (\partial Z_\mu)(\partial Z^\mu) + \sin^2\theta_W (\partial B^\mu)(\partial B_\mu)\\
&\qquad + \sin^2\theta_W (\partial Z_\mu)(\partial Z^\mu) + \cos^2\theta_W (\partial B^\mu)(\partial B_\mu)\\
&\quad = (\partial_\mu Z_\mu - \partial_\nu Z_\mu)(\partial^\mu Z^\mu - \partial^\nu Z^\mu) + (\partial_\mu B_\mu - \partial_\nu B_\mu)(\partial^\mu B^\mu - \partial^\nu B^\mu).
\end{aligned} \tag{4.62}$$

Then the Lagrangian takes the decoupled form

$$\begin{aligned}
\mathscr{L} &= (\partial^\mu \chi_3)(\partial_\mu \chi_3) + \mu^2 \chi_3^2\\
&\quad - \frac{1}{4}\sum_{l=1}^{2}\Big((\partial_\mu W^l_\mu - \partial_\nu W^l_\mu)(\partial^\mu W^{l\mu} - \partial^\nu W^{l\mu}) - g^2 a^2 W^{l\mu} W^l_\mu\Big)\\
&\quad - \frac{1}{4}\left((\partial_\mu Z_\mu - \partial_\nu Z_\mu)(\partial^\mu Z^\mu - \partial^\nu Z^\mu) - (g^2 + g'^2) a^2 Z^\mu Z_\mu\right)\\
&\quad - \frac{1}{4}(\partial_\mu B_\mu - \partial_\nu B_\mu)(\partial^\mu B^\mu - \partial^\nu B^\mu) + \ldots
\end{aligned} \tag{4.63}$$

In the higher terms that we omitted, there are still coupling terms of the fields included. But now we have a unique answer to the question of which fields obtained what mass. We have one scalar field χ_3 and in addition three massive vector particles W^1, W^2 and Z with masses

$$W^1_\mu: \quad m_{W^1} = g\frac{a}{\sqrt{2}} \tag{4.64}$$

$$W^2_\mu: \quad m_{W^2} = g\frac{a}{\sqrt{2}} = m_{W^1} \tag{4.65}$$

$$Z_\mu: \quad m_Z = \sqrt{g^2 + g'^2}\frac{a}{\sqrt{2}} \tag{4.66}$$

$$\chi_3: \quad m_{\chi_3} = \sqrt{-2\mu^2} \tag{4.67}$$

and we have one remaining massless vector particle B_μ. We check for the conservation of degrees of freedom. In the end there are 3 degrees of freedom per 3 massive vector particles, 2 per massless particle and 1 for the Higgs field. This equals the 12 degrees of freedom we had in advance of the spontaneous symmetry breaking, 4 in the iso-doublet ϕ, 6 in the massless gauge bosons W_μ^l and 2 in the massless gauge boson B_μ.

We compare this to the global example from Subsection 3.5.2 where we observed 3 massless Goldstone bosons after the spontaneous breaking of the $SU(2)$ symmetry. Now there appear 3 massive vector bosons as predicted before by the correlation of globally appearing Goldstone bosons and the locally appearing massive gauge bosons. One massive scalar boson remains as in the global case and is again refered to as the Higgs boson. This will be further elaborated in the next chapter.

Table 4.1: Comparison of the Global and Local Symmetry Breaking

Symmetry group	Goldstone mode (global)	Higgs mode(local)
$U(1)$	2 massive scalar bosons ↓ 1 massive scalar boson + 1 massless scalar boson	2 massive scalar bosons + 1 massless vector boson ↓ 1 massive scalar boson + 1 massive vector boson
$U(1)\otimes SU(2)$	4 massive scalar bosons ↓ 1 massive scalar boson + 3 massless scalar bosons	4 massive scalar bosons + 3 massless vector bosons ↓ 1 massive scalar boson + 3 massive vector bosons

5 Summary and Physical Importance

We summarize that the phenomenon of spontaneous symmetry breaking occurs, when a system is invariant under a global or local symmetry group, whereas the vacuum does not share the same full symmetry. In the case of a global symmetry this leads to the Goldstone theorem and the existence of massless particles, the Goldstone bosons, as has been proven. If the Lagrangian is invariant under a symmetry group G and the vacuum state is invariant under $H \subset G$, as many Goldstone bosons appear as symmetries are broken. This is equal to $\dim G/H$. If we constructed as many fields as we have generators in G, the number of massive scalar bosons is $\dim H$. The concept was illustrated in a semiclassic approach with the examples of a $U(1)$ and $SU(2) \otimes U(1)$ symmetry.
In the case of a local symmetry massless gauge fields appear in the Lagrange function. During spontaneous breakdown of the symmetry, some of the massless gauge fields take up the massless Goldstone bosons in order to become massive themselves. The number of massive gauge bosons then equals the number of Goldstone bosons, i.e. $\dim G/H$.

The concept of spontaneous symmetry breaking of gauge symmetries has been a great contribution to many fields of physics. Also the global case my appear in nature, such that the Goldstone theorem finds application as well. It explains the fact that the pion is nearly massless, because the spontaneous breakdown of the chiral symmetry predicts massless pions. Due to an explicit symmetry breaking in the Lagrangian the symmetry becomes *approximate*. The breaking terms are then a perturbation of the symmetric case and cause the pion to obtain a small mass. For a detailed discussion see [8] page 121ff.

One of the major contributions of the local mechanism was in the field of solid state physics, giving a reasonable explanation to the Meissner effect in superconductors. Superconductivity describes the effect that many metals have no resistivity at very low temperatures. The parameter in this example is controlled by the temperature. Recalling any of our examples we set the parameter $\mu = a(T - T_c)$ such that for $T < T_c$ the parameter becomes negative and the symmetry of the Lagrangian is broken for the ground state. It can be shown, that this causes the resistivity to vanish. Due to the Higgs mechanism the photons gain mass at the same time. Then the photons penetrate the solid only up to a certain depth remaining in a border region of the solid. This is the essence of the Meissner effect, saying that in a superconductor the magnetic flux is effectively screened. It is not an effect caused by induction. Consider the metal to be at $T > T_c$ and apply an external electromagnetic radiation. It will penetrate deep into the metal. Lowering the temperature below T_c will spontaneously break the symmetry and immediately ban the magnetic field from the solid, except from a border shell. This shell has the thickness of the allowed penetration depth of the massive photons.

In the fields of particle physics a major improvement in the model of unified weak and electromagnetic interactions was accomplished by the concept of the Higgs mechanism. The Weinberg-Salam [12] model incorporated the following ideas. The Lagrangian for the theory involves terms for massless electrons, muons and neutrinos as well as massless gauge bosons that are responsible for the weak interaction. This Lagrangian is invariant under the internal symmetry group $SU(2)$. In order to obtain massive gauge bosons at lower energies we introduce a scalar field ϕ, called *Higgs field* with non-vanishing vacuum expectation value for low energies. This causes a spontaneous breakdown of symmetry as in the non-abelian example discussed in Section 4.3 giving mass to the electrons, the muons and some gauge fields but not to the photons and neutrinos. The parameter for this model is the energy of the system. For high energies the vacuum keeps the symmetry of the system. However, for lower energies the spontaneous breakdown occurs. In the preceding example the field B_μ remained massless (see (4.63)). This field can now again be identified as the electromagnetic field A_μ. The spontaneous breakdown in the Weinberg-Salam model happens in this examples context in the following way

$$SU(2) \otimes U(1) \rightarrow U_{em}(1). \tag{5.1}$$

In simple words the highly symmetric system with many massless gauge bosons breaks down at low energies to a system symmetric only under the $U(1)$ group of quantum electrodynamic, i.e. a system involving now only one massless particle: the photon.
At CERN in Genf it is the subject to experimentally find the predicted Higgs particle. This will be done at the not yet finished *Large Hadron Collider (LHC)*, the largest and highest energy particle accelerator in the world. It will be possible to accelerate particles for high energy collision experiments. Simulation predicts that the collision of high energy protons will result in a Higgs boson, whose decay signature is unique and can therefore be detected. If however, it is not found in those experiments, there already exist *Higgsless mechanisms* not involving Higgs particles.

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