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Simulations of the F model on planar ϕ^4 Feynman diagrams*

Wolfhard Janke†
Institut für Theoretische Physik, Universität Leipzig,
Augustusplatz 10/11, D-04109 Leipzig, Germany
E-mail: wolfhard.janke@itp.uni-leipzig.de

Martin Weigel
Department of Physics, University of Waterloo,
Waterloo, Ontario, N2L 3G1, Canada
E-mail: weigel@boromir.uwaterloo.ca

The 6-vertex F model on the square lattice exhibits a critical line with central charge $C = 1$, terminating in a critical point of the Kosterlitz-Thouless type. As such, its coupling to two-dimensional quantum gravity by placing it onto the dynamically triangulated random surfaces (DTRS) of the simplicial formulation yields an interesting realization of the limiting case $C = 1$ where the continuum description of quantum gravity plus matter fields in two dimensions breaks down. Technically, since the general 6- and 8-vertex models of statistical mechanics are defined with respect to four-valent lattices, the model has to be coupled to *dynamical quadrangulations*. Generalizing the well-known algorithmic tools for treating dynamical triangulations in a Monte Carlo simulation to the case of these random lattices made of squares, we present extensive numerical results for the critical-point properties of the coupled system, including the matter related as well as the graph related critical exponents of the model.

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†Speaker.

Since field theoretical formulations of Einstein gravity are perturbatively non-renormalizable, constructive approaches toward a quantization of gravity have been an ever more active field of research in the past decades [1]. The dynamical triangulations model in its Euclidean and Lorentzian versions has proved a successful candidate for the construction of such a consistent theory of quantum gravity [2]. The basic idea is to model the quantum fluctuations of space-time by a probabilistic sum over an ensemble of discrete, simplicial manifolds [3]. For the Euclidean case in two dimensions (2D), this ensemble is commonly taken as the set of all gluings of equilateral triangles to a regular, usually closed surface of fixed topology, while counting each of the possible gluings with equal weight. The resulting random-surface model and its simplicial generalisation to higher dimensions are numerically tractable, for instance by Monte Carlo (MC) simulations. In 2D, the use of matrix models and generating-function techniques led to exact solutions for the cases of pure Euclidean gravity [4] and the coupling of certain kinds of matter, such as the Ising model [5], to the surfaces. Furthermore, the critical exponents governing the transitions are conjectured exactly from conformal field theory as functions of the exponents on regular lattices via the so-called KPZ/DDK formula [6]

$$\tilde{\Delta} = \frac{\sqrt{1-C+24\Delta}-\sqrt{1-C}}{\sqrt{25-C}-\sqrt{1-C}}, \tag{1}$$

where Δ is the original scaling weight, $\tilde{\Delta}$ the "dressed" scaling weight upon coupling to gravity and C the central charge of the matter variables. The field-theory ansatz leading to Eq. (1) breaks down for central charges $C > 1$, an effect which has been termed the $C = 1$ "barrier", whereas discrete models of $C > 1$ matter coupled to dynamical triangulations are still well defined. This mismatch of descriptions and its driving mechanism is still one of the less well understood aspects of the dynamical triangulations model.

The 6-vertex model is defined by the Boltzmann weights ω_i of its arrow configurations as sketched in Fig. 1 (for reviews see, e.g., Refs. [7, 8]). On *regular* lattices the 6- and 8-vertex models form one of the most general classes of models of statistical mechanics with discrete symmetry.

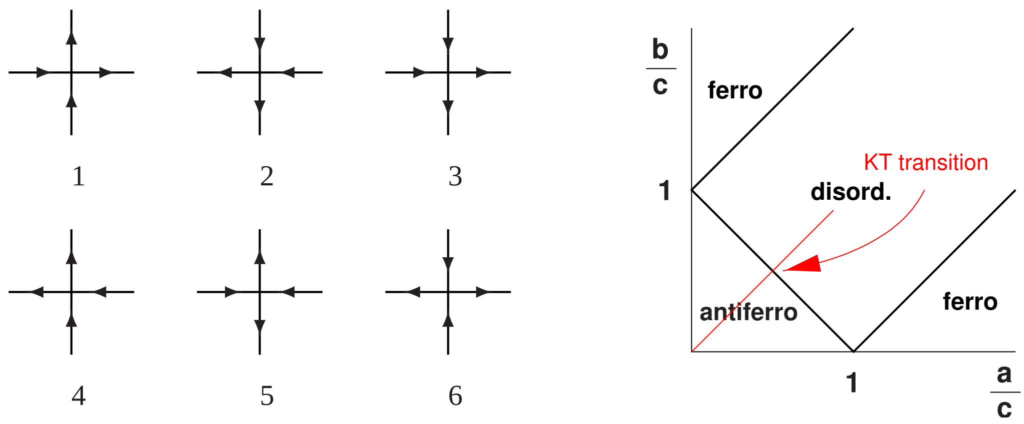

Figure 1: Left: Allowed arrow configurations of the 6-vertex model with weights $\omega_i = \exp(-\varepsilon_i/k_bT)$. Right: Phase diagram of the symmetric 6-vertex model with $a = \omega_1 = \omega_2$, $b = \omega_3 = \omega_4$, and $c = \omega_5 = \omega_6$. The locus of the F model with its Kosterlitz-Thouless (KT) phase transition runs along the diagonal.

Special cases can be mapped onto more well-known Ising and Potts models or graph colouring problems [8]. In 2D, several of these vertex models can be solved exactly, yielding a very rich phase diagram with various transition lines as well as critical and multi-critical points [8]. Hence, coupling this class of models to a fluctuating geometry of the dynamical triangulations type is of obvious interest, both as a prototypic model of statistical mechanics subject to *annealed connectivity disorder* and as a paradigmatic type of matter coupled to 2D Euclidean quantum gravity.

Recently, the use of matrix model methods led to a solution of the thermodynamic limit of a special 6-vertex model, the F model, coupled to planar ϕ^4 graphs [9]. This model corresponds to a $C = 1$ conformal field theory, i.e., it lies on the boundary to the region $C > 1$, where the KPZ/DDK formula (1) breaks down. The locus of the F model is depicted in the phase diagram of Fig. 1 for a (static) square lattice where the model exhibits a Kosterlitz-Thouless phase transition at $\beta_c = \ln 2$ [7, 8]. The same type of transition is predicted on dynamical lattices, and in particular the critical coupling $\beta_c = \ln 2$ should agree with that on the square lattice [9]. Also, a special slice of the 8-vertex model could be analysed via transformation to a matrix model [10]. However, due to intrinsic limitations of the analytical method, these studies can neither reveal the behaviour of the matter related observables and the details of the occurring phase transition nor the fractal properties of the graphs such as, e.g., their internal Hausdorff dimension d_h.

We found it therefore worthwhile to investigate this model by means of numerical MC simulations [11]. Since 6- and 8-vertex models are generically defined on four-valent lattices, instead of considering the more common dynamical triangulations or the dual planar, "fat" (i.e., orientable) ϕ^3 graphs, one has to use an ensemble of dynamical *quadrangulations* or the dual ϕ^4 Feynman diagrams as the geometry to model the coupling of vertex models to quantum gravity. For MC simulations of 2D combinatorial dynamical triangulations or ϕ^3 graphs, an ergodic set of updates is given by the so-called Pachner moves [12]. An adaption of these link-flip moves to simulations of quadrangulations proposed in Ref. [13] is shown in Fig. 2. Via explicit counter-examples it can be shown, however, that these moves do *not* in general constitute an ergodic dynamics for simulations of dynamical quadrangulations. Introducing a second type of link-flip moves, a "two-link flip" (see Fig. 2), we constructed an algorithm for dynamical quadrangulations, which does not show any signs of ergodicity breaking [14, 15, 16]. As expected for any local algorithm, however, the update dynamics exhibits severe slowing down near criticality. To alleviate this problem, we adapted the non-local "baby-universe surgery" method proposed in Ref. [17] for triangulations to quadrangulations [15, 16]. For the vertex model part, we employed the loop-cluster algorithm [18], slightly modified for the case of simulations on random lattices. These algorithmic developments as well as the technical details of the simulational set-up will be discussed elsewhere [16].

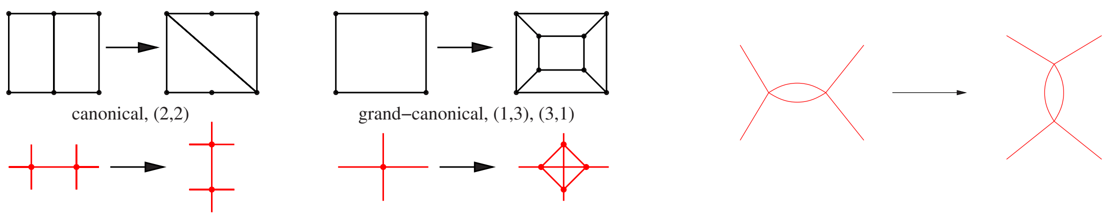

Figure 2: Analog of Pachner moves (left) and the "two-link flip" (right) for ϕ^4 graph simulations.

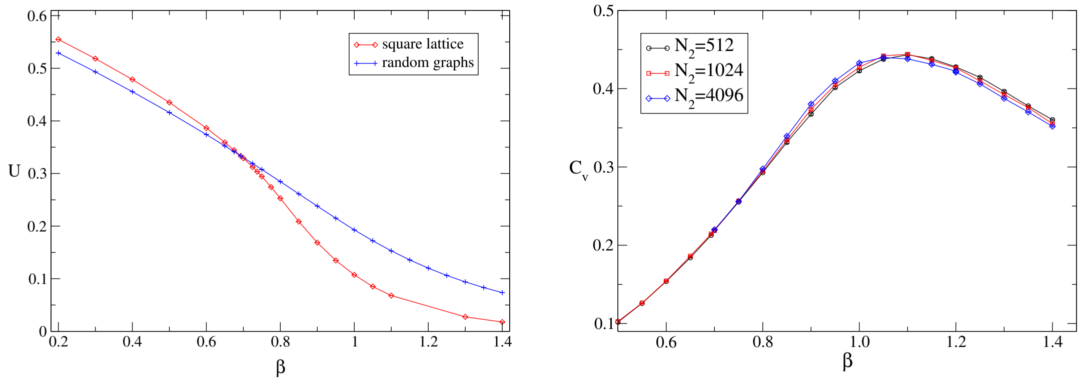

Figure 3: Internal energy U and specific heat C_v.

Among the most easily measurable quantities are the internal energy U and the specific heat C_v. The observed non-scaling of C_v with system size (see Fig. 3) is a first evidence for the expected KT-like transition. At the transition point $\beta = \beta_c = \ln 2$, we find on our largest ϕ^4 random graphs ($N_2 = 65\,536$ vertices) estimates of $U = 0.333355(11)$ and $C_v = 0.2137(12)$. Comparing with the exact critical values for the *square lattice* [7, 8] of $U = 1/3$ and $C_v = 28(\ln 2)^2/45 \approx 0.2989$, we conjecture that the critical internal energy of the F model is not affected by the coupling to random graphs. As shown in Fig. 3, this is specific to the critical point $\beta_c = \ln 2$, where the curves for the two lattice types cross. This probably indicates an additional common symmetry at criticality.

On coupling the vertex model to quantum gravity we expect a renormalization of the critical exponents as prescribed by the KPZ/DDK formula (1), which should also marginally apply to the present limiting case $C = 1$. To find the usual critical exponents from the weights, one assumes that the well-known scaling relations stay valid and thus arrives at $\alpha = \frac{1-2\Delta_\varepsilon}{1-\Delta_\varepsilon}$, $\beta = \frac{\Delta_P}{1-\Delta_\varepsilon}$, $\gamma = \frac{1-2\Delta_P}{1-\Delta_\varepsilon}$, $d_h\nu = \frac{1}{1-\Delta_\varepsilon}$, $2-\eta = (1-2\Delta_P)d_h$, where Δ_ε is the weight of the energy operator and Δ_P denotes that of the scaling operator corresponding to the spontaneous staggered polarisation P_0, which here takes on the rôle of a magnetisation. For the special case of an infinite-order KT phase transition considered here, the usual exponents are not well-defined by power-law singularities, but the finite-size scaling (FSS) exponents $\beta/d_h\nu = \Delta_P$ and $\gamma/d_h\nu = 1-2\Delta_P$ related to the polarisation still have a well-defined meaning. From the exact exponent $\gamma/d_h\nu = 1/2$ for a (static) square lattice with $d_h = 2$, we find the conformal weight $\Delta_P = 1/4$, leading via Eq. (1) for $C = 1$ to $\tilde{\Delta}_P = 1/2$ and hence $\widetilde{\beta/d_h\nu} = \tilde{\Delta}_P = 1/2$, $\widetilde{\gamma/d_h\nu} = 1-2\tilde{\Delta}_P = 0$.

For a numerical check of the exponent $\gamma/d_h\nu$ conjectured by the KPZ/DDK formula, there are the two principal possibilities of considering the FSS of the staggered polarisability χ (analogous to a susceptibility) at its maxima for the finite graphs *or* at the fixed asymptotic transition coupling $\beta_c = \ln 2$. In both cases, by analogy to the square-lattice model [19] we start from an FSS form including a leading effective correction term, namely,

$$\chi(N_2) = A_\chi N_2^{\gamma/d_h\nu}(\ln N_2)^{\omega_\chi}. \tag{2}$$

For the square-lattice one has $\omega_\chi = 2$, whereas for the random-graph model considered here the correction exponent is not known. While in the asymptotic regime both FSS sequences are expected to lead to the same exponents, this is not obvious in the presence of large correction effects for the

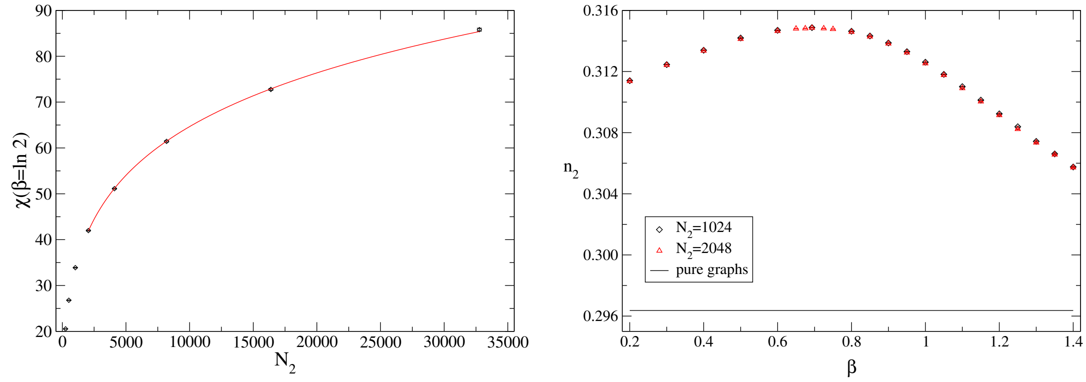

Figure 4: Left: FSS of the polarisability at the asymptotic critical coupling $\beta_c = \ln 2$ together with a purely logarithmic fit (see text). Right: Fraction of loops of length two.

accessible graph sizes (note that the dynamical lattices are highly fractal with $d_h \approx 4$). Indeed, analyses of the maxima data turned out to be very intricate [11]. On the other hand, assuming a purely logarithmic increase of $\chi(N_2)$ as implied by the KPZ/DDK prediction $\gamma/d_h\nu = 0$, the data at β_c up to $N_{2,\max} = 32\,768$ yield good-quality fits already for $N_{2,\min} \gtrsim 512$; for $N_{2,\min} = 2048$ the parameters of this purely logarithmic fit shown in Fig. 4 are $A_\chi = 0.3960(96)$ and $\omega_\chi = 2.295(11)$ with a perfect goodness-of-fit $Q = 0.39$.

For the spontaneous polarisation P_0 (analogous to a magnetisation), the FSS behaviour was found to be qualitatively very similar to that of the polarisability. Here the FSS ansatz is taken as

$$P_0(N_2) = A_{P_0} N_2^{-\beta/d_h\nu} (\ln N_2)^{\omega_{P_0}}. \tag{3}$$

At β_c the result $\beta/d_h\nu = 0.469(15)$ from a fit starting at $N_{2,\min} = 2048$ is consistent with the KPZ/DDK conjecture $\beta/d_h\nu = 1/2$ within about two times the quoted standard deviation. We note that the estimated correction exponents ω_χ and ω_{P_0} differ largely [11] (see also Ref. [19]).

As far as the graph properties are concerned, we first looked at the coordination number distribution. The r.h.s. of Fig. 4 shows the fraction of loops of length two in the graph as a function of β which exhibits a peak at $\beta_0 = 0.6894(54)$, in good agrement with $\beta_c = \ln 2 \approx 0.693$. In fact, this observable, which clearly reflects the matter back-reaction on the graphs, turned out to be much more suitable for locating β_c than more traditional quantities such as the peak location of the polarisability [11].

The string susceptibility exponent γ_s is defined through $Z(N_2) \sim e^{\mu_0 N_2} N_2^{\gamma_s - 3}[1 + \mathscr{O}(1/N_2)]$ for the planar case. By decomposing the graphs into a self-similar tree of "baby universes", the distribution of minBUs of size B can be used to determine γ_s from

$$\langle n_{N_2}(B)\rangle \sim N_2^{2-\gamma_s} [B(N_2 - B)]^{\gamma_s - 2}. \tag{4}$$

This method, originally introduced for triangulations or ϕ^3 graphs [17, 20], has been generalized to ϕ^4 graphs [11]. Pure ϕ^4 graphs yield $\gamma_s = -1/2$ in agreement with universality. For the F model with central charge $C = 1$, the scaling form has again to be augmented with logarithmic corrections. We find that the resulting estimates are compatible with $\gamma_s = 0$ for $\beta \leq \ln 2$ (critical phase) and with $\gamma_s = -1/2$ in the ordered phase, in agreement with the KPZ/DDK conjecture.

Finally we considered also the Hausdorff dimension d_h which previously was found difficult to decide numerically between the contradictory predictions 4.83 and ∞ as $C \to 1$ [21]. By a FSS analysis of the (geometrical) two-point correlation function of the graphs and of their mean extent we obtained $d_h = 4$, independent of β.

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