UNIVERSITÄT LEIPZIG

Institut für Theoretische Physik
Prof. Dr. R. Verch
Dr. M. Hänsel, D. Janssen
Summer Term 2020

Cosmology Problem Sheet 8 (new edit)

Problem 8.1 [4 points]

For $k = 1$ and $p = 0$, there is a static solution to Friedmann's equations with $a(\tau) = 3C/2$, and infinite lifetime, $\tau \in J = \mathbb{R}$. The "space" at every cosmological time is hence the compact S^3 with a constant geometrical radius and volume.

Assuming the matter density is luminous (the "grains of dust" are galaxies), with constant average luminosity in time: Does this cosmological model evade Olbers' paradox? Or does it not?

Problem 8.2 [4 points]

Let S be a two-dimensional spacelike submanifold of a spacetime manifold M with Lorentzian metric $g_{\mu\nu}$. This means,

$$\mathrm{k}_{\mu\nu} v^\mu w^\nu = -g_{\mu\nu} v^\mu w^\nu$$

is a Riemannian metric for S, for any pair of vectors v^μ , w^ν tangent (at the same, arbitrary point) to S.

Supposing that there is a coordinate chart $(y^j)_{j=1,2}$ for S whose chart domain covers S completely (potentially, up to only 1-dimensional pieces), the *metric area* of S is defined as

$$A(S) = \int \sqrt{\mathrm{k}(y^1, y^2)}\, dy^1\, dy^2$$

where the integral is over the 2-dimensional chart range, and $\mathrm{k}(y^1, y^2) = \det(\mathrm{k}_{ij}(y^1, y^2))$ is the determinant of the coordinate expression of the metric. (One can show that this definition of the metric area is independent of the choice of a chart for S.)

Consider an FLRW spacetime with scale factor $a(\tau)$. A star (pointlike) emits radiation at some cosmic time τ_*. At some later time $\tau_\star$, all the lightrays that have emanated from the star at τ_* reach (and cover exactly) a 2-dimensional, spacelike submanifold $S = S_{\tau_*,\tau_\star}$ (of $\{\tau_\star\} \times \Sigma^{(k)}$).

Choosing suitable coordinates, show that the definition of the luminosity distance d_L between the star and an observer receiving radiation from the star at $\tau_\star$ coincides with

$$d_L^2 = \frac{L}{A(S)} ,$$

assuming isotropic light propagation.

Problem 8.3 [8 points]

Verify the luminosity distance / redshift relation up to 2nd order in z,

$$d_L = \frac{z}{H_0}(1 + \frac{1}{2}(1 - q_0)z) + O(z^3)\,,$$

with $H_0 = H(\tau_0)$ and $q_0 = q(\tau_0)$, where

$$q(\tau) = -\frac{\ddot{a}(\tau)}{H(\tau)^2 a(\tau)}$$

is the *deceleration parameter* at cosmic time τ. In a d_L-z-diagram, sketch $d_L = d_L(z)$ for the cases that $\ddot{a}(\tau_0)$ is equal to 0, or larger/smaller than 0.

Problem 8.4 [8 points]

Consider an FLRW spacetime with $k = 0$, $J = (0, \infty)$ and scale factor $a(\tau)$. It will be assumed that there is some $\tau_0 > 0$ such that $a(\tau) \geq a_0 \tau^\beta$ if $\tau \geq \tau_0$ with constants $a_0 > 0$ and $\beta > 1$.

Show the following: If an observer has the worldline $\gamma_q(\tau) = (\tau, q)$, and some galaxy has the worldline $\gamma_{q'}(\tau) = (\tau, q')$, where q and q' are in $\mathbb{R}^3$, then there is for any $\tau > \tau_0$ some $R > 0$ so that, if the geometric distance between $\gamma_q(\tau)$ and $\gamma_{q'}(\tau)$ is as least as large as R, no light signal emitted from $\gamma_{q'}(\tau')$ at any time $\tau' \geq \tau$ can reach any point $\gamma_q(\tau^\circ)$ with $\tau^\circ \geq \tau$. The minimal R fulfilling this condition is called the *Hubble radius* of the observer at cosmic time τ. Heuristically, if the scale factor grows fast enough and the galaxy is far enough from the observer, then the observer receeds too fast for the lightray emanating from the galaxy to the future to still catch up with the observer.

The solutions to the problems are to be handed in by Thu, 18 June 2020, 8 pm, using the moodle-tool for the Cosmology course, see

https://moodle2.uni-leipzig.de/course/index.php?categoryid=2765

You should upload your solution as pdf file. It is perfectly ok if you work out the solutions in hand-written form and scan/photograph them and convert the files into a single pdf file. However, please make sure that the result is very well readable, and that the pdf files aren't excessively large. Please leave some margin space for marking. The marked solutions will be made available in the moodle-tool.