UNIVERSITÄT LEIPZIG

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Cosmology Problem Sheet 6

Problem 6.1 [4 points]

Let M be a 4-dimensional manifold with Lorentzian metric $g_{\mu\nu}$. Define a new metric $\tilde{g}_{\mu\nu} = \lambda g_{\mu\nu}$ where $\lambda > 0$ is a *constant*. How are the Riemann tensor, the Ricci tensor and the scalar curvature of that new metric related to the corresponding quantities of $g_{\mu\nu}$?

Problem 6.2 [6 points]

The Einstein tensor of a Lorentzian metric $g_{\mu\nu}$ is given by

$$G^{(\Lambda)}_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$$

with respect to arbitrary local coordinates (x^μ), where Λ is a cosmological constant. Show that the Einstein tensor has vanishing divergence, i.e.

$$g^{\sigma\lambda} \nabla_\sigma G^{(\Lambda)}_{\lambda\nu} = 0$$

everywhere in spacetime. Show also that the Einstein tensor is symmetric, i.e.

$$G^{(\Lambda)}_{\mu\nu} = G^{(\Lambda)}_{\nu\mu} \,.$$

Problem 6.3 [6 points]

Suppose that w.r.t. coordinates (x^0, x^1, x^2, x^3) of a 4-dimensional Lorentzian manifold the coordinate component functions $g_{\mu\nu}$ have the form

$$g_{00}(x) = 1, \quad g_{11}(x) = g_{22}(x) = g_{33}(x) = -f(x^0)\,,$$

all others $= 0$, where $f(x^0)$ is a smooth, strictly positive function. Suppose that the stress-energy tensor of some material in the spacetime assumes, in these coordinates, the form

$$T_{\mu\nu}(x) = \begin{cases} \varrho(x^0) & \text{if} \quad \mu = \nu = 0 \\ 0 & \text{else} \end{cases}$$

(A) What conditions must $\varrho(x^0)$ fulfill for $T_{\mu\nu}$ to be divergence-free, i.e.

$$g^{\sigma\lambda} \nabla_\sigma T_{\lambda\nu} = 0 \quad ?$$

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(B) Suppose that an observer with worldline $\gamma(t)$ is equipped with an energy measuring device. What is the energy density that the observer measures on the material having the said stress-energy tensor at the spacetime point $\gamma(t)$?

Hint: That is very similar as in Minkowski spacetime; think of the relative velocities of an infinitesimal volume element and the observer, and how they are expressed using the metric.

Problem 6.4 [8 points]

In four-dimensional Minkowski spacetime $(\mathbb{R}^4, \eta)$, consider a mass distribution of the type "dust" (zero pressure), where the infinitesimal dust particles move along geodesics, with tangent vector field u^a. With respect to a given Lorentz system, the initial values for the tangent vector field are given by

$$u^0(x^0 = 0, x^1, x^2, x^3) = \frac{1}{\sqrt{1 - s^2|\underline{x}|^2}}, \quad u^j(x^0 = 0, x^1, x^2, x^3) = \frac{sx^j}{\sqrt{1 - s^2|\underline{x}|^2}} \quad (j = 1, 2, 3)$$

provided that $s|\underline{x}| = s\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} < 1/2$, and $u^\mu(x^0 = 0, x^1, x^2, x^3) = 0$ if $s|\underline{x}| \geq 1/2$. Units are such that $c = 1$ (velocity of light). s is a suitable positive constant (of what physical dimension?), and $\underline{x} = (x^1, x^2, x^3)$.

(a) Determine the vectorfield $u^\mu(x^0, x^1, x^2, x^3)$ on $\mathbb{R}^4$ which is fixed by these conditions. Show that it is time-like and future-directed, and of unit length.

(b) Determine the family of geodesics which are integral curves of the vectorfield u^μ.

(c) At $x^0 = 0$, the mass-density is equal to a constant ϱ_0 all over the spatial volume occupied by the mass distribution. Determine the mass-density $\varrho(x^0, x^1, x^2, x^3)$ such that it is constant over the spatial volume occupied by the mass distribution at coordinate time x^0, and such that the total energy is conserved. Hence, give an explicit formula for the stress-energy tensor $T_{\mu\nu}(x^0, x^1, x^2, x^3)$ of the mass distribution.

(d) Illustrate the situation by means of a spacetime diagram and give an interpretation.

The solutions to the problems are to be handed in by Thu, 04 June 2020, 4 pm, using the moodle-tool for the Cosmology course, see

https://moodle2.uni-leipzig.de/course/index.php?categoryid=2765

You should upload your solution as pdf file. It is perfectly ok if you work out the solutions in hand-written form and scan/photograph them and convert the files into a single pdf file. However, please make sure that the result is very well readable, and that the pdf files aren't excessively large. Please leave some margin space for marking. The marked solutions will be made available in the moodle-tool.