



Cosmology Problem Sheet 5

Problem 5.1 [4 points]

Let the matrix of functions $(g^{\mu\nu})$ be defined as the inverse matrix of $(g_{\alpha\beta})$, i.e. by $g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu}$, where $g_{\alpha\beta}$ are the coordinate component functions of a $\binom{0}{2}$ tensor field. Show that the $g^{\mu\nu}$ are coordinate component functions of a $\binom{2}{0}$ tensor field, by showing that they fulfill the appropriate transformation properties under change of coordinates.

Problem 5.2 [6 points]

Let $\Gamma^{\alpha}_{\beta\gamma}$ and $\bar{\Gamma}^{\lambda}_{\mu\nu}$ be Christoffel symbols with respect to coordinate charts (x^{α}) and (\bar{x}^{λ}) , respectively. Verify the following transformation behaviour of the Christoffel symbols:

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\lambda}}{\partial x^{\alpha}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}} \frac{\partial \bar{x}^{\lambda}}{\partial x^{\alpha}}$$

Problem 5.3 [4 points]

Verify the coordinate expression

$$\frac{d^2}{ds^2} \gamma^{\mu}(s) + \Gamma^{\mu}_{\lambda\varrho}(\gamma(s)) \left(\frac{d}{ds} \gamma^{\lambda}(s) \right) \left(\frac{d}{ds} \gamma^{\varrho}(s) \right) = 0 \quad (s \in S)$$

for the geodesic equation, defined as $\dot{\gamma}^{\mu} \nabla_{\mu} \dot{\gamma}^{\nu} = 0$.

Hint: You may use that one can locally choose coordinates so that pieces of a smooth (non-selfintersecting) curve are “coordinate lines”.

Problem 5.4 [10 points]

Suppose that w.r.t. the coordinates (x^0, x^1, x^2, x^3) of a 4-dimensional manifold the coordinate component functions $g_{\mu\nu}$ of a Lorentzian metric have the form

$$g_{00}(x) = 1, \quad g_{11}(x) = g_{22}(x) = g_{33}(x) = -f(x^0),$$

all others = 0, where $f(x^0)$ is a smooth, strictly positive function.

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Calculate (i.e. express in terms of f and its derivatives, using the coordinate expressions for the quantities given in the lectures):

1. The Christoffel symbols $\Gamma_{\mu\nu}^{\lambda}$,
2. The Riemann tensor $\mathfrak{R}^{\nu}_{\sigma\lambda\mu}$,
3. The Ricci tensor $\text{Ric}_{\sigma\mu}$,
4. The scalar curvature R ,
5. The Einstein tensor $G_{\alpha\beta} = \text{Ric}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$.

The solutions to the problems are to be handed in by Thu, 28 May 2020, 4 pm, using the moodle-tool for the Cosmology course, see

<https://moodle2.uni-leipzig.de/course/index.php?categoryid=2765>

You should upload your solution as pdf file. It is perfectly ok if you work out the solutions in hand-written form and scan/photograph them and convert the files into a single pdf file. However, please make sure that the result is very well readable, and that the pdf files aren't excessively large. Please leave some margin space for marking. The marked solutions will be made available in the moodle-tool.