UNIVERSITÄT LEIPZIG

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Cosmology Problem Sheet 4

Problem 4.1 [8 points]

Let $M \subset \mathbb{R}^N$ be an n-dimensional smooth manifold, and let (ϕ, M_Δ) be a coordinate chart, with $q \in M_\Delta$. The coordinate components of ϕ are denoted by $x^\kappa = \phi^\kappa$.
Suppose that $\phi(q) = (x^1_{(q)}, \ldots, x^n_{(q)})^T$. Show that $\partial_{x^\kappa}|_q$ (as defined in Lecture 11) coincides with the tangent vector $\dot{\gamma}_{(\kappa)}|_q = (d/ds)|_{s=0}\gamma_{(\kappa)}(s)$ of the curve

$$\gamma_{(\kappa)}(s) = \phi^{-1}(x^1_{(q)}, ..., x^\kappa_{(q)} + s, ..., x^n_{(q)})$$

Draw a figure illustrating this.

Problem 4.2 [8 points]

Consider the embedded sphere $S^2 \subset \mathbb{R}^3$ as 2-dimensional submanifold of $\mathbb{R}^3$, with coordinate charts

$$U_1 = \{S^2 \backslash (0,0,1)\}, \quad \phi_1 : \mathbb{R}^3 \supset U_1 \to \mathbb{R}^2, \quad (x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$
$$U_2 = \{S^2 \backslash (0,0,-1)\}, \quad \phi_2 : \mathbb{R}^3 \supset U_2 \to \mathbb{R}^2, \quad (x,y,z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

(a) Calculate the coordinate change map $\Psi = \phi_2 \circ \phi_1^{-1}$ on $\mathbb{R}^2$.

(b) Calculate the tangent vector $\dot{\gamma}\big|_p \in \mathbb{R}^3$ at the point $p = (1,0,0) \in S^2 \subset \mathbb{R}^3$, and calculate in particular its coordinate components with respect to ϕ_1, for the curve

$$\gamma(s) = (\sqrt{1-s^2}, s, 0) \quad (-1/4 < s < 1/4)\,.$$

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Problem 4.3 [8 points]

Consider the 2-dimensional manifold "cone without tip" embedded in $\mathbb{R}^3$,

$$M = \{(y^1, y^2, y^3) \in \mathbb{R}^3 : (y^1)^2 + (y^2)^2 - (y^3)^2 = 0\,,\ y^3 > 0\}\,.$$

Take as coordinate chart

$$\phi(y^1, y^2, y^3) = (y^3, \arcsin(y^2/y^3))\,.$$

(a) Determine the maximal chart domain M_Δ and corresponding chart range Δ subject to the condition

$$\phi(y, 0, y) = (y, 0)$$

for $y > 0$.

(b) For $q = (y, 0, y)$ and $\phi = (x^\mu)_{\mu=1,2} = (x^1, x^2)^T$, determine the basis vectors ∂_{x^1}, ∂_{x^2} in $T_q M$.

The solutions to the problems are to be handed in by Thu, 21 May 2020, 4 pm, using the moodle-tool for the Cosmology course, see

https://moodle2.uni-leipzig.de/course/index.php?categoryid=2765

You should upload your solution as pdf file. It is perfectly ok if you work out the solutions in hand-written form and scan/photograph them and convert the files into a single pdf file. However, please make sure that the result is very well readable, and that the pdf files aren't excessively large. Please leave some margin space for marking. The marked solutions will be made available in the moodle-tool.