UNIVERSITÄT LEIPZIG

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Cosmology Problem Sheet 2

Problem 2.1 - Lorentz transform and redshift [14 points]

Note that Einstein's summation convention is used throughout in the following (implicit summation on doubly appearing indices): $Y^\mu{}_\nu y^\nu = \sum_{\nu=0}^3 Y^\mu{}_\nu y^\nu$. All indices run over the range 0,1,2,3.

Consider an electromagnetic vector potential having

$$\begin{array}{ll} \text{w.r.t. inertial system } S \text{ the coordinate functions} & A_\mu(x)\,, \quad x = (x^\nu) \\ \text{w.r.t. inertial system } \overline{S} \text{ the coordinate functions} & \bar{A}_\lambda(\bar{x})\,, \quad \bar{x} = (\bar{x}^\kappa) \end{array}$$

For a Lorentz transformation $\Lambda = (\Lambda^\alpha{}_\beta)$ with $\bar{x} = \Lambda x$, i.e. $\bar{x}^\kappa = \Lambda^\kappa{}_\nu x^\nu$, it then holds that

$$\bar{A}_\lambda(\bar{x}) = L^\mu{}_\lambda A_\mu(x)\,, \quad \bar{x} = \Lambda x\,,$$

where $L^\mu{}_\lambda$ are components of the inverse matrix $L = \Lambda^{-1}$: $L^\mu{}_\sigma \Lambda^\sigma_\nu = \delta^\mu{}_\nu$ [$= 1$, if $\mu = \nu$, and $= 0$ in all other cases].

(*Warning*: Observe carefully the positioning of indices. Writing Y^μ_ν instead of $Y^\mu{}_\nu$ is ambiguous and can lead to mistakes.)

Assume the Lorentz transformation has the form

$$\Lambda = (\Lambda^\alpha{}_\beta) = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$, and the electromagnetic potential has, w.r.t. S, the form of a plane wave propagating along the x^1-axis, with frequency ω and wave number k, so that

$$A_\mu(x) = a_\mu \cos(\omega x^0 - k x^1 + \varphi_0)$$

where a_μ and φ_0 are real constants.

Show that $\bar{A}_\lambda(\bar{x})$ is also of the form of a plane wave with respect to $\overline{S}$, propagating along the $\bar{x}^1$-axis. Calculate the frequency $\bar{\omega}$ of this plane wave and confirm the Doppler effect formula

$$\bar{\omega} = \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}\,\omega\,.$$

Calculate also the relation between k and $\bar{k}$. Interpret the results and represent the situation in a spacetime diagram.

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Problem 2.2 [6 points]

The Hubble Ultra Deep Field survey has detected about 10,000 galaxies at redshift $z \simeq 5$.

Assuming Minkowskian geometry is valid, how far are these galaxies from the terrestial observer (in 10^x pc)?

Assuming also that the Hubble law is valid for any value of z, what is their recession velocity with respect to an inertial system in which the terrestial observer is approximately at rest?

Problem 2.3 [4 points]

Typical peculiar velocities of galaxies are about a few hundred kilometers per second. The mean distance between large galaxies is about 1 Mpc. How distant must a galaxy be from the terrestial observer for its peculiar velocity to be small compared to its Hubble velocity, taking as value for the Hubble parameter $75km/(s \cdot Mpc)$? (Again, Minkowski geometry is assumed, and velocities are with respect to an inertial system in which the terrestial observer is approximately at rest.)

The solutions to the problems are to be handed in by Thu, 07 May 2020, 4 pm, using the moodle-tool for the Cosmology course, see

https://moodle2.uni-leipzig.de/course/index.php?categoryid=2765

You should upload your solution as pdf file. It is perfectly ok if you work out the solutions in hand-written form and scan/photograph them and convert the files into a single pdf file. However, please make sure that the result is very well readable, and that the pdf files aren't excessively large. Please leave some margin space for marking. The marked solutions will be made available in the moodle-tool.