Cosmology

Summer Term 2020, Lecture 28

Rainer Verch

Inst. f. Theoretische Physik
Universität Leipzig
15 July 2020

During inflation, the dominant part of energy density and pressure is carried by the inflaton field φ. Neglecting the radiation (and matter) contribution in energy density and pressure, the inflationary phase is a solution to the **gravitationally coupled Einstein-Klein-Gordon system**, where the fields to be determined dynamically are $g_{\mu\nu}$, the spacetime metric, and φ, the inflaton field, with some given effective potential V. The coupled system of field equations is

$$\Box\varphi + V'(\varphi) = 0\,, \qquad G^{[g]}_{\mu\nu} = \kappa T^{[g,\varphi]}_{\mu\nu}$$

where $\Box = \nabla^\mu\nabla_\mu$ is the d'Alembert operator of the metric $g_{\mu\nu}$, and the Einstein tensor and the stress-energy tensor carry labels in square brackets to indicate that they refer to $g_{\mu\nu}$ and φ.

Now we assume that we have a solution to the gravitationally coupled Einstein-Klein-Gordon system where the resulting spacetime is an FLRW spacetime with $k = 0$ and a scale factor a together with an inflaton field φ (depending only on time) which corresponds to a slow-roll scenario.

In the following, we consider $\tilde{a}(\eta)$ and $\varphi(\eta)$, i.e. the “background” solutions as functions of conformal time η.

We call this a “background” solution because we now want to consider perturbations around that solution which lead to **deviations from spatial homogeneity and isotropy**. The expectation is that small perturbations (or fluctutations) around a (homogeneous and isotropic) solution during the early inflationary phase lead to density deviations from homogeneity which during cosmic expansion “blow up” and are therefore the seeds for density inhomgeneities which lead to temperature fluctations in the CMB.

Essentially, the CMB temperature released from the plasma at recombination varies a little depending on whether it originates from a place with a little higher or lower energy density (compared to the average), and that corresponds to the small temperature fluctuations in the CMB around its average temperature. The main mechanism describing this behaviour of the CMB temperature dependence on the energy density at the time of its release is the **Sachs-Wolfe-effect**. That would be another lecture but for lack of time, I will skip it here. It is well described in the textbooks.

In describing small perturbations of the spacetime metric around a given one, one needs a way of parametrizing them; they are also subject to a gauge freedom which is a complication. There are three types of metric perturbations which can be separated, they are **scalar**, **vector** and **tensor** type perturbations. Here, we look at the simplest type, the scalar perturbations (as do most textbooks), where we assume that the metric is perturbed by a **Bardeen potential** $\Phi(\eta, x, y, z)$, giving perturbed metrics of the form

$$ds^2(\epsilon) = \tilde{a}^2(\eta)\left((1+2\Phi)d\eta^2 - (1-2\Phi)(dx^2 + dy^2 + dz^2)\right)$$

The idea here is that $\Phi = O(\epsilon)$ with $\epsilon << 1$, so if this metric ansatz is inserted into the Einstein equations – more precisely, the coupled Einstein-Klein-Gordon system – terms corresponding to the same order in ϵ will be equated separately and terms of quadratic or higher order in ϵ will be discarded. This is the ususal perturbation-theoretic approach, as you may have seen in quantum mechanics. The metric perturbation is written in this particular way because it ensures “gauge invariance” at the perturbative level. I will avoid the discussion of the gauge freedom in the perturbations of the metric since its is complicated; see the textbooks for detailed discussion. However, it has the consequence that gauge invariant quantities assume a special form.

In a perturbation of the Einstein-Klein-Gordon system around a "background" solution given by $\tilde{a}$ and φ, there need to be also perturbations around φ:

$$\varphi(\epsilon) = \varphi + \delta\varphi\,, \qquad \text{where} \quad \delta\varphi = \delta\varphi(\eta, x, y, z) = O(\varepsilon)$$

A gauge invariant quantity describing the joint metric & matter perturbation at $O(\epsilon)$ is the **Mukhanov-Sasaki variable**

$$u = u(\eta, x, y, z) = \tilde{a}\delta\varphi + \zeta\Phi$$

where

$$\zeta = \frac{\tilde{a}}{\mathcal{H}}\sqrt{\varrho_\varphi + p_\varphi}\,;$$

here, all quantities refer to the "background" solution, and $\tilde{a}(\eta)\cdot\mathcal{H}(\eta) = d\tilde{a}(\eta)/d\eta$.

Plugging the perturbed metric and perturbed $\varphi(\epsilon)$ into the Einstein-Klein-Gordon system gives for the terms of $O(\epsilon)$ a wave equation for u,

$$\partial_\eta^2 u - \Delta u - \frac{\partial_\eta^2\zeta}{\zeta}u = 0 \qquad (\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2)$$

If u is a solution to the wave equation, one can find the corresponding metric perturbation $\Phi(\eta, x, y, z)$ as the solution to

$$\Delta\Phi = \frac{\mathcal{H}}{2\tilde{a}^2}(\zeta\partial_\eta u - u\partial_\eta\zeta)$$

Therefore, the metric perturbation has properties similar to that of a solution to a wave equation; it is a gravitational wave (of small amplitude) perturbing the background (which is constant in space).

Instead of u, also the **comoving curvature perturbation** is considered:

$$\mathtt{R} = -\frac{u}{\zeta} \quad \Rightarrow \quad \partial_\eta^2\mathtt{R} - \Delta\mathtt{R} + 2\frac{\partial_\eta\zeta}{\zeta}\partial_\eta\mathtt{R} = 0$$

Like u, also $\mathtt{R}$ is a gauge-invariant joint metric-matter perturbation of the Einstein-Klein-Gordon system at $O(\epsilon)$.

The idea is now: u, or equivalently, $\mathtt{R}$, describe small perturbations of matter/geometry during inflation.

Question: How small can they be?

Answer: There is a lower limit set by quantum fluctuations.

E.g. in a cavity, you cannot get rid of "shot noise" in the electromagnetic field due to quantum fluctuations.

This is related to the lowest stationary energy level of the quantized harmonic oscillator = $\hbar\omega/2 > 0$, while classically it is = 0.

In turn, this is related to the quantum mechanical uncertainty relation $\Delta x \cdot \Delta p \geq \hbar/2$.

Thus: Treating u, resp. $\mathtt{R}$ as a *quantized* field, so that $\mathtt{R}(\eta, x, y, z)$ is formally replaced by an *operator* $\mathbf{R}(\eta, x, y, z)$ (in a suitable Hilbert space), it holds that

$$\langle \mathbf{R}(\eta, \boldsymbol{x})^* \mathbf{R}(\eta', \boldsymbol{x}') \rangle_\psi > 0 \qquad (\boldsymbol{x} = (x, y, z))$$

if $(\eta, \boldsymbol{x})$ is close enough to $(\eta', \boldsymbol{x}')$ for any state ψ of the quantum field.

(Actually, the expectation value on the left hand side has to be viewed as a distribution, which allows it to give a more quantitative lower bound, including its dependence on $\hbar$.)

A simple, formal way (which can be made completely rigorous) of quantizing $\mathtt{R}$ is by the ansatz

$$\mathbf{R}(\eta, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3k \left(m_k(\eta) A_{\boldsymbol{k}} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \overline{m_k(\eta)} A^*_{\boldsymbol{k}} \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right)$$

where $k = |\boldsymbol{k}|$, the $m_k(\eta)$ are "mode functions" chosen so that $\mathbf{R}(\eta, \boldsymbol{x})$ satisfies the same wave equation as $\mathtt{R}(\eta, \boldsymbol{x})$, and the $A_{\boldsymbol{k}}$ are formally operators fulfilling the canonical commutation relations:

$$[A_{\boldsymbol{k}}, A_{\boldsymbol{k}'}] = 0 = [A^*_{\boldsymbol{k}}, A^*_{\boldsymbol{k}'}] , \quad [A_{\boldsymbol{k}}, A^*_{\boldsymbol{k}'}] = \hbar\delta(\boldsymbol{k} - \boldsymbol{k}')$$

Remark A completely rigorous treatment of the quantization of fluctuations in inflation, also explaining why u (or $\mathtt{R}$) are the preferred fields to quantize, including an in-depth discussion of the gauge freedom involved, can be found in: T.-P. Hack,"Cosmological Applications of Algebraic Quantum Field Theory in Curved Spacetimes", Springer Briefs in Mathematical Physics, Springer-Verlag, 2016.

One can define

$$\hat{\mathbf{R}}_{\boldsymbol{k}}(\eta) = m_k(\eta)A_{\boldsymbol{k}} + \overline{m_k(\eta)}A^*_{\boldsymbol{k}}$$

which is formally the Fourier transform of $\mathbf{R}(\eta, \boldsymbol{x})$ with respect to $\boldsymbol{x}$.

The simplest types of states $\langle ... \rangle_\psi$ for the quantum field **R** have the form

$$\langle \hat{\mathbf{R}}_{\boldsymbol{k}}(\eta)^* \hat{\mathbf{R}}_{\boldsymbol{k'}}(\eta) \rangle_\psi = \frac{2\pi^2}{k^3} P(k) \delta(\boldsymbol{k} - \boldsymbol{k}')$$

where $P(k) = P_\psi(k, \eta)$ is the **2-correlation power spectrum** at conformal time η.

The idea is now to choose a state $\langle ... \rangle_\psi$ which comes closest to a "vacuum state" for **R** since (in analogy to the quantized harmonic oscillator) one expects that in such a state the above expectation value (and hence, the power spectrum) will deviate in the least possible way from 0 .

For $a(\tau) \sim \mathrm{e}^{H_0 \tau}$ with constant H_0 there is a candidate state, known as the "Bunch-Davies state". One sets this state for **R** at $\eta = \eta_{\mathrm{fin}}$ = conformal time at the end of inflation.

When looking at the relative temperature fluctuations $\Theta = \frac{\delta T}{T}$ in the CMB today, one finds that their directional correlations are approximately given by a Gaussian random distribution with a 2-correlation function

$$C_2(\Theta(\boldsymbol{k}), \Theta(\boldsymbol{k}')) = \frac{2\pi^2}{k^3}\mathcal{P}(k)\delta(\boldsymbol{k} - \boldsymbol{k}')$$

which is also characterized by a power spectrum $\mathcal{P}(\boldsymbol{k})$ at $\eta = \eta_{\mathrm{today}}$.

The key point is: One can establish a connection

$$\mathcal{P}(\boldsymbol{k}) = \mathbb{F}(P(\boldsymbol{k}), * * *)$$

where $\mathbb{F}$ is a function – which depends on quite a number of additional parameters, including the density parameters for the late stage cosmic expansion, and also some phenomenological modelling of the behaviour of electromagnetic radiation during the various processes after inflation, particularly at recombination. Knowing $\mathbb{F}$, one has constraints on $P(\boldsymbol{k})$ and can derive constraints on the inflation timescale. A careful narrative leading to $\mathbb{F}$ in textbooks, or in the review article by Norbert Straumann (highly recommended, see course webpage), typically takes 10^x pages, $1 < x < 2$. This is beyond the scope of these lectures, and would fill most of an advanced cosmology lecture course.