Cosmology

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Nucleosynthesis – continued

The **relative abundace** of nuclei with A nucleons (and Z protons) will be denoted by

$$X_A = \frac{n_A \cdot A}{n_b}$$

where n_b is the number density of baryons (protons or neutrons), counting both those bound in nuclei and those that are not bound, and n_A is the number density of nuclei with A nucleons.

Consequently,

$$n_1 = n_{\mathrm{p}} + n_{\mathrm{n}} \ \text{(unbound)}\,, \qquad \sum_A X_A = 1$$

The number density n_A in equilibrium is given, in the non-relativistic limit, by

$$n_A = \mathrm{g}_A \left(\frac{m_A k_B T}{2\pi\hbar^2} \right)^{3/2} \mathrm{e}^{(\mu_A - m_A)/k_B T}$$

where

- $\mu_A = Z\mu_{\mathrm{p}} + (A - Z)\mu_{\mathrm{n}}$,
- μ_{p}, μ_{n} are the chemical potentials of protons and neutrons
- g_A is a degeneracy factor

Similarly, for the number densities of protons and neutrons one obtains

$$n_{\mathrm{p}} = g_{\mathrm{p}} \left(\frac{m_{\mathrm{p}} k_B T}{2\pi\hbar^2} \right)^{3/2} \mathrm{e}^{(\mu_{\mathrm{p}} - m_{\mathrm{p}})/k_B T} \,, \quad n_{\mathrm{n}} = g_{\mathrm{n}} \left(\frac{m_{\mathrm{n}} k_B T}{2\pi\hbar^2} \right)^{3/2} \mathrm{e}^{(\mu_{\mathrm{n}} - m_{\mathrm{n}})/k_B T}$$

The dominant contribution comes from the exponential factor. Therefore, one can make an approximation (and thereby achieve some simplification) by putting in the $(\ldots)^{3/2}$ factor, respectively,

$$A m_{\mathrm{n}} \ \text{ instead of } \ m_A \,, \qquad m_{\mathrm{p}} \ \text{ instead of } \ m_n$$

Observing also that

$$\mu_A - m_A = Z\mu_{\mathrm{p}} + (A - Z)\mu_{\mathrm{n}} - (Z m_{\mathrm{p}} + (A - Z) m_{\mathrm{n}} - B_A)$$

where B_A is the binding energy of the nucleus with A nucleons (and Z protons), this gives

$$n_A = \frac{g_A}{g_{\mathrm{p}}^Z g_{\mathrm{n}}^{A-Z}} A^{3/2} \left(\frac{m_{\mathrm{n}} k_B T}{2\pi\hbar^2} \right)^{-3(A-1)/2} n_{\mathrm{p}}^Z n_{\mathrm{n}}^{A-Z} \mathrm{e}^{B_A/k_B T}$$

You are urged to check this formula!

Now using $X_A = A \cdot n_A/n_b$ as just introduced, $\mathrm{g_p} = \mathrm{g_n} = 2$, and

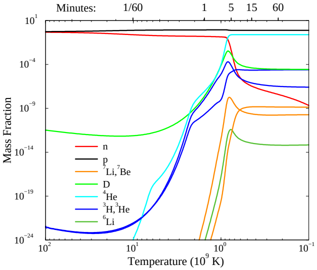
$$n_b = \eta_b \cdot n_\gamma = \eta_b \cdot \frac{2\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar}\right)^3$$

(see previous lecture), then one obtains, with the previous expresion for n_A,

$$X_A = \mathrm{g}_A \zeta(3)^{A-1} \pi^{(1-A)/2} 2^{(3A-5)/2} A^{5/2} \left(\frac{\hbar k_B T}{m_\mathrm{n}}\right)^{3(A-1)/2} {}^{3(A-1)/2}\eta_b^{A-1} X_\mathrm{p}^Z X_\mathrm{n}^{A-Z} \mathrm{e}^{B_A/k_B T}$$

Since the factor η_b is much less than 1, one must have $k_B T \ll B_A$ in order to have a non-negligible contribution for X_A.

One obtains a relative abundance distribution which looks similar to the figure presented on the following slide.



The relative abundances X_A for some light elements. The illustration is taken from S. Burles, K.M. Nollett, M.S. Turner, arXiv:astro-ph/9903300. The relative ratio of He/H $\approx$ 25% comes out nicely; however the abundance of ^{12}C increases too strongly when going to lower temperatures (not represented in the picture). The simple formula for X_A does not properly take the interaction with photons into account and the fact that partially the nuclei that have formed still dissociate under the bombardment by the very abundant photons. Moreover, the effect of the rapidly decreasing density is not modelled appropriately when going to lower temperatures.

Timeline of nucleosynthesis

Prior to onset: $\tau \approx 10^{-2}$ s, $k_B T \approx 10$ MeV

- $e^{\pm}$, $\nu, \overline{\nu}$, γ in thermal equilibrium
- $n_{\mathrm{n}} \approx n_{\mathrm{p}}$
- nuclei formation still extremely low: $X_{\mathrm{D}} \approx 6 \cdot 10^{-12}$, $X_{^3\mathrm{He}} \approx 2 \cdot 10^{-23}$, $X_{^4\mathrm{He}} \approx 2 \cdot 10^{-34}$

Onset: $\tau \approx 1$ s, $k_B T \approx 1$ MeV

- Neutrinos decoupled, $e^+ + e^- \to 2\gamma$ annihilation
- Transformation of neutrons into protons becomes ineffective, $n_{\mathrm{n}}/n_{\mathrm{p}} \approx 1/6$, $X_{\mathrm{p}} \approx 6/7$, $X_{\mathrm{n}} \approx 1/7$
- $X_{\mathrm{D}} \approx 6 \cdot 10^{-12}$, $X_{^3\mathrm{He}} \approx 2 \cdot 10^{-23}$, $X_{^4\mathrm{He}} \approx 2 \cdot 10^{-28}$

Main phase: $\tau \approx 60$ s, $k_B T \approx 0.3$ MeV

- $n_{\mathrm{n}}/n_{\mathrm{p}} \approx 1/7$

End stage: $\tau \approx 180$ s, $k_B T \approx 0.1$ MeV

- $X_{^4\mathrm{He}} \approx 0.25$
- $X_{\mathrm{D}}, X_{^3\mathrm{He}} \approx 10^{-5} \ldots 10^{-3}$
- particle density drops further and nuclear fusion processes become ineffective

Recombination

After the nucleosynthesis, there is for a long time a hot plasma (now basically in the sense it is commonly used), consisting mainly of neutrons, protons, ionized Helium nuclei and electrons together with photons (plus neutrinos that have decoupled from interactions with the plasma).

As the scale factor continues to increase, the temperature of the plasma decreases further. At some stage, the ionization energies of Hydrogen and Helium are reached and the electrons start to bind to the ionized nuclei. This process is, for historical reasons, called **recombination**, although the prefix "re" is somewhat out of place since the electrons haven't been bound to the nuclei at any prior stage.

The basic calculation for estimating the abundance of electrons bound to nuclei is similar as the one used for nucleosynthesis, but at much lower energy scale.

One considers basically the process

$$e^- + \mathrm{p} \longleftrightarrow \mathrm{H} + \gamma$$

In the following: $n_e = n_{e^-}$ is the number density of electrons.

Recombination calculation using the Saha-equation:

One can use $n_\gamma = n_\gamma^{(0)}$ and thus

$$\frac{n_e n_{\mathrm{p}}}{n_{\mathrm{H}}} = \frac{n_e^{(0)} n_{\mathrm{p}}^{(0)}}{n_{\mathrm{H}}^{(0)}} \qquad \text{Saha-eqn}$$

Notation:

$$X_e = \frac{n_e}{n_e + n_{\mathrm{H}}} \qquad \text{number density of free (unbound) electrons}$$

On macroscopic scales, matter is elecrically neutral, so one can assume $n_{\mathrm{p}} = n_e$ and therefore,

$$X_e = \frac{n_{\mathrm{p}}}{n_{\mathrm{p}} + n_{\mathrm{H}}}$$

Using the approximation $m_{\mathrm{p}} \approx m_{\mathrm{H}}$ in the $(\ldots)^{3/2}$ factors of the particle number densities (low-energy limit), the Saha-eqn above takes the form:

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_{\mathrm{H}}} \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} \mathrm{e}^{-\epsilon_0/k_B T}$$

where $\epsilon_0 = m_e - m_{\mathrm{p}} - m_{\mathrm{H}}$ is the binding energy (ionization energy) of Hydrogen.

On neglecting the contribution of Helium (making up 25%), one obtains

$$n_e + n_{\mathrm{H}} = n_p + n_{\mathrm{H}} \approx n_{\mathrm{Bar}} = \eta_b n_\gamma \approx 10^{-9} \left(\frac{k_B T}{\hbar}\right)^3$$

Then the Saha eqn in the form at the bottom the last slide gives at $k_B T \approx 10\,\mathrm{eV} \approx \epsilon_0$:

$$\frac{X_e^2}{1 - X_e} \approx 10^{15},$$

which means $X_e \approx 1$. That means: almost all electrons are still not bound to the nuclei. Only if $k_B T$ becomes significantly smaller than ϵ_0, a larger amount of electrons starts binding to the nuclei.

The Saha eqn is a not very exact since it does not model well the effect that the particle density is still decreasing and therefore the recombination rate is overestimated by the Saha equation.