Cosmology

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Nucleosynthesis

Nucleosynthesis is one of the most important processes in early cosmology – it predicts the abundances of chemical elements, mostly the ratio of Helium vs Hydrogen correctly – and the abundances can be measured (observed) within our local group of galaxies.

The description of the "plasma" again needs to be modified. We want to look at the equlibrium of processes like the transformation of different types of particles,

$$[1] + [2] \longleftrightarrow [3] + [4]$$

where $[x]$ denotes a particle type.

The energy scale (temperature) at which these processes are considered are now comparable to the rest masses of the particles, so we consider the **low energy (or non-relativistic limit** in the distribution function. In this case, however, the equilibrium situation in the transformation process is not only determined by the temperature, but also by the **chemical potentials** of the particle types in the transformation process.

This means that for every particle type, the distribution function is

$$f_{T,\mu}(\mathbf{v}) = \frac{1}{\mathrm{e}^{(E(\mathbf{v})-\mu)/k_B T} \mp 1}$$

with the chemical potential μ.

The negative chemical potential $-\mu$ is the “average energy cost per particle” of a given type to be added to the interacting plasma, from a reservoir of non-interacting particles, at temperature T.

In the non-relativistic limit where $k_B T \ll E(\mathbf{v}) - \mu$ one obtains

$$f_{T,\mu} \approx \mathrm{e}^{\mu/k_B T} f_T \,, \qquad \frac{n_{T,\mu}}{n_T} \approx \mathrm{e}^{\mu/k_B T}$$

for each type of particle, where $n_{T,\mu}$ is the particle density with chemical potential, i.e. if the particle type participates in the interaction process, and n_T $(= n_{T,0})$ is the particle density of the particle type without interaction; both at temprature T.

Now we write $n_{T,\mu}(x)$ for the particle densities of the particle types $x = 1, 2, 3, 4$ in an interaction process as above. They depend on τ through $T = T(\tau)$, so we write instead $n_x^{(\mu)}(\tau) = n_{T(\tau),\mu}(x)$.

Then the particle densities are determined by the **condition of thermal-kinetic equilibrium**, a collection of coupled differential equations for the $n_x^{(\mu)}(\tau)$ of the form

$$\frac{1}{a(\tau)^3}\frac{d(n_1^{(\mu)}(\tau)a(\tau)^3)}{d\tau} = n_1^{(0)}(\tau)n_2^{(0)}(\tau)\langle\sigma v\rangle\left(\frac{n_3^{(\mu)}(\tau)n_4^{(\mu)}(\tau)}{n_3^{(0)}(\tau)n_4^{(0)}(\tau)} - \frac{n_1^{(\mu)}(\tau)n_2^{(\mu)}(\tau)}{n_1^{(0)}(\tau)n_2^{(0)}(\tau)}\right)$$

There are 3 similar equations obtained by suitable interchange of the particle type indices $1, 2, 3, 4$.

Here $\langle \sigma v \rangle = \langle \sigma v \rangle|_\tau$ is called the **thermal average of the interaction cross section at relative partivle velocity** v for the interaction process. (For explanation, see e.g. Chp. 3 in Dodelson’s book.) It is a refinement of σ that we had considered earlier – it was actually σv, but in the relativistic limit, where v was set to be equal to 1 (velocity of light).

Analogously as before, in equilibrium one must have

$$t_{int}^{-1} \approx n_2^{(0)} \langle \sigma v \rangle \gg H \approx t_{exp}^{-1}$$

On the other hand, the left hand side of the differential equation expressing the equlibrium condition for $n_1^{(\mu)}$ is approximately of the order or magnitude of $n_1^{(\mu)} H$, which means that the term in the bracket on the right hand side must be much smaller than 1 in order to have an equilibrium situation. This is idealized to the condition that the expression in that bracket is equal to 0, i.e.

$$\left(\frac{n_3^{(\mu)}(\tau) n_4^{(\mu)}(\tau)}{n_3^{(0)}(\tau) n_4^{(0)}(\tau)} - \frac{n_1^{(\mu)}(\tau) n_2^{(\mu)}(\tau)}{n_1^{(0)}(\tau) n_2^{(0)}(\tau)} \right) = 0$$

This is called the **Saha equation** and is the approximate equilibrium condition for the interaction process $[1] + [2] \longleftrightarrow [3] + [4]$ at non-relativistic energies.

Proton abundance

We will now use the Saha-eqn to get an estimate on the proton abundance, i.e. the ratio $n_{\rm n}/n_{\rm p}$ where $n_{\rm n}$ is the number density of neutrons, and $n_{\rm p}$ is the number density of protons. These are the number densities in the interaction process, with chemical potential, but the superscript μ will be suppressed from the notation. We write $n^{(0)}_{\rm n/p}$ to denote the non-interacting particle densities.

The interaction processes which are relevant are

$$\mathrm{p} + \overline{\nu} \longleftrightarrow \mathrm{n} + e^+ \quad \text{and} \quad \mathrm{p} + e^- \longleftrightarrow \mathrm{n} + \nu$$

In a very simple approach, take

$$n_{\rm n} = n_1 \,, \quad n_{e^+,\nu} = n_2 \,, \quad n_{\rm p} = n_3 \,, n_{e^-,\overline{\nu}} = n_4$$

To make things even simpler, it is also assumed that:

$$n_{e^+,\nu} \approx n^{(0)}_{e^+,\nu} \quad \text{and} \quad n_{e^-,\overline{\nu}} \approx n^{(0)}_{e^-,\overline{\nu}}$$

In other words, the interaction processes do not significantly change the number densities of electrons and positrons and their anti-/neutrinos from what they are without the interaction process. The motivation for this approximation is:

- The $e^{\pm}$ interact strongly with the photons and at the energies of the process, the photon density is still very high (with a number density much larger than that of protons and neutrons) so that the $e^{\pm}$ are still kept in equilibrium by the interaction with the photons.
- Similarly, the number density of the neutrinos is still far higher than that of neutrons and protons, so the interaction process is a small perturbation of the neutrinos' thermal equilibrium distribution without the interaction process.

In this (somewhat crude) approximation, i.e.

$$\frac{n_{e^+,\nu}}{n^{(0)}_{e^+,\nu}} = 1 = \frac{n_{e^-,\overline{\nu}}}{n^{(0)}_{e^-,\overline{\nu}}}$$

the Saha-equation now takes the form

$$n_{\mathrm{n}} = n_{\mathrm{p}} \frac{n^{(0)}_{\mathrm{n}}}{n^{(0)}_{\mathrm{p}}}$$

In the non-relativistic limit one finds

$$\frac{n_{\mathrm{n}}^{(0)}}{n_{\mathrm{p}}^{(0)}} = \left(\frac{m_{\mathrm{n}}}{m_{\mathrm{p}}}\right)^{3/2} \mathrm{e}^{-(m_{\mathrm{n}}-m_{\mathrm{p}})/k_B T} \approx \mathrm{e}^{-Q/k_B T}$$

for $k_B T < Q = m_{\mathrm{n}} - m_{\mathrm{p}}$ where m_{n} and m_{p} are the rest energies of neutron and proton, respectively. (Notice once again that $c = 1$, so the requisite factor c^2 is hidden in the notation.)

With $Q = 1.293\,\mathrm{MeV}$ one can see that, if $k_B T < 1\,\mathrm{MeV}$, then the ratio $n_{\mathrm{n}}/n_{\mathrm{p}}$ drops rapidly.

Actually, the result confirms what one would naively expect: If the ambient thermal energy is less than the internal energy difference between neutron and proton, then the abundance of neutrons compared to protons is suppressed by the factor $\mathrm{e}^{-Q/k_B T}$, representing the energy cost for creating neutrons from protons in the above interaction process.

However, a more detailed consideration shows that the transformation of neutrons into protons becomes inefficient below $k_B T < 0.8\,\mathrm{MeV}$ since the neutrinos decouple from the process at this energy scale, and at $k_B T \approx 0.5$ MeV, the electron-positron annihilation sets in, which makes the process even less efficient and the ratio $n_{\mathrm{n}}/n_{\mathrm{p}}$ doesn’t fall significantly below $1/7$ at $k_B T \approx 0.3$ MeV.

Primordial nucleosynthesis

At $k_B T \approx 1\,\text{MeV}$, the hadronization has been completed and the annihilation processes

$$\text{baryons} + \text{antibaryons} \longrightarrow \text{photons}$$
$$\text{leptons} + \text{antileptons} \longrightarrow \text{photons} + \text{anti-/neutrinos}$$

cease to be efficient, leaving an excess of baryonic matter (protons and neutrons) and leptons, mainly e^-.

As an aside – the reason of this matter/antimatter asymmetry will not be discussed here. It is apparently a standing open question in cosmology. See the literature for further discussion.

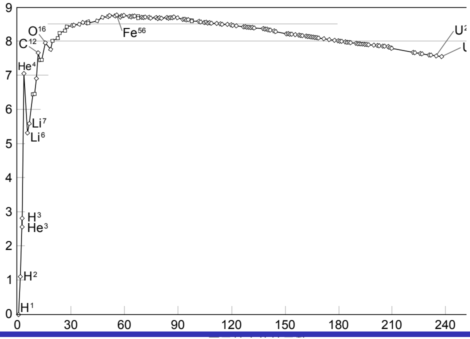
So below $k_B T \approx 1\,\text{MeV}$, the total number of baryons and the total number of photons remains approximately constant, and consequently the number densities of baryons and photons remain approximately constant: $n_{\rm bar}/n_\gamma \approx \rm const.$

Moreover, below $k_B T \approx 1\,\text{MeV}$, protons and neutrons can fuse to form heavier elements which are stable since their binding energy per nucleon is larger than the ambient energy (so the nuclei particles are not separated immediately e.g. by collisions with the highly aboundant photons).

The binding energy B_A for a nucleus of nucleon number A, having Z protons and $A - Z$ neutrons bound together in the nucleus, is given by

$$B_A = |Zm_\mathrm{p} + (A - Z)m_\mathrm{n} - m_A|$$

where m_A is the rest energy of the nucleus and m_n is the rest energy of the neutron, m_p is the rest energy of the proton (mind the c^2 hiding in the convention $c = 1$). The binding energy per nucleon for such a nucleus is then B_A/A. It is illustrated in the following table, taken from wikipedia, "nuclear binding energy". B_A/A is on the vertical axis, A is the horizontal axis.



The diagram shows that ^{4}He is a first local maximum of the binding energy per nucleon and that suggests that ^{4}He should be a highly abundant element created in the primordial nucleosynthesis since

the process $^4\mathrm{He} + \mathrm{p} \longrightarrow X$ is energetically unfavorable

the process $3\,^4\mathrm{He} \longrightarrow {}^{12}C$ is only efficient if the density of Helium is very high

Some estimates that we will consider soon will confirm this.

Using that $\eta_{\rm bar} = n_{\rm bar}/n_\gamma$ can be assumed to be constant from the energy scale relevant for nucleosynthesis onwards, one can gain an estimate on $n_{\rm bar}$ as follows:

At our present cosmic time τ_0, we determine $n_{\rm bar}$ by observation of the material around us (in our local group of galaxies). (There may be systematic errors about that, but it is maily the luminous matter, including gas clouds.)

We determine n_γ at our present cosmic time from the CMB:

$$n_\gamma|_{\tau_0} = \frac{2\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar}\right)^3 , \quad T = T_{CMB} = 2.73\,K$$

One finds form observations at τ_0

$$\eta_{\rm bar} = 5.5 \cdot 10^{-10} \cdot \frac{\Omega_{\rm bar} h^2}{0.02}\,, \quad \Omega_{\rm bar} \approx 0.04\,, \quad h = 0.7$$

At an early cosmological time τ with $k_B T(\tau) < 1\,$MeV, one then has

$$n_{\rm bar}\big|_\tau = \eta_{\rm bar}\; n_\gamma\big|_\tau = \eta_{\rm bar} \frac{2\zeta(3)}{\pi^2} \left(\frac{k_B T(\tau)}{\hbar}\right)^3$$