Cosmology

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The Λ-CDM standard cosmological model shows that there is an accelerated cosmic expansion ($\ddot{a} > 0$) in a “late” epoch of cosmic evolution, including our present time. With $k = 0$ as favoured value, we are therefore in the scenario depicted as II in the table on p 4 of Lecture 18.

When going back to small values of τ, and hence small values of $a(\tau)$, the contribution of radiation to the energy density dominates because of $\varrho_{\mathrm{rad}}(\tau) \sim (a(\tau_0)/a(\tau))^4$ whereas $\varrho_M(\tau) \sim (a(\tau_0)/a(\tau))^3$ for the contribution to the energy density formed by baryonic and cold dark matter.

In that situation, the matter/energy distribution in the Universe is very dense and hot; in fact the energy density is so high that various elementary particle reactions happen at high rates. These processes have characteristic energy thresholds for the types of interactions involved. There are also energy scales at which certain phase transitions set in. They relate to certain time-scales of cosmic expansion, since with increasing cosmic expansion the energy density and pressure of the matter/energy distribution decreases.

In order to study the high energy elementary particle processes, the ideal fluid picture used so far is no longer sufficient and a more detailed description is needed. The material present in the Universe at high energy density will be described by a **distribution function** in the sense of (relativistic) statistical mechanics, and it will be assumed that the distribution is of a Maxwell-Boltzmann form, characteristic of thermal equilibrium at temperature T.

A **distribution function** is a smooth function on the space of velocities at every point in space and time; so for the underlying FLRW spacetime,

$$f : J \times \Sigma^{(k)} \times \mathbb{R}^3 \to \mathbb{R}\,, \quad (\tau, q, \mathbf{v}) \mapsto f(\tau, q, \mathbf{v})$$

is a distribution function, where $\mathbf{v}$ is to be identified with an element in $T_q\Sigma^{(k)}$ (e.g. using one of the charts for $\Sigma^{(k)}$ used before). Therefore, $\mathbf{v}$ is viewed as spatial velocity at the space point q.

Consider, for the moment, Minkowski spacetime. We wish to describe a relativistic gas of Bose or Fermi particles in thermal equlibrium at absolute temperature T. In this case, there is an inertial system with coordinates (x^0, x^1, x^2, x^3) with respect to which the gas is at rest (on average); this is part of the condition of equilibrium (the condition of thermal equilibrium is not Lorentz covariant). We suppose that the particle type is fixed and it has a rest mass-energy $m = mc^2 > 0$; we also use units where $c = 1$.

In this case, the distribution function is

$$f_T(x^0, q, \mathbf{v}) = f_T(\mathbf{v}) = \frac{1}{\mathrm{e}^{E(\mathbf{v})/k_B T \mp 1}}\,, \quad E(\mathbf{v}) = \sqrt{m^2(|\mathbf{v}|^2 + 1)}$$

where $q = (x^1, x^2, x^3)$ is any point in space; "$-$" is for Bosons, "$+$" is for Fermions. k_B is Boltzmann's constant. Note that m is actually mc^2, and velocities are measured in units of c.

This distribution function is independent of (x^0, x^1, x^2, x^3) so it is time-independent and homogeneous in space, as one would expect for a thermal equilibrium situation. Moreover, it depends on $\mathbf{v}$ only in terms of $|\mathbf{v}|$, fitting with an isotropic situation, or state, of the collection of Bosons or Fermions.

The distribution function acquires its meaning by the quantities that can be derived from it.

$$n_T(E_0, \Delta E) = \frac{\mathrm{g}}{(2\pi)^3 \hbar^3} \int_{E_0 \leq E(\mathbf{v}) \leq E_0 + \Delta E} f_T(\mathbf{v})\, d^3 v$$

is the spatial **number density** of particles with total energy (relative to the inertial coordinate system in which the gas is at rest) between E_0 and $E_0 + \Delta E$.

Here, g is the **degeneracy factor** of the type of particle under consideration, this is usually a number characteristic of spin/vector/tensor degrees of freedom for the particle type.

Similarly,

$$n_T = \frac{g}{(2\pi)^3 \hbar^3} \int_{\mathbb{R}^3} f_T(\mathbf{v})\, d^3 v$$

is the (total) spatial **particle number density**,

$$\varrho_T = \frac{g}{(2\pi)^3 \hbar^3} \int_{\mathbb{R}^3} E(\mathbf{v}) f_T(\mathbf{v})\, d^3 v$$

is the spatial **energy density**,

$$p_T = \frac{g}{(2\pi)^3 \hbar^3} \int_{\mathbb{R}^3} \frac{|m\mathbf{v}|^2}{3E(\mathbf{v})} f_T(\mathbf{v})\, d^3 v$$

is the **pressure**

of the Bose or Fermi gas at equilibrium temperature T.

The following limiting cases are usually of interest:

relativistic or high-energy limit: $k_B T \gg m$ (note we use $c = 1$, so here m is actually mc^2, the particle's rest energy)

$$\varrho_T = \begin{cases} \frac{\pi^2}{30\hbar^3} \mathrm{g}(k_B T)^4 & \text{Bosons} \\ \frac{7}{8}\frac{\pi^2}{30\hbar^3} \mathrm{g}(k_B T)^4 & \text{Fermions} \end{cases}$$

$$n_T = \begin{cases} \frac{\zeta(3)}{\pi^2\hbar^3} \mathrm{g}(k_B T)^3 & \text{Bosons} \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2\hbar^3} \mathrm{g}(k_B T)^3 & \text{Fermions} \end{cases}$$

$$p_T = \frac{\varrho_T}{3}$$

(here, ζ denotes the ζ-function)

non-relativistic or low-energy limit: $m \gg k_B T$

$$\varrho_T = m \cdot n_T$$

$$n_T = \mathrm{g} \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} \mathrm{e}^{-m/k_B T}$$

$$p_T = n_T k_B T << \varrho_T$$

Now we carry these ideas over to FLRW spacetime. We define a distribution function $f_T(\tau, q, \mathbf{v})$ for a Bose/Fermi gas “in thermal equilibrium at absolute temperature T at cosmic time τ” by setting

$$f_T(\tau, q, \mathbf{v}) = f_T(\mathbf{v}) = \frac{1}{\mathrm{e}^{E(\mathbf{v})/k_B T \mp 1}}\,, \quad E(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + m^2}$$

under the condition that we use, at every spacetime point (τ, q), coordinates $(x^0 = \tau, x^1, x^2, x^2)$ such that, in these coordinates, $g_{\mu\nu}|_{(\tau,q)} = \eta_{\mu\nu}$ (Minkowski metric), and that $\mathbf{v} = (v^1, v^2, v^3)$ is identified with an element of $T_q\Sigma^{(k)}$ using these coordinates, i.e. with $v^j \partial_{x^j}|_q$. Owing to the spatial homogeneity and isotropy of FLRW spacetime, and of $f_T(\mathbf{v})$, this distribution function describes a homogeneous and isotropic Bose or Fermi gas.

There is an implicit τ-dependence of the distribution function entering via the particular choice of coordinates which depends on τ. But the spatial densities, after integrating on $\mathbf{v}$, depend only on T – where now one allows for a τ-dependence of the absolute temperature, $T = T(\tau)$. In particular, one obtains the same expressions in the relativistic or non-relativistic limits for ϱ_T, n_T and p_T, with $T = T(\tau)$.

The slightly problematic assumption here is that it is meaningful to use the "extremely adiabatic" approximation of an "instantaneous equilibrium state" for a Bose or Fermi gas at a fixed cosmic time even if $a(\tau)$ may be very rapidly varying around τ, i.e $\dot{a}(\tau)$ is very large.

However, accepting that this is a meaningful approximation (and that quantitative limits of its validity can be established when needed), one can extend that formalism a bit, which allows it to later draw some interesting conclusions. We consider a **mixture** of Bose/Fermi gases of different particle species (meaning various degeneracy factors and masses), where each gas component is in an (instantaneous) thermal equilibrium state at some temperature. The various gas components need not interact, and therefore their equilibrium temperatures can be different.

In more detail: We assume that we have a mixture of

N_B Bosonic particle species, labelled by $a = 1, \ldots, N_B$, with masses m_a, degeneracy factors g_a,

N_F Fermionic particle species, labelled by $f = 1, \ldots, N_F$, with masses m_f, degeneracy factors g_f,

each Bosonic particle species forms a gas in thermal equilibrium at temperature $T_b = T_b(\tau)$,

each Fermionic particle species froms a gas in thermal equilibrium at temperature $T_f = T_f(\tau)$.

In the relativistic limit, the mixture of the gases has an **effective temperature** T and an **effective degeneracy factor** g_*, defined by

$$g_* = \sum_{b=1}^{N_B} g_b \left(\frac{T_b}{T}\right)^4 + \frac{7}{8}\sum_{f=1}^{N_F} g_f \left(\frac{T_f}{T}\right)^4 ,$$

so that the total energy density of the mixture is

$$\varrho_T = \sum_{b=1}^{N_B} \varrho_{T_b} + \sum_{f=1}^{N_F} \varrho_{T_f} = \frac{\pi^2}{30\hbar^3} g_* (k_B T)^4$$

Moreover, the total pressure is

$$p_T = \sum_{b=1}^{N_B} p_{T_b} + \sum_{f=1}^{N_F} p_{T_f} = \varrho_T/3\,,$$

assuming the relativistic limit.

Then one can define the **entropy density** of the mixture,

$$s_T = \frac{\varrho_T + p_T}{k_B T} = \frac{4\varrho_T}{3k_B T}$$

which can be expressed as

$$s_T = \frac{2\pi^2}{45\hbar^3}\hat{g}_*(k_B T)^3$$

with the **entropy degeneracy factor**

$$\hat{g}_* = \sum_{b=1}^{N_B} g_b \left(\frac{T_b}{T}\right)^3 + \frac{7}{8}\sum_{f=1}^{N_F} g_f \left(\frac{T_f}{T}\right)^3 .$$

A hypothesis of important consequences is “the total entropy of matter in the Universe is constant”, motivated by viewing the Universe as a “closed system”. (That is not really correct because the matter degrees of freedom interact with the spacetime metric degrees of freedom which therefore “exchange” entropy, but in certain regimes, such an exchange can be considered as a small effect.) A more precise version of the hypothesis is the **principle of constant entropy** in the form

$$s_T a(\tau)^3 = \mathrm{const} \qquad (T = T(\tau))$$

which provides a constraint on the behaviour $T(\tau)$ of the effective temperature of mixtures.

In a mixture of Bose and Fermi gases, some particle species typically interact with others. In a static spacetime, they are therefore held in thermal equilibrium by the interaction, and their temperatures agree. In a FLRW spacetime, if the timescale of the interaction t_{int} between two particle species is much smaller than the expansion time scale t_{exp} of the spacetime metric, then the interaction between the particle species can hold them in thermal equilibrium, and their temperatures agree.

As measure for the expansion time scale, one usually chooses in the early, "radiation dominated" phase of the Universe,

$$t_{exp} = \frac{1}{H(\tau)} = 2\tau \quad \text{(for radiation with } a(\tau) = \tau^{1/2}\text{)}$$

If particle species labelled by A and B interact, then an estimate for the interaction time scale in the relativistic limit is

$$t_{int} = \frac{1}{n \cdot \sigma}$$

where σ is the **interaction cross section** for the interaction process between the particle types A and B, and n is the lesser of the particle number densities of the particle types.

If $a(\tau) = \tau^{1/2}$, then $n \sim \tau^{-3/2}$. Moreover, σ will decrease with decreasing energy density of the particle types partcipating in the interaction and therefore will decrease with increasing τ. Thus $t_{int} \gtrsim \tau^{3/2}$ so there will be some τ_{dec} with $t_{int} \gtrsim t_{exp}$ for $\tau \geq \tau_{dec}$. One says that τ_{dec} is the **decoupling time** at which the particle types A and B **decouple from joint thermal equilibrium**.

We will shortly look more closely at an example where this happens, involving electrons, neutrinos and photons.