Cosmology

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Solutions to the Friedmann equations for $\Lambda \neq 0$

A common feature of the solutions to the Friedmann equations with $\Lambda = 0$ is that they all have a *decelerating expansion*, i.e. $\ddot{a} < 0$.

However, a behaviour pointing at an *accelerating expansion*, $\ddot{a} > 0$, is observed at various epochs of the evolution in the Universe, as indicated e.g. by the observed relation between redshift and luminosity distance – as we will discuss in more detail soon. A way to model this effect is to consider various types of matter, and a non-vanishing cosmological constant $\Lambda \neq 0$.

In this section, we will keep the simple matter models of “dust” or “radiation” and look at solutions to the Friedmann equations in this case for $\Lambda \neq 0$. To recall,

$$3\left(\frac{\dot{a}^2 + k}{a^2}\right) + \Lambda = \kappa\varrho \qquad \text{(1st Friedmann eqn)}$$

$$\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} + \Lambda = -\kappa p \qquad \text{(2nd Friedmann eqn)}$$

Rewriting the 2nd Friedmann equation,

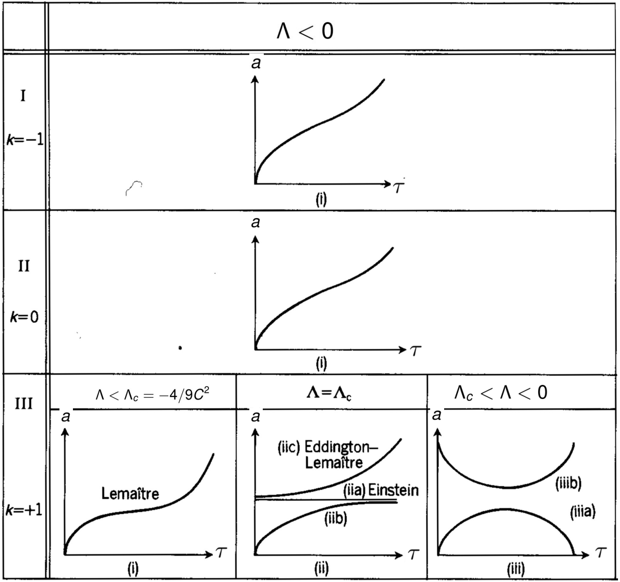
$$\frac{2}{a}\ddot{a} = -\frac{\dot{a}^2}{a^2} - k - \kappa p - \Lambda$$

one can see the effect that a non-vanishing Λ has, particularly if

$\Lambda < 0$: Then the right hand side may become positive – if $|\Lambda|$ is large enough – leading to the possibility that $\ddot{a} > 0$. Writing $p_\Lambda = p + \Lambda$, one can say that $\ddot{a} > 0$ is caused by a "negative effective pressure" $p_\Lambda < 0$, in comparison to a situation without a cosmological constant appearing explicitly in the Friedmann equations. A system with negative pressure gains internal energy when expanding.

$\Lambda > 0$: In this case, there is an additional negative contribution on the right hand side, enhancing the deceleration of the scale factor expansion.

In the next slides, the qualitative behaviour of solutions to Friedmann's equations for "dust" ($p = 0$) is illustrated. Further comments appear on the slides thereafter. The illustrations are taken from the book: R. d'Inverno, *Introducing Einstein's Relativity*, Clarendon Press, Oxford, 1992. Note that the opposite sign convention for Λ is used in d'Inverno's book. This has been taken into account and the figures have been re-labelled accordingly in this presentation.


Λ < 0
I
k=−1
a
τ
(i)
II
k=0
a
τ
(i)
III
k=+1
Λ < Λc = −4/9C²
a
Lemaître
τ
(i)
Λ=Λc
a
(iic) Eddington–
Lemaître
(iia) Einstein
(iib)
τ
(ii)
Λc < Λ < 0
a
(iiib)
(iiia)
τ
(iii)

Comments on the figures in the previous slide:

For $k = -1, 0$, if $\Lambda < 0$, there is initially an interval of τ-values where $\ddot{a}(\tau) < 0$, after which $\ddot{a}$ changes sign and becomes positive. That is also the case if $k = 1$ provided that $|\Lambda|$ is large enough; more precisely, if $\Lambda < \Lambda_c = -4/9C^2$ where $C = \kappa\varrho a^3/3 > 0$ is the constant parametrizing the Friedmann equations for "dust". In all these cases, labelled as (i) in the diagram, the asymptotic behaviour is

$$a(\tau) \sim \tau^{2/3} \ \text{ for } \ \tau \to 0\,, \qquad a(\tau) \sim \mathrm{e}^{\sqrt{|\Lambda/3|}\tau} \ \text{ for } \ \tau \to \infty$$

In the case $k = 1$, there are further possibilities. For $\Lambda = \Lambda_c$, three different types of solutions may occur:

(iia) is the constant solution $a(\tau) = a_c = 3C/2$. The possibility of obtaining a *static* cosmological spacetime with a *constant* scale factor was a motivation for Einstein to introduce the cosmological constant since at the time (prior to the observation of redshifted galaxies by Hubble), a scale factor that is time-dependent didn't appear to be an attactive cosmological scenario. A Universe with constant a doesn't have a finite past lifetime but is "eternal", without "big bang" scenario.

(iib) a solution which starts out a 0 and grows with rapidly decreasing $\dot{a}$ so that $a(\tau) \to a_c$ as $\tau \to \infty$.

(iic) is a strictly growing solution with $a(\tau) \underset{\tau \to 0}{\longrightarrow} a_c$ and $a(\tau) \underset{\tau \to \infty}{\sim} \mathrm{e}^{\sqrt{|\Lambda/3|}\tau}$

Finally, for $k = 1$, there is also the case $\Lambda_c < \Lambda < 0$. There are two possibilities:

(iiia) is very similar to the case $\Lambda = 0$.

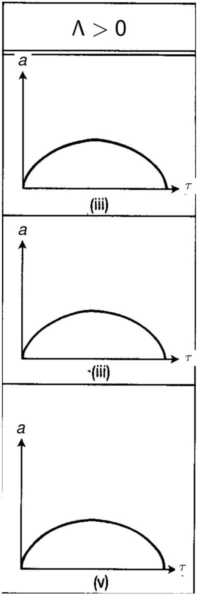
(iiib) is a behaviour of the scale factor which different from the cases considered before. $a(\tau)$ is defined for all $\tau \in \mathbb{R}$; it first contracts until it reaches a minimal value and than expands again. Such a scenario is called "big bounce". It may accomodate the effects of a "big bang" scenario – i.e. a hot, dense phase in the past – depending on parameters, but it avoids an initial singularity where $a \to 0$, and has an infinite lifetime to past and future.

In the following slide, the behaviour of the Friedmann equations for $\Lambda > 0$ is sketched. As mentioned before, $\Lambda > 0$ enhances the deceleration of the scalae factor expansion. In fact, it does so with a strong effect so that in all cases $k = 1, 0, -1$ one finds $a(\tau)$ first expanding until it reaches a maximum value, and then contracting, where the graphs are symmetric with respect to the maximum. I.e. one finds qualitatively the same behaviour as in the case $\Lambda = 0$, $k = 1$, with the same asymptotic behaviour at the boundaries of the intervals on which the $a(\tau)$ are defined and smooth.

k = -1

k = 0

k = 1



We now give a sketch about how a negative Λ influences the luminosity-distance relation, assuming $k = 0$ throughout for simplicity.

The *deceleration parameter* at some cosmic reference time τ_0 is defined by

$$q_0 = - \left. \frac{a\ddot{a}}{\dot{a}^2} \right|_{\tau=\tau_0}$$

Then one can write a Taylor expansion for $a(\tau_\star)$ around τ_* as

$$a(\tau_\star) = a(\tau_*) \left(1 + x + \frac{q_*}{2} x^2 + O(x^3) \right) , \quad x = (\tau_\star - \tau_*) H(\tau_*) , \quad q_* = q_0|_{\tau_0 = \tau_*}$$

adopting units so that $c = 1$.

In our considerations of the redshift, the equations

$$1 + z = \frac{a(\tau_\star)}{a(\tau_*)} \quad \text{and} \quad d_L = |r_\star - r_*| a(\tau_\star) \quad (k = 0)$$

were exact, but $|r_\star - r_*| \approx |\tau_\star - \tau_*| a(\tau_\star)$ was only an approximation (in linear order in $|\tau_\star - \tau_*|$).

An approximation to higher order gives

$$d_L = \frac{1}{a(\tau_*) H(\tau_*)} x + \frac{1}{2H(\tau_*)} x^2 + O(x^3)$$

Expanding z similarly to 2nd order in x gives

$$z = x - \frac{q_*}{2} x^2 + O(x^3)$$

One can revert this relation and express x in terms of z up to terms of quadratic order in z. Inserting this into the approximate expression of $d_L = d_L(x)$ gives an expression of d_L in z up to 2nd order.

However, in doing this, it is customary to interchange the roles of $\tau_\star$ and τ_*, i.e. to expand around $\tau_\star = \tau_0$, our "present time" at which we observe the redshift z and $H(\tau_\star) = H_0$ on "nearby" galaxies. The result is

$$d_L = d_L(z) = \frac{z}{H_0}\left(1 + \frac{1}{2}(1 - q_0)z\right) + O(z^3)$$

If $\ddot{a}(\tau_0) > 0$, this means $q_0 < 0$, and hence the quadratic term in z has a positive coefficient. In contrast, if $\ddot{a}(\tau_0) < 0$, the coefficient of the z^2-term is smaller; potentially negative. The distinct behaviour is visible in an d_L vs z diagram. A quadratic increase of the graph after an initial almost linear part indicates an epoch of $\ddot{a} > 0$ and therefore, a negative cosmological constant (at least in this approximation). We will soon look at a more sophisticated version of that argument.