# Cosmology

## Summer Term 2020, Lecture 17

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## Luminosity distance and redshift

Recall the previously given definitions:

- $L$ = radiation energy emitted by the source (galaxy $*$ at $r_*$ in the previous lecture) per unit of time: **absolute luminosity**
- $\ell$ = radiation energy received from the source by the observer (galaxy $\star$ at $r_\star$) per unit of time: **apparent luminosity**

Then the **luminosity distance** is *defined* as

$$d_L = \sqrt{\frac{L}{4\pi\ell}}$$

Assuming that the light emission is isotropic, one obtains with the coordinates as in the previous lecture:

$$\ell = \frac{L}{4\pi(r_\star - r_*)^2 a(\tau_\star(1))^2(1+z)^2} \quad \Rightarrow \quad d_L = |r_\star - r_*| a(\tau_\star(1))(1+z)$$

For $k = 0$, the luminosity distance is the spatial geodesic distance of the galaxies at the cosmic time when the emitted radiation is received, multiplied by the redshift factor $1 + z$.

If $|r_\star - r_*|$ is sufficiently small, one obtains (see previous lecture)

using the notation $\tau_* = \tau_*(1)\,, \quad \tau_\star = \tau_\star(1)$ :

$$|r_* - r_\star| \approx \int_{r_*}^{r_\star} \frac{dr}{\sqrt{1-kr^2}} = \int_{\tau_*}^{\tau_\star} \frac{d\tau}{a(\tau)} \approx \frac{|\tau_\star - \tau_*|}{a(\tau_\star)} \qquad \text{and}$$

$$1 + z = \frac{a(\tau_\star)}{a(\tau_*)} \approx \frac{a(\tau_\star)}{a(\tau_\star) - \dot{a}(\tau_\star)|\tau_\star - \tau_*|} \approx 1 + \frac{\dot{a}(\tau_\star)}{a(\tau_\star)}|\tau_\star - \tau_*|$$

Combining these two equations yields the **luminosity-distance relation** (for sufficiently small $|r_\star - r_*|$, i.e. sufficiently small $d_L$)

$$z \approx \frac{\dot{a}(\tau_\star)}{a(\tau_\star)} d_L = H(\tau_\star) d_L$$

with the **Hubble parameter**

$$H(\tau) = \frac{\dot{a}(\tau)}{a(\tau)}$$

Assuming $\dot{a} > 0$, the FLRW cosmological spacetime models reproduce the observed Hubble relation $z = H_0 d_L$ in a natural way. (Note that units are chosen so that $c = 1$.) However the factor of proportionality between redshift $z$ and luminosity distance $d_L$ is not a constant, but at function $H(\tau)$ changing in $\tau$.

**Solutions to the Friedmann equations, $\Lambda = 0$**

Now we look at solutions to Friedmann's equations and their behaviour. First, we assume a vanishing cosmological constant, $\Lambda = 0$. This value of the cosmological constant is favoured by local observations. In particular, other values of $\Lambda$ are in conflict with the Newtonian limit of Einstein's equations. Hence, should $\Lambda$ actually be different from 0 (it is very difficult to test gravity at sub-microscopic scales, because the other interactions are dominating in that regime), it would have to be very small, at least if Einstein's equations apply in the same form at all scales – which is their common interpretation, leaving hypothetical quantum gravity aside.

As mentioned, Einstein's equations must be furnished with a matter model. Partially, this is done by assuming that the stress energy tensor assumes the ideal fluid form. There is then also an equation of state, $p = f(\varrho, T)$, relating pressure and energy density in thermal equilibrium at absolute temperature $T$. Therefore, the idea here is that (near-to) equilibrium states of an ideal fluid are considered – very much in line with the assumption that the matter distribution is homogeneous and isotropic.

**"Dust", $p = 0$**

The simplest equation of state is $p = 0$, pictured as "dust", i.e. a swarm of particles which do not interact with each other (or only very weakly) of are far enough distant from each other so that interaction is negligible. E.g., a thin gas, or galaxies or even galaxy clusters with large distances between them.

The Friedmann equations then simplify to

$$\left(\frac{\dot{a}^2 + k}{a^2}\right) = \kappa\varrho \quad \text{and} \quad \frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} = 0$$

As pointed out previously, the fact that the stress-energy tensor is divergence-free implies that (since $p = 0$)

$$\frac{d}{d\tau}(\varrho a^3) = 0$$

(cf. Problem 7.2). Inserting this into the 1st Friedmann equation leads to a one-parametric family of differential equations for $a(\tau)$,

$$a(\tau) > 0\,, \qquad \dot{a}(\tau)^2 - \frac{C_0}{a(\tau)} + k = 0$$

where $C_0 = \kappa\varrho(\tau)a(\tau)^3/3 > 0$ is a parameter which is constant in $\tau$.

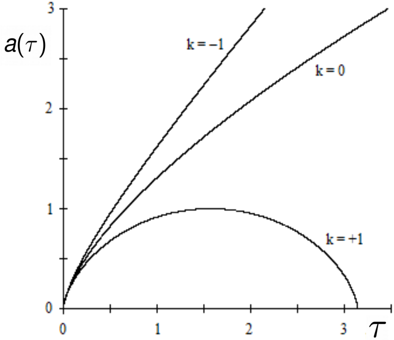
For any value $C_0 > 0$, the differential equation possesses general analytic solutions:

$$k = +1: \qquad a(u) = \frac{1}{2}C_0(1 - \cos(u))\,, \qquad \tau(u) = \frac{1}{2}C_0(u - \sin(u))$$

$$k = 0: \qquad a(\tau) = \left(\frac{9C_0}{4}\right)^{1/3} \tau^{2/3}$$

$$k = -1: \qquad a(u) = \frac{1}{2}C_0(\cosh(u) - 1)\,, \qquad \tau(u) = \frac{1}{2}(\sinh(u) - u)$$

The qualitative behaviour of the solutions is sketched in the figure, assuming that initial conditions are chosen so that $\lim_{\tau \to 0+} a(\tau) = 0$



$\underline{k=1}$ $a(\tau)$ has the form of a cycloide which is symmetric with respect to a $\tau_{\max}$ where it assumes its absolute maximum value $a_{\max}$. $a(\tau)$ is strictly growing until it reaches its maximum value past which the scale factor is strictly decreasing. In other words, in this scenario, the scale of the Universe initially expands up until a maximal value is reached, and then it contracts again. The maximal lifetime of a Universe with such a scale factor is $2\tau_{\max}$. Behaviour for $\tau \to 0$ is $a(\tau) \sim \tau^{2/3}$.

$\underline{k=0}$ $\dot{a}(\tau) > 0$ for all $\tau > 0$, i.e. the scale factor of the Universe is always expanding. However, the expansion rate is decreasing; it holds that $\ddot{a}(\tau) < 0$. The asymptotic behaviour is

$$a(\tau) \sim \tau^{2/3} \text{ for } \tau \to 0\,; \qquad a(\tau) \to \infty \;\text{ and }\; \dot{a}(\tau) \to 0 \;\text{ for }\; \tau \to \infty$$

$\underline{k=-1}$ Again $\dot{a}(\tau) > 0$ and $\ddot{a}(\tau) < 0$. The behaviour of $a(\tau)$ is qualitatively very similar to the case $k = 0$, with a different asymptotic behaviour of $\dot{a}(\tau)$ as $\tau \to \infty$:

$$a(\tau) \sim \tau^{2/3} \text{ for } \tau \to 0\,; \qquad a(\tau) \to \infty \;\text{ and }\; \dot{a}(\tau) \to 1 \;\text{ for }\; \tau \to \infty$$

**“Radiation”, $p = \varrho/3$**

Radiation can thought of as becoming a more and more dominant form of energy when the scale factor of the Universe is so small that the density is higher than the density of the interior of a star like the sun. Quite generally, the higher the energy density, the more energy is carried by the “lighter” degrees of freedom, or put differently, the contribution of the rest mass energies becomes small compared to the kinetic energy. In Problem 7.4$*$ you have been asked to give an argument for this equation of state for electromagnetic radiation.

With the equation of state $p = \varrho/3$, the Friedmann equations thake the form

$$\left(\frac{\dot{a}^2 + k}{a^2}\right) = \kappa\varrho \qquad \text{and} \qquad \frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} = -\kappa\varrho/3$$

As the stress-energy tensor is divergence-free, it holds that $d(\varrho a^3)/d\tau + pd(a^3)/d\tau = 0$ (cf. Problem 7.2), and with the equation of state $p \sim \varrho/3$ for radiation, the Friedmann equations yield that

$$\Gamma_0 = \frac{\kappa\varrho a^4}{3}$$

is a $\tau$-independent constant.

In consequence, one obtains a one-parametric family of differential equations for $a(\tau)$,

$$a(\tau) > 0\,, \qquad \dot{a}(\tau)^2 - \frac{\Gamma_0}{a(\tau)^2} + k = 0$$

where $_0 = \kappa\varrho(\tau)a(\tau)^4/3 > 0$ is a parameter which is constant in $\tau$.

For every $\Gamma_0 > 0$, there are the following general solutions, assuming $\tau > 0$:

$$k = +1: \qquad a(\tau) = \sqrt{\Gamma_0}\left(1 - \left(1 - \frac{\tau}{\sqrt{\Gamma_0}}\right)^2\right)^{1/2}$$

$$k = 0: \qquad a(\tau) = (4\Gamma_0)^{1/4}\sqrt{\tau}$$

$$k = -1: \qquad a(\tau) = \sqrt{\Gamma_0}\left(\left(1 + \frac{\tau}{\sqrt{\Gamma_0}}\right)^2 - 1\right)^{1/2}$$

The qualitative behaviour for the cases $k = 1, 0, -1$ is very similar to that for the case of “dust”. In particular, for $k = 1$, $a(\tau)$ is again symmetric w.r.t. a $\tau_{\max}$ where it assumes its maximal value $a_{\max}$, and $a(\tau) \to \infty$ as $\tau \to \infty$ for $k = 0, -1$. The other asymptotics are:

$$a(\tau) \sim \tau^{1/2} \text{ as } \tau \to 0 \; (k = 1, 0, -1)\,; \quad \dot{a}(\tau) \to 0 \; (k = 0)\,, \;\; \dot{a}(\tau) \to 1 (k = -1) \text{ for } \tau \to \infty$$

Note that $\varrho \sim a^{-4}$, implied by the equation of state $p = \varrho/3$, is consistent with the redshift effect upon cosmic expansion: At cosmic time $\tau$, for electromagnetic radiation one can write

$$\varrho(\tau) = n(\tau)\varepsilon(\tau) = \frac{n(\tau_0)a(\tau_0)^3}{a(\tau)^3} 2\pi\hbar\nu(\tau) = \frac{n(\tau_0)a(\tau_0)^3}{a(\tau)^3} \cdot \frac{c}{\lambda(\tau)}$$

where $\tau_0$ is any fixed cosmic reference time and

$$\begin{aligned} n(\tau) &= \text{ average numer of photons per unit volume at time } \tau \\ \varepsilon(\tau) &= \text{ average photon energy at time } \tau \\ \nu(\tau) &= \text{ average photon frequency at time } \tau \\ \lambda(\tau) &= \text{ average photon wavelength at time } \tau \end{aligned}$$

According to the redshift effect, $\lambda(\tau_0)/\lambda(\tau) = a(\tau_0)/a(\tau)$, and hence

$$\varrho(\tau) = \frac{n(\tau_0)a(\tau_0)^3}{a(\tau)^3} \cdot \frac{c}{\lambda(\tau_0)} \cdot \frac{\lambda(\tau_0)}{\lambda(\tau)} = \mathcal{C}\frac{c}{a(\tau)^4}$$

with the $\tau$-independent constant $\mathcal{C} = 2\pi\hbar n(\tau_0)a(\tau_0)^4/\lambda(\tau_0)$.

For both matter models, “dust” or “radiation”, the solutions to the Friemann equations have the following common features:

- initial expansion of the Universe, and depending on $k$, expansion followed by contraction ($k = 1$) or eternal expansion to the future ($k = 0, -1$). Im particular, the scale factor is not time-independent.
- there is a finite past lifetime of the Universe with an initial point of cosmic time (here always chosen as $\tau = 0$); the FLRW spacetime metric breaks down at $\tau = 0$ because $a(\tau) = 0$.
- since $\varrho a^3$ = constant for “dust”, or $\varrho a^4$ = constant for “radiation”, the energy density diverges as $\tau \to 0$: This is confirms the view of the “big bang” scenario: If the Universe has expanded to the future (up to our present time), it contracts if we go back to the past, and in the far past, it must have been very dense and hot. As mentioned before, the CMB is one of the best arguments for that point of view.

It should be noted that this holds under the assumption $\Lambda = 0$. If $\Lambda \neq 0$, there could be different scenarios, depending on the value of $\Lambda$. As indicated before, observations at large distances point at a value of $\Lambda$ that is different from 0.