Cosmology

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The notation for the spacetime metrics $g^{(k)}_{\mu\nu}$ of the FLRW cosmological spacetimes ($k = -1, 0, 1$) is usually given in terms of the *metric line elements*. For the case $k = 0$, it is typically written

$$ds^2_{(k=0)} = d\tau^2 - a(\tau)^2(dx^2 + dy^2 + dz^2)$$

where (x, y, z) are Cartesian coordinates of $\mathbb{R}^3$.

In the other cases, i.e. $k = \pm 1$,

$$ds^2_{(1)} = d\tau^2 - a(\tau)^2 \left(d\psi^2 + \sin^2(\psi)(d\theta^2 + \sin^2(\theta)d\phi^2)\right)$$
$$ds^2_{(-1)} = d\tau^2 - a(\tau)^2 \left(d\psi^2 + \sinh^2(\psi)(d\theta^2 + \sin^2(\theta)d\phi^2)\right)$$

in the 3-dimensional spherical polar coordinates, resp. spherical hyperbolic coordinates.

The terminology “metric line element” has some historical roots – see textbooks on GR for explanation. It is a convenient and customary way of denoting the coordinate expression of a metric with respect to particular coordinates. If the coordinates are chosen so that the coordinate expression of the metric diagonalizes, like here, it is a very useful notation.

To indicate how it is used: Eg. in the case $k = 0$, one reads $d\tau^2$ as a symmetric $\binom{0}{2}$ tensor field fulfilling

$$d\tau^2(\partial_\tau \otimes \partial_\tau) = 1\,, \quad d\tau^2(\partial_\tau \otimes \partial_x) = d\tau^2(\partial_x \otimes \partial_\tau) = 0\,, \quad d\tau^2(\partial_x \otimes \partial_x) = 0$$

and similarly upon replacing x by y or z. Analogously, dx^2 is a symmetric $\binom{0}{2}$ tensor field fulfilling

$$\begin{aligned} dx^2(\partial_x \otimes \partial_x) = 1\,, &\quad dx^2(\partial_\tau \otimes \partial_x) = 0\,, \quad dx^2(\partial_\tau \otimes \partial_\tau) = 0 \\ dx^2(\partial_x \otimes \partial_y) = 0\,, &\quad dx^2(\partial_y \otimes \partial_y) = 0\,, \quad dx^2(\partial_y \otimes \partial_z) = 0 \end{aligned}$$

and similarly upon replacing y by z.

Analogously one has, e.g., at the coordinate point $q = (\tau, \psi, \theta, \phi)$

$$\left[d\tau^2 - a(\tau)^2 \left(d\psi^2 + \sin^2(\psi)(d\theta^2 + \sin^2(\theta)d\phi^2)\right)\right] (\partial_\phi|_q \otimes \partial_\phi|_q) = -a(\tau)^2 \sin^2(\psi) \sin^2(\theta)$$

Another choice of coordinates for the FLRW spacetimes is also often in use: It is given for all cases $k = -1, 0, 1$ by (τ, r, θ, ϕ).

τ (cosmological time) and (θ, ϕ) (spherical polar coordinates of the S^2) are as before.

r is a radial coordinate: for $k = -1, 0$, $r \in (0, \infty)$, while for $k = 1$, $r \in (0, 1)$, with

$$ds^2_{(k)} = d\tau^2 - a(\tau)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right) \qquad (k = -1, 0, 1)$$

It is a different way of choosing coordinates on $\Sigma^{(k)}$.

The FLRW cosmological spacetimes $(M^{(k)} = J \times \Sigma^{(k)}, g^{(k)}_{\mu\nu})$ which are possible as solutions to Einstein's equations with matter described as a homogeneous and isotropic ideal fluid are therefore very much restricted. What remains undetermined so far are

- The value of $k = -1, 0, 1$
- The smooth function $a : J \to (0, \infty)$, called the **scale factor** of an FLRW spacetime. (Note that the interval $J \subset \mathbb{R}$ isn't determined either.)

The Friedmann Equations

For any choice of $k = -1, 0, 1$, the scale factor $a(\tau)$ is determined by the requirement that the spacetime $(M^{(k)} = J \times \Sigma^{(k)}, g^{(k)}_{\mu\nu})$ with the metric line element

$$ds^2_{(k)} = d\tau^2 - a(\tau)^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \right) \qquad (k = -1, 0, 1)$$

be a solution to Einstein's equations

$$G^{(\Lambda)}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

where the stress-energy tensor on the right hand side is of the form of a homogeneous and isotropic ideal fluid,

$$T_{\mu\nu} = (\varrho + p) u_\mu u_\nu - p g_{\mu\nu} \quad \text{with} \quad u^\mu = (\partial_\tau)^\mu$$

Recall that $G_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor.

Using the form of the metric above together with the assumed form of the stress-energy tensor, Einstein's equations assume the much simpler form **Friedmann's equations**, a system of differential equations coupling $a(\tau)$, $\varrho(\tau)$ and $p(\tau)$ – owing to spatial homogeneity, ϱ and p can only depend on τ, not on (r, θ, ϕ):

$$3\left(\frac{\dot{a}^2 + k}{a^2}\right) + \Lambda = \kappa\varrho \qquad \text{(1st Friedmann eqn)}$$

$$\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} + \Lambda = -\kappa p \qquad \text{(2nd Friedmann eqn)}$$

where a dot denotes differentiation with respect to τ.

Showing that Einstein's equations assume the form of Friedmann's equations under the assumptions stated is a classical exercise problem for any student in cosmology.

Remarks

(i) Besides the constant $k = -1, 0, 1$, the cosmological constant Λ enters as another – at this point, undetermined – constant into the Friedmann equations.

Caution: In cosmology, many authors use **the opposite sign convention** for Λ, i.e. they write $-\Lambda$ instead of Λ as used here. So there is another sign convention that needs to be checked with the literature.

(ii) Assuming k and Λ given, the Friedmann equations in this form are underdetermined because there are no relations between ϱ and p. They have to be supplied, as a further specification of the matter model at hand, by an **equation of state** in the form $p = f(\varrho, T)$ with a suitable function of ϱ and T (absolute temperature) characterizing a (near to) thermal equilibrium situation. The most common choices in cosmology are:

- $p = 0$. This form of matter is called **dust** and it is appropriate for the picture that galaxy clusters are viewed as the "grains of a dust cloud".
- $p = \varrho/3$. This form of matter is called **radiation**. It is appropropriate at the very early stages of the cosmic evolution were electromagnetic radiation can be considered as the main energy-carrying component of the cosmic inventory.

(iii) Furthermore, to find an explicit solution, it is necessary to specify *initial conditions* at some cosmic time τ_0.

Once equations of state of the form "dust" or "radiation" are assumed, one can further simplify Friedmann's equations, by eliminating ϱ, p from the equations, which makes it possible to determine concrete solutions (even analytically) from initial conditions. We will return to this shortly.

Scale factor and Redshift

Here, we discuss the meaning of the **scale factor** $a(\tau)$ appearing in the FLRW spacetime metric: It scales spatial lengths, depending on cosmic time τ.

To see this formally, let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ be any two points in $\Sigma^{(k)}$.

Then consider for any cosmic time τ a smooth curve $s \mapsto \gamma_\tau(s) = (\tau, \boldsymbol{\gamma}(s)) \in J \times \Sigma^{(k)}$, $s \in [s_0, s_1]$, where $s \mapsto \boldsymbol{\gamma}(s)$ is a smooth curve in $\Sigma^{(k)}$ connecting $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$. Thus, the γ_τ are copies of a "curve in space" at various values of cosmic time τ.

The metric length of these curves, for different τ, is given by

$$\ell(\gamma_\tau) = \int_{s_0}^{s_1} \left| g^{(k)}\left(\frac{d}{ds}\gamma_\tau(s), \frac{d}{ds}\gamma_\tau(s)\right) \right|^{1/2} ds = a(\tau) \int_{s_0}^{s_1} \left(h^{(k)}\left(\frac{d}{ds}\boldsymbol{\gamma}(s), \frac{d}{ds}\boldsymbol{\gamma}(s)\right) \right)^{1/2} ds$$

where the form of the FLRW metric $g^{(k)}_{\mu\nu} = d\tau_0 d\tau_0 - a(\tau)^2 h^{(k)}_{ij}$ as in Lec 15, slide 11, was used.

Therefore,

$$\frac{\ell(\gamma_\tau)}{\ell(\gamma_{\tau_0})} = \frac{a(\tau)}{a(\tau_0)}$$

for any values τ and τ_0 of cosmic time, showing that the scale parameter scales lengths in space over time.

Correspondingly, $a(\tau)^3$ is the scaling factor for a fixed coordinate space volume over cosmic time τ.

Using either the Friedmann equations, or more generally, the requirement that the stress-energy tensor be divergence-free, $\nabla^\mu T_{\mu\nu} = 0$, one obtains in the FLRW spacetimes

$$\frac{d}{d\tau}(\varrho a^3) + p\frac{d}{d\tau}(a^3) = 0$$

With $a(\tau)^3 \sim V_\tau$ = spatial unit volume at cosmic time τ,

$V_\tau \varrho = E_\tau =$ energy in unit volume at cosmic time τ obtains, implying

$$\frac{d}{d\tau}E_\tau + p\frac{d}{d\tau}V_\tau = 0\,, \qquad \text{i.e. } dE + pdV = 0$$

In other words, we see the validity of a conservation law analogous to the 1st law of thermodynamics.

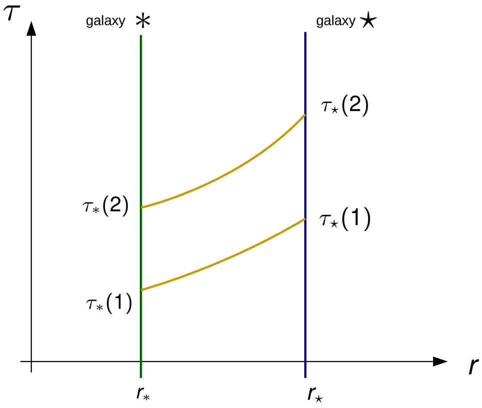
We now aim at relating the scale factor $a(\tau)$ and the redshift. To this end, consider the following situation: We have two galaxies, $*$ and $\star$. We assume that the galaxies have the following worldlines with respect to the (τ, r, θ, ϕ) coordinates

- $*$ $\quad \tau \mapsto (\tau, r_*, \theta_0, \phi_0)$
- $\star$ $\quad \tau \mapsto (\tau, r_\star, \theta_0, \phi_0)$

That means, the galaxies have the same angular coordinates, but different radial coordinates, r_* and $r_\star$. We assume for convenience (but without restriction of generality) that $r_* < r_\star$.

Then we consider the situation that in short succession, two light signals (lightrays) are sent from galaxy $*$ to galaxy $\star$ where they are registered. At galaxy $*$, the first light signal is emitted at $\tau_*(1)$ and the second at $\tau_*(2)$. The light signals are registered in galaxy $\star$ at $\tau_\star(1)$ and $\tau_\star(2)$. We are interested in the relation between $\Delta\tau_* = \tau_*(2) - \tau_*(1)$ and $\Delta\tau_\star = \tau_\star(2) - \tau_\star(1)$. These are the proper time differences of the light signals emitted/received in the galaxies.

The following picture illustrates the situation described in the previous slide. The worldlines of the galaxies are parallel to the τ-axis (cosmic time) and remain at their fixed r-coordinates. The lightlike geodesics of the light signals are sketched as the golden lines. They appear here curved, assuming that $a(\tau)$ is growing in τ.



Let $s \mapsto \gamma(s) = (\tau(s), r(s), \theta_0, \phi_0)$, $s \in [s_*, s_\star]$ be a lightlike geodesic describing the worldline of a light signal travelling from galaxy $*$ to galaxy $\star$, at fixed angular coordinates. Then by the form of the FLRW metric, the condition that the geodesic is lightlike means

$$\tau'(s)^2 - a(\tau(s))^2 \frac{r'(s)^2}{1 - kr(s)^2} = 0 \quad (s \in [s_*, s_\star])$$

where the prime means differentiation with respect to the affine parameter s. We have, by assumption, $\tau'(s) > 0$ and $r'(s) > 0$, and hence

$$\frac{\tau'(s)}{a(\tau(s))} = \frac{r'(s)}{\sqrt{1 - kr(s)^2}} \,, \qquad \text{implying}$$

$$\int_{\tau(s_*)}^{\tau(s_\star)} \frac{d\tau}{a(\tau)} = \int_{s_*}^{s_\star} \frac{\tau'(s)}{a(\tau(s))}\, ds = \int_{s_*}^{s_\star} \frac{r'(s)}{\sqrt{1 - kr(s)^2}}\, ds = \int_{r_*}^{r_\star} \frac{dr}{\sqrt{1 - kr^2}}$$

Therefore,

$$\int_{\tau_*(1)}^{\tau_\star(1)} \frac{d\tau}{a(\tau)} = \int_{r_*}^{r_\star} \frac{dr}{\sqrt{1 - kr^2}} = \int_{\tau_*(2)}^{\tau_\star(2)} \frac{d\tau}{a(\tau)}$$

This now implies, noting that $\tau_*(2) = \tau_*(1) + \Delta\tau_*$ and $\tau_\star(2) = \tau_\star(1) + \Delta\tau_\star$,

$$\int_{\tau_*(1)+\Delta\tau_*}^{\tau_\star(1)+\Delta\tau_\star} \frac{d\tau}{a(\tau)} - \int_{\tau_*(1)}^{\tau_\star(1)} \frac{d\tau}{a(\tau)} = 0$$

This yields

$$\frac{\Delta\tau_\star}{\Delta\tau_*} = \frac{a(\tau_\star(1))}{a(\tau_*(1))} + O(\Delta\tau_*)$$

Therefore, if one sets

$$\Delta\tau_\star \sim \lambda' = \text{ wavelength of light signals registered in galaxy } \star$$
$$\Delta\tau_* \sim \lambda = \text{ wavelength of the light signals registerd in galaxy } *$$

one obtains in the "short wave length limit" (i.e. the wavelength of light signals between the galaxies is very much smaller than the spatial distance of the galaxies),

$$\frac{\lambda'}{\lambda} = \lim_{\Delta\tau_* \to 0} \frac{\Delta\tau_\star}{\Delta\tau_*} = \frac{a(\tau_\star(1))}{a(\tau_*(1))}$$

Assuming $\dot{a} > 0$, our previous assumption $\tau' > 0$ (as a function of the affine parameter) now implies the occurrence of a **redshift**,

$$z = \frac{\lambda' - \lambda}{\lambda} = \frac{a(\tau_\star(1))}{a(\tau_*(1))} - 1 > 0$$

between the wavelength λ of a light signal at its emission from galaxy $*$, and the wavelength λ' of the light signal when it is registerd in the galaxy $\star$.

Therefore, the FLRW spacetime models describe the redshift effect of light received from remote galaxies if $\dot{a} > 0$. It is also obvious that the redshift factor z increases when the galaxies are further apart because increasing $r_\star$ implies increasing $\tau_\star(1)$.

It should be noted that the redshift factor is in general not constant in τ since $\frac{a(\tau_\star(1))}{a(\tau_*(1))}$ depends on the cosmic time $\tau_*(1)$ at which the light signal is being emitted.