Cosmology

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The stress-energy-tensor

$$T_{\mu\nu} = (\varrho + p)u_\mu u_\nu - p g_{\mu\nu}$$

describes the energy content of an ideal fluid in a spacetime $(M, g_{\mu\nu})$ in a near-equilibrium state.

It is characterized by the *energy density* ϱ and the *isotropic pressure* p, both smooth functions on spacetime, with *non-negative* real values. Note also that there are no energy or momentum fluxes or vorticities since an ideal fluid has zero viscosity and u^μ is the tangent vector field of the worldlines of the infinitesimal volume elements of the fluid.

Generally, a stress-energy tensor should be *symmetric*, $T_{\mu\nu} = T_{\nu\mu}$, and *divergence-free*, $\nabla^\mu T_{\mu\nu} = 0$. That latter condition is the counterpart of the continuity equation which implies that no mass/energy/momentum is locally spontaneously created or annihilated. It is a significant constraint on the behaviour of ϱ and p.

Furthermore, the energy density of the (macroscopic) matter distribution should always and everywhere be non-negative, i.e. $T_{\mu\nu} v^\mu v^\nu \geq 0$ for any timelike vector at any point in spacetime. This is referred to as the *weak energy condition*. It constrains the possibilities for equations of state. There are other, more restrictive forms of energy conditions – please see the textbooks on GR for further discussion.

We summarize some general features of Einstein's field equations and their solutions – some not directly (but indirectly) pertaining to cosmology.

* Einstein's field equation's are 2nd order, hyperbolic, non-linear partial differential equations in the $g_{\mu\nu}$. They are difficult to solve, even for simple matter models such as a perfect fluid. Analytic solutions are, in general, only known in cases of high symmetry, such as spherical symmetry or axisymmetry of the mass distribution. Prominent examples are the Schwarzschild solution, which is the solution to Einstein's equations outside of a spherically symmetric, static mass distribution and which approaches the Newtonian gravity field with growing distance from the mass distribution. There are also the Kerr-Newmann generalizations describing the solutions to Einstein's equations outside of rotating and charged mass distributions.
* For local gravitational effects such as described by the Schwarzschild or Kerr-Newmann solutions to Einstein's equations, the value $\Lambda =$ is preferred. However, on larger distance scales, observations indicate that a vanishing cosmological constant is actually not the preferred value, as we will see.
* Owing to the general properties of the stress-energy tensor, in particular the weak energy condition or its variants, gravity is always attractive in solutions to Einstein's equations. Therefore, inhomogeneities in a mass distribution will be enhanced in time.

* For $\Lambda = 0$, an approximately spherically symmetric, isolated mass distribution will undergo gravitational collapse once the energy density is sufficiently high, forming a black hole surrounded by an “event horizon”. There is no way to send any signals from the region within the event horizon to the outside (roughly, the exterior region of the black hole). The “singularity theorems” in GR, pioneered by Stephen Hawking, Roger Penrose and others, state that quite generally, solutions to Einstein’s equations will undergo gravitational collapse and create “singular behaviour” of the spacetime geometry, like diverging scalar curvature, when an isolated mass distribution has become sufficiently concentrated, even without any symmetry assumptions. It is an open question if such “singularities” are always surrounded by an event horizon (“cosmic censorship conjecture”).
* Again for $\Lambda = 0$, the (near-to-spherically-symmetric) black-hole solutions of Einstein’s equations show a relationship between the Schwarzschild radius r_S of a black hole (corresponding to the horizon radius) and the black hole mass m: $r_S = 2\mathrm{G}m/c^2 \, (= \kappa c^2 m/4\pi)$. Therefore, for isolated mass distributions, there is an a priori upper limit for the mass density in GR given by that relationship. Furthermore, black holes cannot split, and if they merge, the event horizon area of the merged object is at least as large as the sum of the horizon areas of the initial black holes (“Bardeen-Carter-Hawking Theorem”).

* The FLRW cosmological spacetimes that we will look at soon also have a high degree of symmetry *with respect to space*, and that is instrumental for being able to solve Einstein's equations with perfect fluid matter explicitly. In those FLRW solutions, apart from some exceptional cases, there results for most values of Λ a *finite past of the Universe*. That means that the spacetime becomes singular when following it to the past for a sufficiently large but finite duration of time. It is worth noting that the singularity theorems of Hawking and Penrose arrive at a similar conclusion for solutions to Einstein's equations without symmetry assumptions: Under very general conditions, there must be timelike geodesics whose past lifetime, measured by any geodesic parameter, is finite. Again, please see the textbooks on GR for more details on this theme which cannot be discussed here any further.

The FLRW homogeneous-isotropic cosmological spacetimes

The FLRW cosmological spacetime models provide a very simplified picture of the spacetime geometry of the Universe at *very large scales*, i.e. larger than the inhomogeneity scale, so that, at every moment of (global) time, the matter distribution in space can be idealized as homogenoues and isotropic. Furthermore, the matter distribution is described as an ideal fluid. Basically, for the case of “dust”, i.e. zero pressure, that amounts to picturing galaxy clusters as “grains of dust”.

(1) By assumption, on a cosmological spacetime $(M, g_{\mu\nu})$, the stres-energy tensor of the matter distribution is

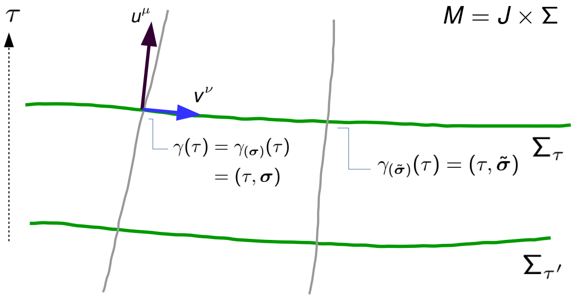
$$T_{\mu\nu} = (\varrho + p)u_\mu u_\nu - p g_{\mu\nu}$$

In the just mentioned picture of “dust” with dust grains = galaxy clusters, one may assume that their worldlines γ (with $\dot{\gamma}^\mu(q) = u^\mu(q)$) don’t intersect. Moreover, since we assume that there is spatial homogeneity and isotropy, there should be no rotational movements in the matter distribution. As the γ are timelike geodesics, this means they are hypersurface-orthogonal.

In technical terms, that means that $M = J \times \Sigma$ where J is an open interval – there is global time coordinate, τ, with values in J. Σ is a 3-dimensional manifold such that, for any $\tau \in J$,

$$\Sigma_\tau = \{\tau\} \times \Sigma \qquad \text{is an acausal, 3-dim submanifold of } M\,,$$
$$g_{\mu\nu} u^\mu v^\nu|_{\gamma(\tau)} = 0 \quad \text{for any vector } v^\nu \text{ which is tangential to } \Sigma_\tau \text{ in } \gamma(\tau)$$

Here, Σ_τ is called *acausal* in $(M, g_{\mu\nu})$ if, for any pair of points $q = (\tau, \boldsymbol{\sigma})$ and $\tilde{q} = (\tau, \tilde{\boldsymbol{\sigma}})$ in Σ_τ (where $\boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}} \in \Sigma$) there is no causal curve in Σ_τ connecting q and $\tilde{q}$.



Therefore, τ is a global time coordinate which is also a common affine parameter of all the worldlines (timelike geodesics) $\gamma(\tau) = \gamma_{(\boldsymbol{\sigma})}(\tau)$ of the "infinitesimal volume elements" $\hat{=}$ "dust grains" $\hat{=}$ galaxy clusters of the matter distributed in the Univserse. The swarm of these worldlines can be smoothly parametrized by the points $\boldsymbol{\sigma}$ of the "model space manifold" Σ.

The time coordinate τ is called **cosmological time**. At each cosmological time τ, the Σ can be viewed as the "space in which the matter distribution is instantaneously at rest".

(2) The matter distribution in Σ_τ is for every τ assumed to be homogeneous and isotropic. Since $(M, g_{\mu\nu})$ is to be a solution of Einstein's equations with that matter distribution, also the all restrictions $g_{\alpha\beta}|_{\Sigma_\tau}$ of the spacetime metric to any Σ_τ should be homogeneous and isotropic. Note all tangent vectors v^ν of Σ_τ are spacelike since Σ_τ is assumed as acausal. Therefore, $g_{\alpha\beta}|_{\Sigma_\tau}$ is, up to an overall minus sign, a Riemannian metric on Σ_τ.

Fixing τ, the **homogeneity** of the Riemannian manifold $(\Sigma_\tau, g_{\alpha\beta}|_{\Sigma_\tau})$ can be defined as follows:

Given any pair of points q and $\tilde{q}$ in Σ_τ, there is an *isometry* F of $(\Sigma_\tau, g_{\alpha\beta}|_{\Sigma_\tau})$ so that $F(q) = \tilde{q}$.

Here, an **isometry** F of $(\Sigma_\tau, g_{\alpha\beta}|_{\Sigma_\tau})$ is a C^∞, bijective (invertible and onto) map $F : \Sigma_\tau \to \Sigma_\tau$ which has the property that it leaves the metric length of all smooth curves $\gamma : S \to \Sigma_\tau$ unchanged, $\ell(\gamma) = \ell(F \circ \gamma)$.

Therefore, regarding the space geometry that the spacetime metric $g_{\mu\nu}$ induces on Σ_τ by its restriction $g_{\alpha\beta}|_{\Sigma_\tau}$, no point in Σ_τ is preferred. Note that this is for any *fixed* τ, or in other words, the maps F depend on τ. It is not required to have the same F for all τ.

For fixed τ, the **isotropy** of the Riemannian manifold $(\Sigma_\tau, g_{\alpha\beta}|_{\Sigma_\tau})$ can be defined a follows:

For any point q in Σ_τ and any tangent vector $v^\nu = \dot{\lambda}^\nu|_{q=\lambda(0)}$ of Σ_τ at q with a smooth curve $\lambda : (-\varepsilon, \varepsilon) \to \Sigma_\tau$, so that $g_{\mu\nu} v^\mu v^\nu = -1$, the following holds:

Given any other tangent vector w^ν of Σ_τ at q with $g_{\mu\nu} w^\mu w^\nu = -1$, there is an isometry of $(\Sigma_\tau, g_{\alpha\beta}|_{\Sigma_\tau})$ so that

$$F(q) = q \quad \text{and} \quad w = \frac{d}{ds} F(\lambda(s))\Big|_{s=0}$$

In other words, at every point q, any “unit” tangent vector v^μ of Σ_τ can be “rotated” into any other “unit” tangent vector w^μ of Σ_τ by an isometry. In that sense, there are no geometrically preferred directions in Σ_τ at any of its points.

The homogeneity and isotropy with respect to space as formulated so far are an expression of the *Copernicanian priciple*: The “local laws of physics over small time scales are everywhere the same”. We have already alluded to this earlier as an important ingredient for the interpretation of signals from remote stellar objects, in particluar with reagrd to redshift an distance determination.

The conditions on spatially homogeneous and isotropic cosmological spacetimes are very restrictive and reduce the possibility drastically. Work pioneered by Friedmann and Lemaître, and completed by Robertson and Walker, has led to the following result:

Any cosmological spacetime $(M, g_{\mu\nu})$ complying with the conditions (1) and (2) above (i.e. it is a solution to Einstein's equation with a homogeneous and isotropic mass distribution modelled as an ideal fluid) must be of the form

$$M = J \times \Sigma^{(k)}\,, \quad g_{\mu\nu} = d\tau_0 d\tau_0 - a(\tau)^2 h^{(k)}_{ij} \quad (k = -1, 0, 1; \;\; i, j = 1, 2, 3)$$

with certain 3-dim manifolds $\Sigma^{(k)}$ that will be introduced below, and with a smooth function $a : J \to (0, \infty)$.

The equation for the metric is to be read as follows:

On choosing local coordinates $(y^0 = \tau, y^1, y^2, y^3)$ for M, where τ is the coordinate for the interval J, and (y^1, y^2, y^3) are local coordinates for $\Sigma^{(k)}$, then

$$g_{00} = g(\partial_\tau, \partial_\tau) = 1\,, \quad g_{0j} = 0\,, \quad g_{ij}|_{(\tau,\boldsymbol{\sigma})} = -a(\tau)^2 h^{(k)}_{ij}|_{\boldsymbol{\sigma}} \quad (i, j = 1, 2, 3\,, \; \boldsymbol{\sigma} \in \Sigma^{(k)})$$

where $h^{(k)}_{ij}$ is a Riemannian metric on $\Sigma^{(k)}$. In other words, the $(\Sigma^{(k)}, h^{(k)}_{ij})$ $(k = -1, 0, 1)$ are 3-dimensional Riemannian manifolds – exactly, the following:

$k = 0$ $\Sigma^{(0)} = \mathbb{R}^3$ with the standard Euclidean metric as $h^{(0)}_{ij}$, i.e. $h^{(0)}_{ij} = \delta_{ij}$ when the (y^1, y^2, y^3) are the standard Euclidean coordinates.

$k = 1$ $\Sigma^{(1)} = S^3 = \{(a, b, c, d) \in \mathbb{R}^4 : a^2 + b^2 + c^2 + d^2 = 1\}$, the "3-sphere", with the Riemannian metric $h^{(1)}_{ij}$ that is induced by the Euclidean metric on the ambient $\mathbb{R}^4$ by the natural embedding of S^3 into $\mathbb{R}^4$.
Using as local coordinates $(y^1, y^2, y^3) = (\psi, \theta, \phi)$ the spherical polar coordinates for S^3, one has

$$h^{(1)}_{11} = 1\,, \quad h^{(1)}_{22} = \sin^2(\psi)\,, \quad h^{(1)}_{33} = \sin^2(\psi)\sin^2(\theta)\,, \quad h^{(1)}_{ij} = 0 \ \text{ if } \ i \neq j$$

$k = \text{-}1$ $\Sigma^{(1)} = H^3 = \{(a, b, c, d) \in \mathbb{R}^4 : a^2 - b^2 - c^2 - d^2 = 1\,,\ a > 0\}$ is the "upper part" of the unit hyperboloid embedded into $\mathbb{R}^4$. The Riemannian metric $h^{(-1)}_{ij}$ is induced by the Minkowski metric on $\mathbb{R}^4$ by the natural embedding of H^3 into $\mathbb{R}^4$.
Using as local coordinates $(y^1, y^2, y^3) = (\psi, \theta, \phi)$ the hyperbolic spherical coordinates for H^3, one has

$$h^{(-1)}_{11} = 1\,, \quad h^{(-1)}_{22} = \sinh^2(\psi)\,, \quad h^{(-1)}_{33} = \sinh^2(\psi)\sin^2(\theta)\,, \quad h^{(-1)}_{ij} = 0 \ \text{ if } \ i \neq j$$