Cosmology

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General Relativity: Kinematical Principle

The *kinematical principle* of General relativity consists of the following elements:

- Spacetime (= the "catalogue of all events") is described by $(M, g_{\mu\nu})$, a 4-dimensional manifold with a Lorentzian metric. (Already before we have referred to any such $(M, g_{\mu\nu})$ as a spacetime.)
- In the absence of other than gravitational interactions:

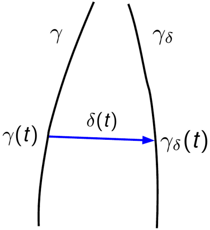
  $\star$ The worldlines of material particles (pointlike idealized) are *timelike geodesics*

  $\star$ The worldlines of light signals are *lightlike geodesics*.

As already indicated when the geodesic equation made its appearance, the effects of a gravitational field on test objects – the material point particles and light signals mentioned are, in this sense, test objects which are assumed not to act as a source of gravitational fields – are thereby described via a non-constant metric; more precisely, by non-vanishing Christoffel symbols. In fact, a spacetime $(M, g_{\mu\nu})$ is called **flat** if the Riemann tensor vanishes, i.e. if $\mathfrak{R}^{\nu}{}_{\sigma\lambda\mu} = 0$. One can show (again, a popular exercise) that this implies that the manifold M can be covered by coordinate charts in each of which the Christoffel symbols vanish, $\Gamma^{\lambda}_{\mu\nu} = 0$.

Therefore, gravitational effects in the sense that, no matter which coordinates are chosen, timelike or lightlike geodesics will deviate from straight coordinate lines, are related to the presence of a non-vanishing Riemann tensor, that is to say, non-trivial curvature.

Moreover, the gravitational deflection of test particles or light rays in an external gravitational field can be understood as an effect of curvature of the spacetime $(M, g_{\mu\nu})$. To this end, consider two very closely neighbouring timelike geodesics γ and γ_δ. Here, $\delta(t)$ is the (spacelike) minimal geodesic distance between $\gamma(t)$ and $\gamma_\delta(t)$, while t is the proper time parameter of the geodesic γ, i.e. $g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu = 1$.



Using appropriate coordinates, it can be shown that

$$\frac{1}{\delta}\left(\gamma_\delta^\mu(t) - \gamma^\mu(t)\right) = j^\mu(t) + O(\delta)$$

in the limit $\delta \to 0$, where j^μ is a vectorfield along γ which is γ-orthogonal, i.e. $g_{\mu\nu} j^\mu \dot{\gamma}^\nu = 0$. (Very roughly, j^μ is indicated by the blue arrow in the previous figure.)

$$v^\mu(t) = \frac{d}{dt} j^\mu(t) \quad \text{describes the velocity}$$

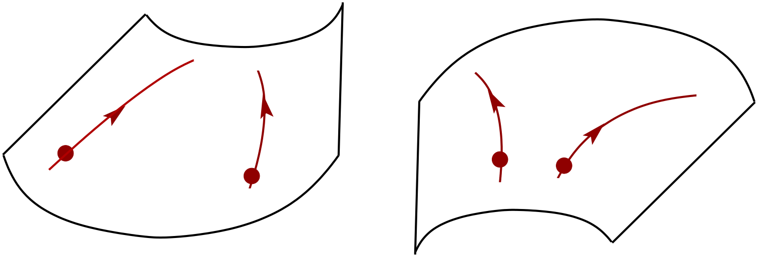
$$a^\mu(t) = \frac{d^2}{dt^2} j^\mu(t) \quad \text{describes the acceleration}$$

of "infinitesimally neighbouring" geodesics towards/away from each other, and it holds that

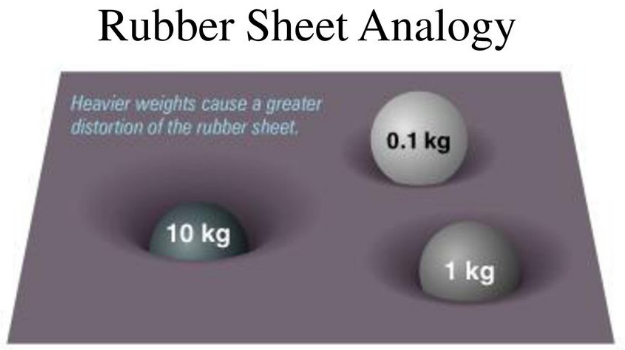
$$a^\nu = \mathfrak{R}^\nu{}_{\sigma\lambda\mu} \dot{\gamma}^\sigma \dot{\gamma}^\lambda j^\mu \quad (+\, O(\delta)) \quad \text{“Jacobi's eqn of geodesic deviation”}$$

in other words, the "relative acceleration of infinitesimally neighbouring geodesics" or "tidal forces" are expressed by the Riemann tensor of the spacetime metric in which the test particles are placed. This is very much in the spirit of the comparison with electrodynamics mentioned earlier: The geodesic equation is akin to the Lorentz force equation on a test charge in an external electromagnetic field, the curvature resembles the field strength and the metric the potentials.

There is a geometric analogy shown in the cartoon below. Two marbles on a curved surface move towards each other or away from each other, depending on the curvature (where tacitly the picture assumes a gravitational field acting from top to bottom of the page along the vertical axis). The situation on the left, where the marbles approach each other, is commonly referred to a positive curvature, whereas on the right, there is negative curvature, so that the marbles recede from one another.



However, this doesn’t yet take into account that point like masses – and photons – do not only react to external graviational fields, but they are also themselves “sources of curvature”. Taking up on the curved surface analogy of the previous slide, the curvature would be sourced by the masses present in spacetime. The more massive they are, the more positive curvature they create, and consequently the gravitational attraction becomes stronger. This is depicted in the cartoon below.



General Relativity: Dynamical Principle

The following dynamical principle incorporates the just mentioned ideas.

The spacetime metric of the spacetime $(M, g_{\mu\nu})$ is not given a priori but is dynamically coupled with the energy-matter content in spacetime; it is determined dynamically as a solution to **Einstein's field equations** of gravity:

$$G^{(\Lambda)}_{\mu\nu} = \kappa T_{\mu\nu}\,, \qquad \text{where}$$

$$G^{(\Lambda)}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} \quad \text{with} \quad G_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \quad \textbf{Einstein tensor}$$

$$T_{\mu\nu} = \textbf{stress-energy-tensor} \text{ of all matter/radiation in spacetime}$$

$$\kappa = \frac{8\pi \mathrm{G}}{c^4} \quad \text{with } \mathrm{G} = \text{Newton's gravitational constant, if } \; T_{00} \; \text{ is energy density}$$

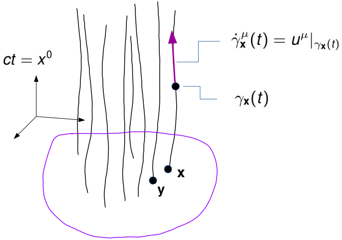
Furthermore, the real parameter Λ is called the **cosmological constant**. Its appearence signalizes that there is a freedom of choosing a parameter in Einstein's equations, which in some way has the flavour of a freedom in choosing a zero-point of energy.

One important point here is the underlying assumption that there is a tensor field $T_{\mu\nu}$ which specifies the mass-energy content of all matter and radiation in spacetime, called the *stress-energy-tensor*. This is (in classical GR, without regard for quantum matter) considered as in the description of macroscopic media in the sense of continuum mechanics/hydrodynamics.

We will here look at a particular, but very general way of describing a (continuous, macroscopic) matter distribution in spacetime and its associated stress energy tensor; it is also the way matter distributions are usually described in standard cosmology. Quite generally, in order to have a more complete picture of $T_{\mu\nu}$ and Einstein's field equations, one would have to specify a **matter model**. A matter model describes also the degrees of freedom of the matter and their interactions, other than gravitational. For most purposes in cosmology (and also in GR, more generally), one works in a thermodynamic limit where the non-gravitational interactions are subsumed by an *equation of state* which characterizes close-to-equilibrium situations of certain classes of matter.

Once a matter model is chosen, Einstein's equation's (together with eqns of motion due to other interactions) are to be viewed as an initial value problem. At some initial (global) time, specify matter distribution and spacetime metric consistently (together with first time-derivatives); then find solutions to Einstein's equations compatible with the initial conditions.

The picture below is a sketch of a continuous matter distribution – best thought of as a cloud of gas or a big liquid "drop". At some initial time ($t = 0$), the matter distribution occupies some volume, indicated as the area encircled by the purple line (in the picture, one space dimension is suppressed). Sketched are the worldlines of "infinitesimal volume elements" of the matter distribution, they are parametrized by the continuous spacepoints **x** (one has been denoted **y** for distinction) within the initially occupied volume. At every point $\gamma_{\mathbf{x}}(t)$ along such a worldline, there is the tangent vector $u^\mu|_{\gamma_{\mathbf{x}}}(t) = \dot{\gamma}^\mu_{\mathbf{x}}(t)$. We assume that the worldlines are timelike and that t is a common proper time parameter for all of them, so that $g_{\mu\nu}u^\mu u^\nu = 1$. In other words, the worldlines $\gamma_{\mathbf{x}}$ are the integral curves of the timelike vectorfield u^μ.



In the situation just described, the matter distribution is called an **ideal fluid** in equilibrium if

- the $\gamma_{\mathbf{x}}$ are geodesics
- the stress-energy tensor has the form

$$T_{\mu\nu}(q) = (\varrho + p)(q)u_\mu(q)u_\nu(q) - p(q)g_{\mu\nu}(q)$$

  at any spacetime point q within the mass distribution
  (i.e. $q = \gamma_{\mathbf{x}}(t)$ for some t, $\mathbf{x}$).
  Recall $u_\mu = g_{\mu\nu}u^\nu$.
  The function ϱ is called the **energy density** and the function p is called the **isotropic pressure** of the ideal fluid.
  Upon choosing at $q = \gamma_{\mathbf{x}}(t)$ a coordinate chart so that $\partial_{x^0}|_q = u|_q$ with $g(\partial_{x^\mu}|_q, \partial_{x^\nu}|_q) = \eta_{\mu\nu}$, the stress energy tensor has the form an ideal fluid takes in special relativity for the case that the "infinitesimal volume element" is "momentarily at rest" with respect to the chosen coordinates at q. Therefore, there appear no current-like quantities in this description of the fluid: It is described in its "co-moving rest frame", or what comes closest to it in general relativity. Note, however, that the energy density and pressure may vary in time and space.