Cosmology

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Geodesics

Let $(M, g_{\mu\nu})$ be a spacetime and let ∇ be the covariant derivative of $g_{\mu\nu}$. Furthermore, let $\gamma : s \mapsto \gamma(s) \in M$ (s in an interval S) be a smooth curve.

γ is called an **affinely parametrized geodesic** if

$$\dot{\gamma}^\mu \nabla_\mu \dot{\gamma}^\nu = 0 \qquad \text{all along the curve } \gamma$$

Strictly speaking, this geodesic equation is abuse of notation, since $\dot{\gamma}^\mu$ isn't a smooth vectorfield on M. However, in this equation, one can think of $\dot{\gamma}^\mu$ as being replaced by any smooth vectorfield v^μ on M which agrees with $\dot{\gamma}^\mu$ along γ ($v|_{\gamma(s)} = \dot{\gamma}(s)$ for all $s \in S$); there are many such vector fields. Since the covariant derivative is only taken "in the direction of $\dot{\gamma}$", the geodesic equation does not depend on how $\dot{\gamma}^\mu$ is extended to a smooth vectorfield away from the curve γ, and therefore there is actually no ambiguity upon writing the geodesic equation in the form as above.

In a coordinate chart (ϕ, M_Δ), writing $\gamma^\mu(s) = x^\mu(\gamma(s))$ where as usual, $x^\kappa = \phi^\kappa$ denote the chart's coordinate components, the geodesic equation taskes the form

$$\frac{d^2}{ds^2}\gamma^\mu(s) + \Gamma^\mu_{\lambda\varrho}(\gamma(s)) \left(\frac{d}{ds}\gamma^\lambda(s)\right) \left(\frac{d}{ds}\gamma^\varrho(s)\right) = 0 \quad (s \in S)$$

It is worth mentioning that the geodesic equation given here in the "affinely parametrized" form depends on the curve parametrization and that it is preserved exactly under affine re-parametrizations of the form $s \mapsto \tilde{s} = as + b$ with $a, b \in \mathbb{R}$, $a \neq 0$. For more details on this point, please refer to textbooks on GR.

To address the significance of the goedesic equation, think of $(M, g_{\mu\nu})$ as being Minkowski spacetime. Choosing the x^μ as inertial coordinates, the metric is the constant Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$. Since it is constant (independent of the coordinate positions $(x^0, \dots, x^3)^T$), and as the Christoffel symbols are formed from derivatives of the metric w.r.t. the x^μ, we see that $\Gamma^\mu_{\lambda\varrho} = 0$ everywhere on Minkowski spacetime (in inertial coordinates). Thus, in inertial coordinates on Minkowski spacetime, the geodesic equation reduces to

$$\frac{d^2}{ds^2}\gamma^\mu(s) = 0$$

and therefore, *on Minkowski spacetime, geodesics are straight lines in inertial coordinates*.

The following observations may be drawn from this:

- For a more general metric $g_{\mu\nu}$, geodesics are the generalization of "straight lines" with respect to the geometry described by the metric. We will soon see this point of view corroborated by the fact that geodesics characterize curves of extremal metric length.
- For Minkowski spacetime, consider a curve $\gamma^0(t) = t$, $\gamma^k(t) = x^k(t)$ $(k = 1, 2, 3)$ where t is the inertial time coordinate. We think of $\boldsymbol{x}(t) = (x^1(t), x^2(t), x^3(t))^T$ as the space trajectory of a test particle, assuming non-relativistic velocities with respect to the chosen inertial coordinate system. If the particle is subject to a conservative, static force $\boldsymbol{F}(\boldsymbol{x}) = -\mathbf{grad}\, U(\boldsymbol{x})$, then Newton's equation is

$$\frac{d^2}{dt^2}\gamma^k(t) + \frac{1}{m}\delta^{kj}\frac{\partial}{\partial x^j}U(\gamma(t))\left(\frac{d\gamma^0(t)}{dt}\right)\left(\frac{d\gamma^0(t)}{dt}\right) = 0$$

since $d\gamma^0(t)/dt = 1$. This shows that there is a close formal relationship between the geodesic equation and Newton's equation and that derivatives of the potential may be identified with the Christoffel symbols of a suitable, non-constant (in inertial coordinates) metric to describe gravitational effects. At this point, the identification $d\gamma^0(t)/dt = 1$ may appear a bit arbitrary, but this turns out as the right ansatz – again, I must refer to textbooks on GR for considerable further discussion.

Let again $(M, g_{\mu\nu})$ be a spacetime, and let $\gamma : [a, b] \to M, \quad s \mapsto \gamma(s)$ be a smooth curve. ($a < b$ are real numbers.)

The **length** of the curve γ (w.r.t. the metric $g_{\mu\nu}$) is defined as

$$\ell(\gamma) = \int_a^b |g_{\mu\nu}(\gamma(s))\dot{\gamma}^\mu(s)\dot{\gamma}^\nu(s)|^{1/2}\, ds$$

It is a popular exercise to show that this quantity is independent of the parametrization of the curve.

Geodesics are curves of extremal length

Let p, q be to points in M. A smooth curve $\gamma : [a, b] \to M$, with $\gamma(a) = p$ and $\gamma(b) = q$ is said to *connect* p and q. A standard result in differential geometry (usually explained and (sort of) proved in most GR textbooks) states the following (which applies to Riemannian as well as Lorentzian metrics):

Let p and q be two points in M. Then a smooth curve $\overset{\times}{\gamma} : [a, b] \to M$ connecting p and q *locally* extremalizes the length $\ell(\gamma)$ among all smooth curves γ connecting p and q if and only if $\overset{\times}{\gamma}$ is – possibly up to a non-affine re-parametrization – a geodesic (i.e. it fulfills the geodesic equation after a suitable re-parametrization).

Note the qualifier *locally* extremalizes – this has to do the fact that in general, there are many geodesics connecting (separate) points q and p in M.

(Think e.g. of the cylinder $\mathbb{R} \times S^1$ with its natural Riemannian metric which agrees with the Euclidean metric on $\mathbb{R}^2$ when "unrolling" the cylinder; if $p = (h_1, \phi_1)$ and $q = (h_2, \phi_2)$ with $h_1 < h_2$ are two points on the cylinder, there are infinitely many geodesics connecting the points: "Screwlines" with constant inclination, wrapping multiple times around the cylinder. They are geodesics because their "unrolled" counterparts are straight lines.)

More precisely, it means: There is an open neighbourhood N of (the image of) $\overset{\times}{\gamma}$ so that either $\ell(\overset{\times}{\gamma}) \geq \ell(\gamma)$ or $\ell(\overset{\times}{\gamma}) \leq \ell(\gamma)$ for all smooth curves $\gamma : S \to N$ connecting p and q (S any closed interval). This is similar to Hamilton's principle of extremal action in mechanics.

However, locally, geodesics connecting pairs of points are unique: One can show that every point $q' \in M$ possesses open neighbourhoods N so that, whenever $p, q \in N$, there is a <u>unique</u> geodesic $\overset{\times}{\gamma}$ in N (up to affine re-parametrization) connecting p and q. Any such neighbourhood is called a *convex normal neighbourhood* (of q').

If a geodesic $\overset{\times}{\gamma}$ connects two points p and q in a convex normal neighbourhood N, one has:

- If the metric $g_{\mu\nu}$ is Riemannian:

  $\overset{\times}{\gamma}$ minimizes the length $\ell(\gamma)$ among all curves γ in N connecting p and q.

- If the metric $g_{\mu\nu}$ is Lorentzian (i.e. $(M, g_{\mu\nu})$ is a spacetime), there are the following cases:

  - If $\overset{\times}{\gamma}$ is lightlike, then it is the only lightlike curve in N connecting p and q.
  - If $\overset{\times}{\gamma}$ is timelike, then it maximizes the length $\ell(\gamma)$ among all timelike curves γ in N connecting p and q.
  - If $\overset{\times}{\gamma}$ is spacelike, then it minimizes the length $\ell(\gamma)$ among all spacelike curves γ in N connecting p and q.

Running ahead a little bit – the worldlines of lightsignals ("lightrays") will be described as lightlike geodesics in GR. Therefore, to describe a gravitational lensing effect in GR which leads to multiple imaging – as is observed in astronomy, and is significant in certain contexts of cosmology – the spacetime point of the source p and of the observer q cannot lie in a convex normal neighbourhood of the spacetime metric.

Curvature

Again, our standing assumption is that $(M, g_{\mu\nu})$ is a spacetime.

Consider any coordinate chart with coordinate component functions x^μ.

The coordinate components of 2nd (and even higher order) coordinate derivatives on vector fields v^ν commute, i.e. it holds that

$$\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x^\mu}v^\nu = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\lambda}v^\nu$$

But in general, for 2nd covariant derivatives, this will not be the case: Typically (except for very special cases),

$$\nabla_\lambda(\nabla_\mu v^\nu) \neq \nabla_\mu(\nabla_\lambda v^\nu)$$

It turns out that the difference of the order-interchanged 2nd covariant derivatives are linear in v^ν and can be expressed by a tensor field, the *Riemannian curvature tensor*

The Riemannian curvature tensor, properties and descendants

There is a $\binom{1}{3}$ tensor field $\mathfrak{R}$ on M so that

$$\mathfrak{R}^\nu{}_{\sigma\lambda\mu}v^\sigma = \nabla_\lambda(\nabla_\mu v^\nu) - \nabla_\mu(\nabla_\lambda v^\nu)$$

for all vectorfields v^ν on M. This tensor field is called the **Riemannian curvature tensor** of the metric $g_{\mu\nu}$ (or simply **Riemann tensor**).

In any coordinate chart, it holds that

$$\mathfrak{R}^{\nu}{}_{\sigma\lambda\mu} = \frac{\partial}{\partial x^{\lambda}}\Gamma^{\nu}_{\sigma\mu} - \frac{\partial}{\partial x^{\mu}}\Gamma^{\nu}_{\sigma\lambda} + \Gamma^{\nu}_{\lambda\beta}\Gamma^{\beta}_{\sigma\mu} - \Gamma^{\nu}_{\mu\beta}\Gamma^{\beta}_{\sigma\lambda}$$

- Please check the formulas with the literature, as they are prone to typing errors.
- When doing so, please note that there are *different conventions* in use for defining the Riemannian curvature tensor. One deviating convention w.r.t. the one used here is to define the Riemannian curvature tensor with a minus sign (i.e. the negative of our definition). Another deviation is to put the index σ on which the vectorfield index is contracted as the rightmost lower index (whereas we have put it leftmost). Unfortunately, practically all possible conventions are in use, so **check carefully which conventions for curvature quantities are used by the authors!** (Unfortunately, the authors do not always indicate which conventions they are using.)

The Riemannian curvature tensor fulfills some important identities:

(i) $\mathfrak{R}^{\nu}{}_{\sigma\lambda\mu} = -\mathfrak{R}^{\nu}{}_{\sigma\mu\lambda}$

(ii) $\mathfrak{R}^{\nu}{}_{\sigma\lambda\mu} + \mathfrak{R}^{\nu}{}_{\lambda\mu\sigma} + \mathfrak{R}^{\nu}{}_{\mu\sigma\lambda} = 0$ (1st Bianchi identity)

(iii) $\nabla_{\varrho}\mathfrak{R}^{\nu}{}_{\sigma\lambda\mu} + \nabla_{\lambda}\mathfrak{R}^{\nu}{}_{\sigma\mu\varrho} + \nabla_{\mu}\mathfrak{R}^{\nu}{}_{\sigma\varrho\lambda} = 0$ (2nd Bianchi identity)

There are two further, important curvature quantities which can be formed from the Riemannian curvature tensor upon contraction:

$$\mathrm{Ric}_{\sigma\mu} = \mathcal{R}_{\sigma\mu} = \mathfrak{R}^{\lambda}{}_{\sigma\lambda\mu} \qquad \text{the **Ricci tensor**}$$

$$R = g^{\sigma\mu}\mathcal{R}_{\sigma\mu} \qquad \text{the **scalar survature**}$$

Note:

- $\mathrm{Ric}_{\sigma\mu}$ and $\mathcal{R}_{\sigma\mu}$ are synonymous notations for the Ricci tensor. The latter is usually not used for blackbord presentations since in handwriting, $\mathfrak{R}$ and $\mathcal{R}$ are very similar. Many authors also use $R_{\sigma\mu}$ to denote the Ricci tensor.
- Again, different sign convention are in use: Some authors define the Ricci tensor as the negative of our definition. So once more, check the sign conventions when comparing formulas!