Cosmology

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Given a spacetime $(M, g_{\mu\nu})$, one can distinguish smooth curves $\gamma : s \mapsto \gamma(s) \in M$ (s in an interval S) according to the properties of their tangent vectors $\dot\gamma(s) = \dot\gamma|_{\gamma(s)}$, in complete analogy to Minkowski spacetime:

(a) γ is *timelike* if $g_{\mu\nu}\dot\gamma^\mu\dot\gamma > 0$
(b) γ is *spacelike* if $g_{\mu\nu}\dot\gamma^\mu\dot\gamma < 0$
(c) γ is *lightlike* if $g_{\mu\nu}\dot\gamma^\mu\dot\gamma = 0$ and $\dot\gamma \neq 0$
(d) γ is *causal* if $g_{\mu\nu}\dot\gamma^\mu\dot\gamma \geq 0$ and $\dot\gamma \neq 0$

Note that these conditions are supposed to hold at every point along the curve γ, i.e. for all curve parameters $s \in S$.

One could also write $g(\dot\gamma, \dot\gamma)$ in an index-free notation instead of $g_{\mu\nu}\dot\gamma^\mu\dot\gamma$.

Similarly, one can define timelike, spacelike, lightlike and causal tangent vectors at any point in the spacetime.

Some authors use the terminology *null* instead of *lightlike*.

Covariant derivative

Given a manifold M (n-dimensional) one would like to define derivatives of tensor fields X.

If one starts by defining derivatives of the coordinate components of a tensor field with respect to a coordinate chart (ϕ, M_Δ), like e.g.

$$\frac{\partial}{\partial x^\mu} X^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s}(\phi^{-1}(x)) \quad (x \in \Delta)$$

there arises the problem that such a definition depends on the choice of the coordinate chart in that such expressions do not transform as the coordinate components of a tensor field under a change of the coordinate chart.

Similarly, the definition of derivatives of tensor fields should not depend on the particular embedding of an n-dimensional manifold into an ambient $\mathbb{R}^N$.

Once a manifold is endowed with a metric, one can define a unique concept of a derivative of tensor fields which is not dependent on choices of coordinate charts or embeddings: The *covariant derivative* associated to a metric.

We continue with the definition of several quantities which are important for introducing the covariante derivative.

Let $(M, g_{\mu\nu})$ be a spacetime, and let (ϕ, M_Δ) be a coordinate chart; the coordinate component functions are, as usual, denoted by $x^\kappa (= \phi^\kappa)$.

A $\binom{2}{0}$ tensor field $\tilde{g} = \tilde{g}^{\mu\nu}$ on M will be defined by defining the coordinate component matrix $(\tilde{g}^{\mu\nu})$ as the inverse matrix of the coordinate component matrix $(g_{\mu\nu})$ of the spacetime metric so that

$$\tilde{g}^{\mu\sigma} g_{\sigma\nu} = \delta^\mu{}_\nu$$

is the definining condition of $\tilde{g}$ in any coordinate chart.

One can show that with this definition, the $\tilde{g}^{\mu\nu}$ indeed transform like the coordinate components of a $\binom{2}{0}$ tensor field (a popular exercise problem).

It is customary to write $g^{\mu\nu}$ instead $\tilde{g}^{\mu\nu}$, and surprisingly this leads, in practice, not to much confusion (once one is experienced enough with the index notation).

Using the tensor fields $g_{\mu\nu}$ and $g^{\mu\nu}$, one can “raise and lower indices” of tensor fields, e.g. convert a vector field v^ν in a covector field $\omega_\mu = g_{\mu\nu} v^\nu$ by contraction; similarly, convert a covector field ξ_λ into a vector field $w^\sigma = g^{\sigma\lambda}\xi_\lambda$, or e.g. a $\binom{1}{2}$ tensor field $B^\tau{}_{\varrho\delta}$ into a $\binom{1}{2}$ tensor field $B_\mu{}^\nu{}_\delta = g_{\mu\tau} g^{\nu\varrho} B^\tau{}_{\varrho\delta}$. Mind the order of the indices here !!

Let again $(M, g_{\mu\nu})$ be a spacetime, and let (ϕ, M_Δ) be a coordinate chart, with coordinate component functions $x^\kappa (= \phi^\kappa)$.

Given the coordinate chart, the **(generalized) Christoffel symbols** of the metric $g_{\mu\nu}$ with respect to the coordinate chart are given by

$$\Gamma^\sigma_{\lambda\rho} = \frac{1}{2} g^{\sigma\alpha} \left(\frac{\partial}{\partial x^\rho} g_{\lambda\alpha} - \frac{\partial}{\partial x^\alpha} g_{\lambda\rho} + \frac{\partial}{\partial x^\lambda} g_{\rho\alpha} \right)$$

Caution: Despite the notation, the $\Gamma^\sigma_{\lambda\rho}$ **are not the coordinate components of a $\binom{1}{2}$ tensor field**! In that sense, they depend on the choice of a coordinate chart.

It follows from the symmetry of the metric, $g_{\mu\nu} = g_{\nu\mu}$, that the Christoffel symbols are also symmetric in the lower indices (in any coordinate chart):

$$\Gamma^\sigma_{\lambda\rho} = \Gamma^\sigma_{\rho\lambda}$$

The definition of the Christoffel symbols seems a bit ad hoc. However, it turns out that the failure of the $\Gamma^\sigma_{\lambda\rho}$ to transform as a tensor field can be used to compensate the failing tensor transformation property of expressions like $\frac{\partial}{\partial x^\mu} X^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s}(\phi^{-1}(x))$. This is basically what is behind the definition of the covariant derivative, appearing next.

Definition and properties of the covariant derivative

As before $(M, g_{\mu\nu})$ is a spacetime, and (ϕ, M_Δ) is an arbitrary coordinate chart for M, with $x^\kappa = \phi^\kappa$ denoting the coordinate component functions. $\Gamma^\sigma_{\lambda\rho}$ are the Christoffel symbols in the coordinate chart.

(i) Let v^μ be a vector field on M. Then the functions

$$Y^\nu{}_\mu(q) = \frac{\partial}{\partial x^\mu} v^\nu(\phi^{-1}(x))\Big|_{x=\phi(q)} + (\Gamma^\nu_{\mu\varrho} v^\varrho)(q) \quad (q \in M_\Delta))$$

transform like the coordinate component functions of a $\binom{1}{1}$ tensor field under change of the coordinate chart.

(ii) One writes

$$\nabla_\mu v^\nu = Y^\nu{}\mu$$

which is actually a slight abuse of notation since instead of $\nabla_\mu v^\nu$ one should write $(\nabla v)^\nu{}_\mu$ (or $(\nabla v)_\mu{}^\nu$, at this point it is a matter of convention) – because there is no sense in which ∇_μ could be viewed as the coordinate components of a covector field. Nevertheless, the notation $\nabla_\mu v^\nu$ is customary.

(iii) Therefore, ∇ is a linear map taking vectorfields v on M to $\binom{1}{1}$ tensor fields ∇v and in any local coordinate chart, the coordinate components of ∇v are

$$\nabla_\mu v^\nu \ \ (= (\nabla v)^\nu{}_\mu\,) \ = \frac{\partial}{\partial x^\mu} v^\nu + \Gamma^\nu_{\mu\varrho} v^\varrho$$

This map ∇ is called the **covariant derivative** of the metric $g_{\mu\nu}$ (on vectorfields). As mentioned before, the failures of $\frac{\partial}{\partial x^\mu} v^\nu$ and of $\Gamma^\nu_{\mu\varrho} v^\varrho$ to transform (individually) as components of a $\binom{1}{1}$ tensor field just compensate, so that the definition of the covariant derivative renders $\nabla_\mu v^\nu$ as the coordinate components of a $\binom{1}{1}$ tensor field.

(iv) The definition of the covariant derivative of a metric can be extended to tensor fields of any degree. To begin, in any coordinate chart one defines

$$\nabla_\mu f = \frac{\partial}{\partial x^\mu} f$$

for smooth, real-valued functions $f : M \to \mathbb{R}$ – the "tensor fields of degree $\binom{0}{0}$". Then one proceeds by requiring that the covariant derivative on tensor fields should fulfill the Leibniz rule, commute with contractions and that the metric should be "covariantly constant" with respect to ∇, i.e. $\nabla_\sigma g_{\mu\nu} = 0$.

(v) For covectorfields ω_ν this means, concretely,

$$\nabla_\mu \omega_\nu = \frac{\partial}{\partial x^\mu}\omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda$$

ie. $\nabla\omega$ is a $\binom{0}{2}$ tensor field with the coordinate components given in the previous equation.

(vi) More generally, a $\binom{r}{s}$ tensor field X is mapped under the covariant derivative to a $\binom{r}{s+1}$ tensor field ∇X whose general coordinate components are given by

$$\begin{aligned}
\nabla_\mu X^{\lambda_1\dots\lambda_r}{}_{\nu_1\dots\nu_s} &= \frac{\partial}{\partial x^\mu} X^{\lambda_1\dots\lambda_r}{}_{\nu_1\dots\nu_s} \\
&\quad + \sum_{\hat{r}=1}^{r} \Gamma^{\lambda_{\hat{r}}}_{\hat{\lambda}\mu} X^{\lambda_1\dots\hat{\lambda}\dots\lambda_r}{}_{\nu_1\dots\nu_s} \qquad (\hat{\lambda} \text{ at } \hat{r}\text{-th position}) \\
&\quad - \sum_{\hat{s}=1}^{s} \Gamma^{\hat{\nu}}_{\nu_{\hat{s}}\mu} X^{\lambda_1\dots\lambda_r}{}_{\nu_1\dots\hat{\nu}\dots\nu_s} \qquad (\hat{\nu} \text{ at } \hat{s}\text{-th position})
\end{aligned}$$

(vii) The properties

$$\nabla_\sigma g_{\mu\nu} = 0 \qquad \text{the metric is "covariantly constant"}\,,$$
$$\nabla(X \otimes Y) = (\nabla X) \otimes Y + X \otimes (\nabla Y) \qquad \text{Leibniz rule for tensor fields (any degree)},$$
$$\nabla_\mu(\omega_\varrho v^\varrho) = (\nabla_\mu \omega_\varrho)v^\varrho + \omega_\varrho(\nabla_\mu v^\varrho) \qquad \text{commutativity with contractions}$$
$$\text{(also for all tensor field degrees)}$$

can then be shown to hold for the covariant derivative of a metric.

Furthermore, the covariant derivative is *torsion-free*, which means that $\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$ for all smooth functions $f : M \to \mathbb{R}$, as a consequence of having required $\nabla_\mu f = \frac{\partial}{\partial x^\mu} f$.

One can show that there is a unique covariant derivative having all these properties — which is given in (i) above.

In the mathematical literature, the covariant derivative of a metric is also called the *Levi-Civitá derivative*.