Cosmology
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Coordinates and transformation rules

Let (ϕ, M_Δ) be a local coordinate chart for the n-dimensional smooth manifold M.
Denote the coordinate component functions by $x^\kappa = \phi^\kappa$, and let $q \in M_\Delta$.
Define a basis $(dx^1|_q, \ldots, dx^n|_q)$ of $T_q^* M$ by setting

$$dx^\kappa|_q(\dot{\gamma}|_q) = \left.\frac{d}{ds}\right|_{s=0} x^\kappa(\gamma(s))$$

for any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = q$.
The dual basis is a basis of $T_q M$ (note that for any finite-dimensional vector space, there is a canonical identification of the double dual space with the space itself), denoted by $(\partial_{x^1}|_q, \ldots, \partial_{x^n}|_q)$; it is defined by

$$dx^\kappa|_q(\partial_{x^\lambda}|_q) = \delta^\kappa{}_\lambda$$

Notation: Usually, the $|_q$, indicating at which point in M_Δ these basis elements are "affixed", is dropped from the notation – but it should be kept in mind that they depend on q.

Then, any $A = A_q \in (T^r_s)_q M$ can be expanded in these bases, with expansion coefficients, or *coordinate components*,

$$A^{\kappa_1,\ldots,\kappa_r}{}_{\lambda_1,\ldots,\lambda_s} = A(dx^{\kappa_1}, \ldots, dx^{\kappa_r}, \partial_{x^{\lambda_1}}, \ldots, \partial_{x^{\lambda_s}})$$

i.e. one has

$$(\star) \qquad A = \sum_{\kappa_\ldots,\lambda_\ldots} A^{\kappa_1,\ldots,\kappa_r}{}_{\lambda_1,\ldots,\lambda_s} \partial_{x^{\kappa_1}} \otimes \cdots \otimes \partial_{x^{\kappa_r}} \otimes dx^{\lambda_1} \otimes \cdots \otimes dx^{\lambda_s}$$

At this point, it is customary in the GR literature to introduce the **summation convention** (going back to Einstein): Instead of $(\star)$, drop the summation sign and instead write simply

$$A = A^{\kappa_1,\ldots,\kappa_r}{}_{\lambda_1,\ldots,\lambda_s} \partial_{x^{\kappa_1}} \otimes \cdots \otimes \partial_{x^{\kappa_r}} \otimes dx^{\lambda_1} \otimes \cdots \otimes dx^{\lambda_s}$$

so the summation convention means: *Doubly appearing indices (one of them an upper index, the other a lower index) are understood as being summed over* (unless otherwise indicated).

Thus, expressions like $Y^\kappa{}_\lambda y^\lambda$ actually mean $\sum_\lambda Y^\kappa{}_\lambda y^\lambda$, etc.

The summation convention will from now on be applied without explicit notice (unless indicated otherwise).

The next step is to consider the relation of coordinate components of tangent tensors defined with respect to different local coordinates. Thus, let (ϕ, M_Δ) and $(\overline{\phi}, M_{\overline{\Delta}})$ be two local coordinate charts for the n-dimensional smooth manifold M, and let $q \in M_\Delta \cap M_{\overline{\Delta}}$.

Denote the coordinate component functions by $x^\kappa = \phi^\kappa$ and $\overline{x}^\kappa = \overline{\phi}^\kappa$. The coordinate change maps are $\Psi = \phi \circ \overline{\phi}^{-1}$ and $\Psi^{-1} = \overline{\phi} \circ \phi^{-1}$ (defined on suitable open subsets of $\mathbb{R}^n$).

$D\Psi = (D\Psi^\kappa{}_\lambda)$ Jacobi-matrix of Ψ

$D\Psi^{-1} = (D\Psi^{-1\,\alpha}{}_\beta)$ Jacobi-matrix of Ψ^{-1}

This is usually written in the form

$$D\Psi^\kappa{}_\lambda = \frac{\partial x^\kappa}{\partial \overline{x}^\lambda}\,, \qquad D\Psi^{-1\,\alpha}{}_\beta = \frac{\partial \overline{x}^\alpha}{\partial x^\beta}$$

Then it holds that

$$\partial_{x^\lambda} = \frac{\partial \overline{x}^\kappa}{\partial x^\lambda}\partial_{\overline{x}^\kappa}\,, \qquad \partial_{\overline{x}^\beta} = \frac{\partial x^\alpha}{\partial \overline{x}^\beta}\partial_{x^\alpha}$$

$$d\overline{x}^\alpha = \frac{\partial \overline{x}^\alpha}{\partial x^\beta}dx^\beta\,, \qquad dx^\lambda = \frac{\partial x^\lambda}{\partial \overline{x}^\kappa}d\overline{x}^\kappa$$

The coordinate components of tangent tensors transform accordingly under a change of local coordinates.

E.g. if

$$A = A^{\kappa}{}_{\alpha\sigma}\partial_{x^\kappa} \otimes dx^\alpha \otimes dx^\sigma = \overline{A}^{\kappa}{}_{\alpha\sigma}\partial_{\overline{x}^\kappa} \otimes d\overline{x}^\alpha \otimes d\overline{x}^\sigma$$

then the coordinate components transform according to

$$\overline{A}^{\kappa}{}_{\alpha\sigma} = \frac{\partial \overline{x}^\kappa}{\partial x^\varrho}\frac{\partial x^\xi}{\partial \overline{x}^\alpha}\frac{\partial x^\delta}{\partial \overline{x}^\sigma}A^{\varrho}{}_{\xi\delta}$$

Tensorfields can then be defined as follows: An $\binom{r}{s}$ -**tensorfield** is a map which assigns to any $q \in M$ an $A_q \in (T^r_s)_q M$ with the property that in any local coordinate chart, the coordinate component functions

$$q \mapsto A^{\kappa_1,\ldots,\kappa_r}{}_{\lambda_1,\ldots,\lambda_s}(q) \quad \text{of } A_q \text{ are } C^\infty$$

This means, in the local coordinate chart (ϕ, M_Δ), that all the real-valued coordinate component functions

$$z \mapsto A^{\kappa_1,\ldots,\kappa_r}{}_{\lambda_1,\ldots,\lambda_s}(\phi^{-1}(z)) \quad (z \in \Delta \subset \mathbb{R}^n) \text{ are } C^\infty$$

(A mathematically fully satisfactory definition of tensor fields would require introducing tangent tensor bundles, and their smooth sections, but we will not do this here.)

$\binom{1}{0}$-tensorfields are also called *vectorfields*, and $\binom{0}{1}$-tensorfields are called *covectorfields* or *1-form fields*. In particular, for any local coordinate chart (ϕ, M_Δ) with coordinate component functions $x^\kappa = \phi^\kappa$,

$q \mapsto \partial_{x^\kappa}|_q \quad (\kappa = 1, \ldots, n)$ are vectorfields,

$q \mapsto dx^\kappa|_q \quad (\kappa = 1, \ldots, n)$ are covectorfields.

However, they are in general not defined on all of M, but only for $q \in M_\Delta$. Therefore, they are usually called *local co/vectorfields*.

It is also customary in GR to write $\binom{r}{s}$-tensorfields by writing their "general coordinate components", i.e. if A is an $\binom{1}{2}$-tensorfield, then one denotes it by $A^\kappa{}_{\alpha\sigma}$ instead of A. This has the advantage that the tensor type is always visible in the notation. Note that it should be written as $A^\kappa{}_{\alpha\sigma}$ and not, e.g. as $A^\kappa_{\alpha\sigma}$ since that can lead to ambiguities when carrying out operations on tensor fields like tensor products or contractions.

For tensor products, if e.g. A is a $\binom{1}{2}$-tensorfield and B is a $\binom{2}{2}$-tensorfield, then $C = A \otimes B$ is a $\binom{3}{4}$-tensorfield, and in coordinate components one has

$$C^{\alpha\varrho\sigma}{}_{\beta\delta\xi\tau} = A^\alpha{}_{\beta\delta} B^{\varrho\sigma}{}_{\xi\tau}$$

Note that here all index symbols need to be different, as otherwise summation over doubly appearing indices would be implied, or the expression is ambiguous.

The general coordinate component notation, or "index notation", for tensor fields has another advantage: Contractions can be written very simply and efficiently.

Suppose that v is a vectorfield and ω is a covectorfield. Then $\omega(v)$ is a function on the manifold, obtained by evaluating, at every $q \in M$, $\omega|_q \in T_q^*M$ on $v|_q \in T_qM$: $\omega(v)|_q = \omega|_q(v|_q)$. In the coordinate component notation, this can be very simply written as

$$\omega(v) = \omega_\kappa v^\kappa$$

and that is really what it is – keeping the summation convention in mind.

At this point, the advantage of the abstract index notation is not fully visible. But one can also think of $\omega(v)$ as forming the trace on $\omega \otimes v$: The coordinate components of $\omega \otimes v$ are $\omega_\kappa v^\lambda$, and forming the trace means summing over the diagonal elements of that component matrix, giving $\omega_\kappa v^\kappa$. Since it is a trace, it is already clear that it is a quantity which is independent of the dual/basis with respect to which ω and v are expanded (at every point q). This an example of a **contraction**, in this case of a $\binom{1}{1}$-tensorfield. It can be generalized to $\binom{r}{s}$-tensorfields if $r \geq 1$ and $s \geq 1$: E.g. suppose that $B^{\varrho\sigma}{}_{\xi\tau}$ is a $\binom{2}{2}$-tensorfield, then one can contract e.g. with respect to the first upper and the second lower index, to yield the $\binom{1}{1}$-tensorfield $C^\sigma{}_\xi = B^{\varrho\sigma}{}_{\xi\varrho}$.

Metric on a manifold

Suppose that M is an n-dimensional smooth manifold. Then a **metric** on M is a $\binom{0}{2}$-tensorfield $g_{\mu\nu}$ on M (using index notation) with the properties

- Symmetry: $g_{\mu\nu} = g_{\nu\mu}$
- Non-degeneracy: $g_{\mu\nu} v^\mu v'^\nu = 0$ for all vectorfields v'^ν implies $v^\mu = 0$

Clearly, non-degeneracy can also be expressed as $\det(g_{\mu\nu}) \neq 0$, so the coordinate component matrix of a metric is invertible (in any local coordinate chart, and for all $q \in M$).

If one wishes to write the symmetry property more abstractly without the use of indices, then it would read $g|_q(v|_q, v'|_q) = g|_q(v'|_q, v|_q)$ at every point $q \in M$ for all vectorfields v and v' on M. Similarly, the non-degeneracy means that, whenever v' is a vectorfield which is nowhere zero ($v'|_q \neq 0$ for all $q \in M$), then the covectorfield given by

$$v \mapsto g(v, v') \qquad \text{is nowhere zero} \quad \text{(i.e. there is some } v \text{ with } g(v, v') \neq 0 \text{ at any } q\text{)}.$$

As the coordinate component matrix $(g_{\mu\nu})$ of a metric is symmetric and invertible at every point $q \in M$, its normal form is a diagonal matrix with diagonal entries taking the values ± 1, and all off-diagonal entries equal to 0,

$$\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)\,, \qquad \epsilon_j = \pm 1$$

For a metric $g_{\mu\nu}$ on a manifold, the numbers $n_\pm =$ (number of the $\epsilon_j = \pm 1$) are constant over the manifold (as a consequence of the smoothness of the metric, being a tensorfield).

The pair of numbers n_+, n_- is therefore a characteristic feature of the metric, called the *signature* of the metric. Note that $n_+ + n_- = n$ (manifold dimension).

The most important cases (certainly for physics, but largely also in mathematics) are:

- $n_+ = n$, $n_- = 0$. In this case the metric $g_{\mu\nu}$ is called a **Riemannian metric**
- $n_+ = 1$, $n_- = n - 1$. In this case the metric $g_{\mu\nu}$ is called a **Lorentzian metric**

Remark. Of course, whenever one can make a choice of sign, then around 50% of the authors make opposite choices. That is to say, a metric with signature $n_+ = n - 1$, $n_- = 1$ is also called a Lorentzian metric. For the purposes of GR, it doesn't matter which choice of Lorentzian signature is made – the results don't depend on that choice, but there is a fair number of sign changes that one needs to keep track of when comparing the formulas referring to different signature conventions.

I will choose the signature convention $n_+ = 1$, $n_- = n - 1$ since it has the advantage that energies are always naturally positive, which is an advantage in cosmology. That said, many textbooks on GR actually use the opposite signature convention. Thus, whenever reading textbooks or articles related to GR or cosmology, **always check carefully which signature convention is used by the authors**.

Spacetime

Any pair $(M, g_{\mu\nu})$ where M is a 4-dimensional smooth manifold and $g_{\mu\nu}$ is a Lorentzian metric on M is called a **spacetime**.

A particular spacetime is, e.g., $M = \mathbb{R}^4$, with the metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ with respect to any inertial coordinate system for M: This is the Minkowski spacetime of special relativity.

Therefore, we see in what way the concept of spacetime generalizes special relativity: The manifold M of "events" (the markers of points in space and time) can be more general than $\mathbb{R}^4$ (but it is still 4-dimensional, corresponding to 1 dimension for a time-coordinate and 3 dimensions for space-coordinates), and the metric may vary from any spacetime point to another.

Futhermore, given any $q \in M$, one can find a local coordinate chart (ϕ, M_Δ) with $q \in M_\Delta$ so that the coordinate matrix $(g_{\mu\nu}(q))$ at that point takes the normal form, i.e.

$$(*) \qquad g_{\mu\nu}(q) = \eta_{\mu\nu}$$

so that, at the point q, one recovers special relativity upon use of that particular coordinate chart.

Caution: In general, equation $(*)$ will only hold at the single point q, not for all q of an open set with the coordinate components $g_{\mu\nu}$ of the metric referring to the same, fixed coordinate chart. We will see the reason for this a bit later.