Cosmology

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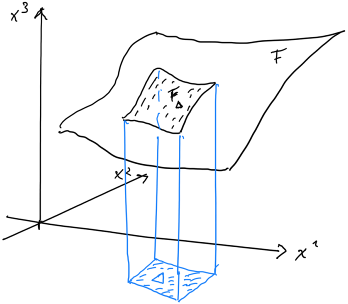
Manifolds

A simple and intuitive example of a manifold is a curved 2-dimensional surface embedded in an ambient $\mathbb{R}^3$



In this particular example, the patch $F_\Delta \subset F$ can be represented as

$$F_\Delta = \{q = (x^1, x^2, x^3)^T \in \mathbb{R}^3 \ : \ (x^1, x^2)^T \in \Delta \subset \mathbb{R}^2 , \ \ x^3 = f(x^1, x^1)\}$$

with some C^∞ (infinitely often differentiable) function $\quad f : \Delta \to \mathbb{R}$

In this manner, the points $(x^1, x^2)^T \in \Delta \subset \mathbb{R}^2$ characterize uniquely the points in $F_\Delta \subset F$, and the "projection map"

$$F_\Delta \to \Delta\,, \quad q = (x^1, x^2, f(x^1, x^2))^T \mapsto (x^1, x^2)^T$$

renders a "coordinatization" of points $q \in F_\Delta$ by coordinates $(x^1, x^2)^T \in \Delta$.

The existence of such (local) coordinatization maps is essential for the description of manifolds.

In what follows, manifolds will be introduced as "manifolds embedded in an ambient $\mathbb{R}^N$". From a certain mathematical point of view, that is not the most desirable way of introducing manifolds. However it is (i) no loss of generality, (ii) entirely sufficient for our purposes and (iii) in some ways more economical (shorter) and more intuitive.

Notation: Coordinate vectors should be written as column vectors, so we will denote them as "transposed row vectors" (which is more convenient for typing — and doesn't occupy so much space):

$$(x^1, \dots, x^n)^T = \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{pmatrix}$$

Definition of a manifold

Let $N \in \mathbb{N}$, and let $M \in \mathbb{R}^N$. Then M is called an *n-dimensional smooth manifold* ($n \leq N$) if the following holds:

(i) There is a finite covering $\{M_{\Delta(j)}\}_{j=1,\ldots,m}$ of M: $m \in \mathbb{N}$ and

$$M_{\Delta(j)} \subset M\,, \qquad \bigcup_{j=1,\ldots,m} M_{\Delta(j)} = M$$

The $M_{\Delta(j)}$ are also required to be open subsets of M. This means that for any j there is some open subset $D(j)$ of $\mathbb{R}^N$ so that $M_{\Delta(j)} = M \cap D(j)$.

[Note: One could generalize the definition by only demanding that the covering is locally finite. But for the purposes of these lectures, a finite covering is sufficient.]

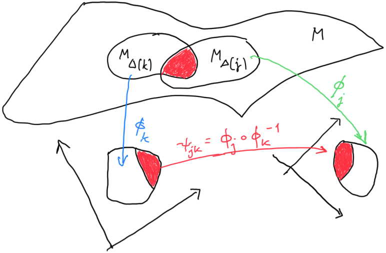
(ii) The $\Delta(j)$ are open subsets of $\mathbb{R}^n$, and for every index j there are bijective, continuous maps

$$\begin{aligned} \phi_{(j)} :& M_{\Delta(j)} \to \Delta(j) \\ & q \mapsto \phi_{(j)}(q) = (\phi^1_{(j)}(q), \ldots, \phi^n_{(j)}(q))^T \end{aligned}$$

The $(\phi_{(j)}, M_{\Delta(j)})$ are called *local coordinate charts* for the manifold M. For manifolds other than $M = \mathbb{R}^n$, the $M_{\Delta(j)}$ are in general proper subsets of M and then there are several of the $M_{\Delta(j)}$ having a non-void intersection. In that case, as part of the definition of a manifold, it is required that the local coordinate charts are C^∞-*compatible*. That means, whenever $(\phi_{(j)}, M_{\Delta(j)})$ and $(\phi_{(k)}, M_{\Delta(k)})$ have $M_{\Delta(j)} \cap M_{\Delta(k)} \neq \emptyset$, then the *coordinate change maps*

$$\Psi_{jk} = \phi_{(j)} \circ \phi_{(k)}^{-1} : \phi_{(k)}(M_{\Delta(k)} \cap M_{\Delta(j)}) \to \phi_{(j)}(M_{\Delta(k)} \cap M_{\Delta(j)})$$

are required to be bijective and C^∞.



There are other continuous, bijective maps $\phi : M_\Delta \to \Delta \subset \mathbb{R}^n$ which are also C^∞*-compatible* with every $(\phi_{(j)}, \Delta(j))$ whenever $M_\Delta \cap M_{\Delta(j)} \neq \emptyset$, and are not equal to any of the $(\phi_{(j)}, M_{\Delta(j)})$ (which define the manifold M in the first place). Then any such (ϕ, M_Δ) is also called a *local coodinate chart* for M.

Notation: If (ϕ, M_Δ) is any local coordinate chart for M, then it is customary to write

$$(\phi^1(q), \dots, \phi^n(q))^T = (x^1(q), \dots, x^n(q))^T$$

i.e. the *coordinate component functions* (sometimes simply called *coordinates*) of a local coordinate chart are typically denoted by x^κ instead of ϕ^κ. However, the notation for coordinate component functions is highly context-dependent: A coordinate function with the significance of a radial coordinate would typically be denoted by r, and a coordinate function with the significance of an angular coordinate by θ... etc.

Let M be an n-dimensional smooth manifold and let $S = (a, b)$ denote an open real interval. Then a map $\gamma : S \to M$ $(\subset \mathbb{R}^N)$ is called a *smooth parametrized curve* in M if, for any local coordinate chart (ϕ, M_Δ) for M, the curve

$$\phi \circ \gamma : S_\gamma \to \Delta \subset \mathbb{R}^n$$

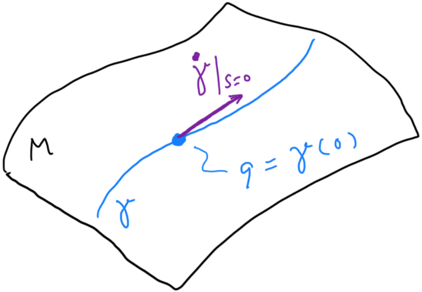
is a smooth curve (on any subinterval S_γ of S so that $\gamma(s) \in M_\Delta$ for all $s \in S_\gamma$).

It is not difficult to see that the smoothness properties of a curve $\gamma : S \to M \subset \mathbb{R}^N$ are fulfilled if and only if $\gamma : S \to \mathbb{R}^N$ is a C^∞ map into the ambient $\mathbb{R}^N$ wherein M is embedded.

For every smooth curve γ in M there is the concept of the tangent vector: Let $q \in M$ and let $\gamma : (-\varepsilon, \varepsilon) \to M$ (for some $\varepsilon > 0$) be a smooth curve such that $\gamma(0) = q \in M$. Then the derivative

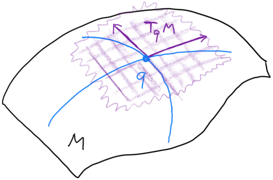
$$\dot{\gamma}|_{s=0} = \left.\frac{d}{ds}\right|_{s=0} \gamma(s)$$

is called the *tangent vector of* γ *at* q.



Any such a tangent vector $\dot{\gamma}|_{s=0}$ is in a natural way a vector in the ambient $\mathbb{R}^N$ which is viewed as being "affixed" at the point q.

Then one collects all tangent vectors of M at the point q into the *tangent space* T_qM of M at q. This set forms in a natural way an n-dimensional subspace of $\mathbb{R}^N$ and is spanned by the tangent vectors of all smooth curves at q.



The tangent spaces at any point have the dimension n and therefore, T_qM is isomorphic, as a vector space, to $\mathbb{R}^n$: $T_qM \simeq \{(q, v) : v \in \mathbb{R}^n\}$.
However, for different points in the manifold at which they are affixed, tangent vectors $(q, v) \in T_qM$ and $(p, z) \in T_pM$ are regarded as different and unrelated for $p \neq q$ (even if $v = z$ as elements of $\mathbb{R}^n$).

Starting from the tangent spaces $T_q M$ of a manifold M, one can build associated tensor spaces.

The *cotangent space* of M at q, denoted by $T_q^* M$, is the space of all (real-) linear maps

$$\omega_q : T_q M \to \mathbb{R}$$

Every cotangent space it isomorphic to $\mathbb{R}^n$. However, also here:

$$T_q^* M \simeq \{(q, \xi) : \xi \in \mathbb{R}^n\}$$

i.e. for $q \neq p$, the spaces $T_q^* M$ and $T_p^* M$ are regarded as distinct, and unrelated.

Extending the definition, one can form

$$(T_s^r)_q M = \underbrace{TqM \otimes \cdots \otimes T_q M}_{r} \otimes \underbrace{T_q^* M \otimes \cdots \otimes T_q^* M}_{s}$$

the space of *r-fold contravariant and s-fold covariant tangent tensors* of M in q.

Once again, the *tangent tensor spaces* $(T_s^r)_q M$ and $(T_s^r)_p M$ are unrelated for $q \neq p$.

The definition of the tensor product spaces – should you be unfamiliar with it – can be given as follows:

$A_q \in (T^r_s)_q M$ is a multilinear map

$$A_q : \underbrace{T^*_q M \times \cdots \times T^*_q M}_{r} \times \underbrace{T_q M \times \cdots \times T_q M}_{s} \to \mathbb{R}$$

where multilinearity means that

$$\begin{aligned} & A_q(\omega^{(1)}, \ldots, \omega^{(j)} + \lambda \omega'^{(j)}, \ldots, \omega^{(r)}, \dot\gamma_{(1)}, \ldots, \dot\gamma_{(s)}) \\ & \quad = A_q(\omega^{(1)}, \ldots, \omega^{(j)}, \ldots, \omega^{(r)}, \dot\gamma_{(1)}, \ldots, \dot\gamma_{(s)}) + \lambda A_q(\omega^{(1)}, \ldots, \omega'^{(j)}, \ldots, \omega^{(r)}, \dot\gamma_{(1)}, \ldots, \dot\gamma_{(s)}) \end{aligned}$$

for any $\lambda \in \mathbb{R}$, and similarly for all other entries.

$(T^r_s)_q M$ is in a canonical manner a real-linear space, isomorphic to $\mathbb{R}^{n(r+s)}$.

The operation of the *tensor product* $\otimes : (T^r_s)_q M \times (T^{r'}_{s'})_q M \to (T^{r+r'}_{s+s'})_q M$ is given by

$$\begin{aligned} & (A_q \otimes A'_q)(\omega^{(1)}, \ldots, \omega^{(r)}, \omega'^{(1)}, \ldots \omega'^{(r')}, \dot\gamma_{(1)}, \ldots, \dot\gamma_{(s)}, \dot\gamma'_{(1)}, \ldots, \dot\gamma'_{(s')}) \\ & \quad = A_q(\omega^{(1)}, \ldots\ldots, \omega^{(r)}, \dot\gamma_{(1)}, \ldots, \dot\gamma_{(s)}) A'_q(\omega'^{(1)}, \ldots, \omega'^{(r')}, \dot\gamma'_{(1)}, \ldots, \dot\gamma'_{(s')}) \end{aligned}$$