Cosmology

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Relativistic Doppler Effect

Suppose that an inertial system $\overline{S}$ is moving with respect to another one, S, with a velocity v along the x^1 direction (of S).

Let an event (like the emission of a flash of light) have coordinates

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{w.r.t.} \ \ S \qquad \text{and} \qquad \begin{pmatrix} \overline{x}^0 \\ \overline{x}^1 \\ \overline{x}^2 \\ \overline{x}^3 \end{pmatrix} \quad \text{w.r.t.} \ \ \overline{S}$$

Presently, we consider the setting of special relativity; the x^0 and $\overline{x}^0$ coordinates are the time-coordinates multiplied with c (velocity of light): $x^0 = ct$, $\overline{x}^0 = c\overline{t}$ in the inertial systems. Our sign convention for the Minkowski metric η is

$$\eta(w,z) = w^0 z^0 - w^1 z^1 - w^2 z^2 - w^3 z^3$$

According to special relativity,

$$\overline{x}^0 = \gamma(x^0 - \frac{v}{c}x^1)\,, \qquad \overline{x}^1 = \gamma(x^1 - \frac{v}{c}x^0)$$

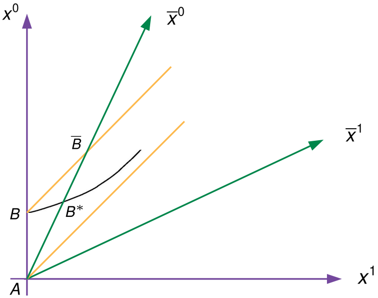
$$\overline{x}^k = x^k \quad (k = 2,3)\,, \qquad \gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}$$

The Lorentz transformation relating the inertial coordinates of S and $\overline{S}$ can also be written in the form

$$\begin{pmatrix} \overline{x}^0 \\ \overline{x}^1 \\ \overline{x}^2 \\ \overline{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

c is the velocity of light in vacuum. Commonly it is set equal to 1, so all velocities are given in units of c.

The Doppler effect is illustrated on the next slide in a spacetime diagram, where in fact we use the convention that $c = 1$.



Spacetime diagram illustrating the Doppler effect. $c = 1$ so lightrays are at 45° inclination. The experimenter located at $x^1 = x^2 = x^3 = 0$ in S emits light pulses at events A and B with proper time difference $\Delta x^0 = 1$, they propagate as the golden lightrays shown. The experimenter in $\overline{S}$ is at rest at $\overline{x}^1 = \overline{x}^2 = \overline{x}^3 = 0$ and records the arrival of the light pulses. At event A with $x^0 = 0 = \overline{x}^0$ the experimenters happen to be coincident. The experimenter in $\overline{S}$ records the light pulse emitted at B in the event $\overline{B}$. However, the event corresponding to $\Delta\overline{x}^0 = 1$ in proper time from the event A in $\overline{S}$ is at B^* where the $\overline{x}^0$-axis intersects the hyperbola $\eta(x, x) = 1 = \eta(\overline{x}, \overline{x})$. This is prior to $\overline{B}$ on the $\overline{x}^0$ axis.

In the further analysis of the spacetime diagram, we ignore the x^2, x^3 and $\overline{x}^2, \overline{x}^3$ coordinates as they always remain equal to 0.

Coordinates of A in S and $\overline{S}$:

$$\begin{pmatrix} x^0(A) \\ x^1(A) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \overline{x}^0(A) \\ \overline{x}^1(A) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Coordinates of B in S:

$$\begin{pmatrix} x^0(B) \\ x^1(B) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Next determine the coordinates of $\overline{B}$ in S:

$$\begin{pmatrix} x^0(\overline{B}) \\ x^1(\overline{B}) \end{pmatrix} \quad \text{is the intersection of the line } \tau \mapsto \begin{pmatrix} 1 \\ v \end{pmatrix} \cdot \tau \,,$$

i.e. the $\overline{x}^0$-coordinate axis, with the lightray

$$\lambda \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \lambda + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{emitted at B}$$

The condition of intersection determines

$$\tau = \frac{1}{1-v} \quad \text{and} \quad \lambda = v\tau \,, \qquad \text{hence}$$

$$\begin{pmatrix} x^0(B) \\ x^1(B) \end{pmatrix} = \begin{pmatrix} \frac{1}{1-v} \\ \frac{v}{1-v} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \overline{x}^0(\overline{B}) \\ \overline{x}^1(\overline{B}) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{1+v}}{\sqrt{1-v}} \\ 0 \end{pmatrix}$$

The frequency of light pulse emission in S is

$$\nu = \frac{1}{x^0(B) - x^0(A)} = \frac{1}{x^0(B)} = 1 \,,$$

The frequency of light pulse recording in $\overline{S}$ is

$$\overline{\nu} = \frac{1}{\overline{x}^0(\overline{B}) - \overline{x}^0(A)} = \frac{1}{\overline{x}^0(\overline{B})} = \frac{\sqrt{1-v}}{\sqrt{1+v}}$$

Summarizing the previous discussion, we have derived the

Doppler frequency shift formula in special relativity:

Assume light signals are emitted with angular frequency ω in an inertial system S. If another inertial system $\overline{S}$ moves relative to S with constant velocity v in the (fixed) direction of propagation of the light signals, then the light signals are recorded in $\overline{S}$ with angular frequency

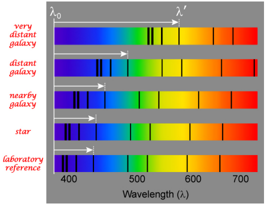
$$\overline{\omega} = \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}\,\omega$$

Note: We have reinserted c, the velocity of light, to facilitate comparison with the literature.

The velocity v can be positive or negative. If $v > 0$ then $\overline{S}$ moves away from S, if $v < 0$ then $\overline{S}$ moves towards S. Correspondingly, recorded light is redshifted in $\overline{S}$ if $v > 0$, and blueshifted if $v < 0$.

Redshift

The redshift effect in cosmology is the effect that spectral lines of distant stellar objects appears redshifted: The gaps between characteristic emission spectra of chemical elements are broadened, and correspondingly the spectral lines are shifted to higher wavelengths, as illustrated in the cartoon (taken from physics.stackexchange):



To conclude that the redshift is due to a recession velocity relative to the observer on Earth, one must, of course, assume that the spectral lines in the rest system of the stellar object are the same as would be observed on Earth, in Earth's rest system. This is part of the **Copernicanian principle** already mentioned before.

Suppose a spectral line has on Earth the (angular) frequency $\omega^{(0)}$. If the same spectral line is emitted in the rest system S of a stellar object, and if the Earth is moving with velocity v away from the stellar object, then the frequency of the spectral line emitted from the stellar object is, according to the Doppler shift formula, seen on Earth (at rest in $\overline{S}$) with the frequency

$$\overline{\omega}^{(0)} = \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}\,\omega^{(0)}$$

Using wavelegths $\lambda = 2\pi c/\omega$ instead of frequencies,

$$\overline{\lambda}^{(0)} = \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}}\,\lambda^{(0)}$$

The relative shift in wavelength,

$$z = \frac{\overline{\lambda}^{(0)} - \lambda^{(0)}}{\lambda^{(0)}} = \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}} - 1 = \frac{v}{c} + O((v/c)^2)$$

is called the **redshift** of the spectral line.

It is, in fact, independent of the wavelength $\lambda^{(0)}$.

The Hubble law – distance-redshift relation

In his observations $\sim$ 1927-29, Edwin Hubble has found a linear relation between the redshift z of spectral lines and luminosity distance d_L for galaxies in the neighbourhood of our galaxy:

$$c \cdot z = H_0 \cdot d_L \qquad (z \leq 0.1)$$

Here, $d_L = d$ is he **luminosity distance** appearing in the relation between absolute luminosity L and apparent luminosity ℓ of the stellar object (lecture 5, p 27). For small $z \leq 0.1$, H_0 is to good approximation constant, and called the **Hubble constant**. Its value is

$$H_0 = 71 \pm 6 \frac{\text{km}}{s \cdot Mpc} \approx h \cdot 100 \frac{\text{km}}{s \cdot Mpc}$$

where $h = 0.7$ is a dimensionless constant in common use in cosmology (not to be confused with Planck's constant).

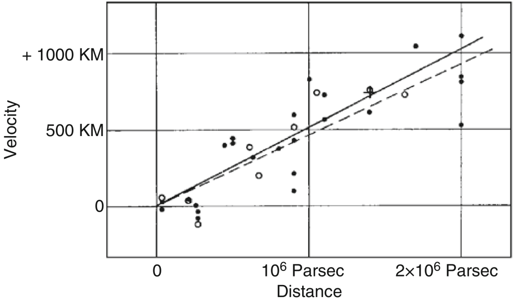


Fig. 1.13 The original 1929 version of the Hubble diagram shows the radial velocity of galaxies as a function of their distance. The reader may notice that the velocity axis is labeled with erroneous units—of course they should read km/s. While the radial (escape) velocity is easily measured by means of the Doppler shift in spectral lines, an accurate determination of distances is much more difficult; we will discuss methods of distance determination for galaxies in Sect. 3.9. Hubble has underestimated the distances considerably, resulting in too high a value for the Hubble constant. Only very few and very close galaxies show a blueshift, i.e., they move towards us; one of these is Andromeda (=M31). Adapted from: E. Hubble 1929, *A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae*, Proc. Nat. Academy Sciences 15, No. 3, March 15, 1929, Fig. 1

The original diagram showing recession velocity (redshift) vs distance for nearby galaxies by E. Hubble, with comments, taken from Peter Schneider's textbook *Extragalactic astronomy and Cosmology*.

Redshift as distance indicator

If the relation between luminosity distance and redshift is assumed to be valid for more distant stellar objects than nearby galaxies, it can serve at a distance indicator if there is no way to determine the absolute luminosity of the stellar object.

Doing so is not without problems since for larger distances (higher z), there is no more a linear relation between z and d_L. This is an indication for “dark energy”, or the Λ-term in the ΛCDM standard cosmological model. We will come back to this in a while.

For the time being – where we assume that the Universe can be modelled by Minkowski spacetime up to the largest observable scales – we take it that the Hubble law (or if needed, a corrected relation between luminosity distance and redshift) is valid for any value of z.

Cosmic distance ladder

What then results is the cosmic distance ladder – the point is that there are different methods of distance determination which are effective at different distance scales. The higher scale methods use the lower scale methods as input, and are calibrated at the overlaps of the scale boundaries, as described before.