UNIVERSITÄT LEIPZIG

Institut für Theoretische Physik

Prof. Dr. R. Verch

Dr. J. Zahn

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Quantum Field Theory in Curved Spacetime
Problem Sheet 8

Problem 8.1 [12 points]

We use the notation of Problem 7.2. Consider the Heisenberg time evolution of the harmonic oscillator with frequency $\omega = \sqrt{k/m}$, i.e.,

$$\hat{q}(t) = \cos\omega t \hat{q} + (km)^{-\frac{1}{2}} \sin\omega t \hat{p}, \tag{0.1}$$

$$\hat{p}(t) = \cos\omega t \hat{p} - (km)^{\frac{1}{2}} \sin\omega t \hat{q}. \tag{0.2}$$

1. Show that
$$\hat{\phi}(q,p)(t) = \hat{\phi}(q(t), p(t)) \tag{0.3}$$
with
$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & -(km)^{\frac{1}{2}} \sin\omega t \\ (km)^{-\frac{1}{2}} \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{0.4}$$

2. Show that a pure quasifree state with two-point function
$$\mu_\xi((q,p),(\tilde{q},\tilde{p})) = \frac{1}{2\xi} q\tilde{q} + \frac{\xi}{2} p\tilde{p} \tag{0.5}$$
is invariant under time-evolution, i.e.,
$$\mu_\xi((q(t),p(t)),(\tilde{q}(t),\tilde{p}(t))) = \mu((q,p),(\tilde{q},\tilde{p})), \tag{0.6}$$
if and only if
$$\xi = (km)^{\frac{1}{2}}. \tag{0.7}$$
We refer to this state as the vacuum state.

3. We prepare the system in the vacuum state, but temporarily perturb the harmonic oscillator by changing $k \to k'$ (hence also $\omega \to \omega'$) for a time interval $[0, T = \frac{\pi}{2\omega'}]$. Show that after the perturbation has been switched off again, the system is in the state
$$\mu_\xi((q(T),p(T)),(\tilde{q}(T),\tilde{p}(T))) = \mu_{\xi'}((q,p),(\tilde{q},\tilde{p})) \tag{0.8}$$
with $\xi' = \frac{k'}{k}\xi$ and ξ given by (0.7).

4. Recall that, for ξ arbitrary,
$$\pi_\xi(\hat{q}) = \frac{1}{\sqrt{2\xi}}(a_\xi^\dagger + a_\xi), \tag{0.9}$$
$$\pi_\xi(\hat{p}) = i\frac{\sqrt{\xi}}{\sqrt{2}}(a_\xi^\dagger - a_\xi), \tag{0.10}$$

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where π_ξ is the GNS representation corresponding to the state μ_ξ. Assuming that there is an isometry $U : \mathcal{F}_{\xi'} \to \mathcal{F}_\xi$ intertwining the representations, i.e.,

$$U\pi_{\xi'}(\cdot)U^{-1} = \pi_\xi(\cdot), \tag{0.11}$$

show that it follows that

$$Ua_{\xi'}U^{-1} = Ca_\xi + Da_\xi^\dagger, \tag{0.12}$$

with constants C, D to be determined in terms of ξ, ξ'. Show that

$$|C|^2 - |D|^2 = 1. \tag{0.13}$$

5. Let $\Omega_{\xi'}$ be the vacuum state in the ξ' representation. Show that the state $\Phi = U\Omega_{\xi'}$ can be characterized by

$$(Ca_\xi + Da_\xi^\dagger)\Phi = 0. \tag{0.14}$$

Solve this explicitly in the Fock (particle number) basis.

6. Upon trivially identifying the Fock spaces $\mathcal{F}_{\xi'}$ and $\mathcal{F}_\xi$, i.e., equating the coefficients in the particle number basis, the relation (0.12) reads

$$UaU^{-1} = Ca + Da^\dagger. \tag{0.15}$$

Show that the operator U is implemented by

$$U = e^{r(aa - a^\dagger a^\dagger)/2} \tag{0.16}$$

with some real r to be determined in terms of C, D.

Problem 8.2* [8 extra points]

Let $\mathcal{H}$ be a Hilbert space, with a Hamilton operator H defined and self-adjoint on a dense domain $D(H)$ in $\mathcal{H}$. Suppose that there is an orthonormal basis $\{\psi_n\}$ in $D(H)$ of eigenvectors of H with corresponding eigenvalues $\epsilon_n \geq 0$. (The index n runs from 1 to $\dim(\mathcal{H})$, which may be finite or infinite.) Suppose that, for any $\beta > 0$, it holds that $Z_\beta = \sum_n \mathrm{e}^{-\beta\epsilon_n}$ exists. This implies, in particular, that $\varrho_\beta = \mathrm{e}^{-\beta H}$ is a trace-class operator. Then consider the Gibbs state, or canonical ensemble,

$$\langle A \rangle_\beta = \frac{1}{Z_\beta}\mathrm{Tr}(\varrho_\beta A) \qquad (A \in \mathsf{B}(\mathcal{H}))$$

at inverse temperature $\beta > 0$. ($\mathrm{Tr}(X)$ denotes the trace of a trace-class operator X.)

(1) Taking as observable algebra $\mathcal{A} = \mathsf{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$, the time-evolution of the system (in the Heisenberg picture) is given by the 1-parametric group $\{\alpha_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathcal{A}$ defined by

$$\alpha_t(A) = \mathrm{e}^{itH} A \mathrm{e}^{-itH} \qquad (A \in \mathsf{B}(\mathcal{H}))$$

Show that the Gibbs state $\langle\,.\,\rangle_\beta$ is a KMS state on $\mathcal{A}$ at inverse temperature β with respect to the automorphism group $\{\alpha_t\}_{t\in\mathbb{R}}$.

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(2) Let A and B be in $\mathcal{A}$ and consider the functions

$$\varphi(t) = \langle A\alpha_t(B)\rangle_\beta\,, \qquad g(t) = \langle \alpha_t(B)A\rangle_\beta \qquad (t \in \mathbb{R})$$

Show that

$$\hat{\varphi}(k) = \mathrm{e}^{\beta k}\hat{g}(k) \qquad (k \in \mathbb{R})$$

where the hat means the Fourier transform. Show that, if $B = A = A^*$, this implies

$$\hat{\varphi}(k) = \mathrm{e}^{\beta k}\hat{\varphi}(-k)$$

Note: (i) the Fourier transforms of φ and g are, strictly speaking, distributions, although one can formally pretend that they are functions.
(ii) the relation stated for the Fourier transforms of φ and g (for any A, B in $\mathcal{A}$) is implied by the KMS condition, and actually (but this is more difficult and requires techniques relating Fourier transform and analyticity like the Paley-Wiener Theorem, see Sec. IX in the 2nd Vol of Reed and Simon) one can show that it is equivalent to the KMS condition.